How Humans Learn to Think Mathematically

With applications to the Calculus

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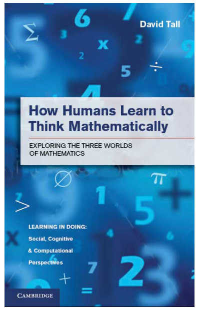
1

The presentation includes recent developments of the theory of three worlds of mathematics as applied to the calculus. It blends together the nature of mathematics and the way that mathematical thinking is processed in the brain and related to emotional and social aspects. This is not the full story. It is focused on what may be possible speaking to a seminar of mathematicians in a limited time.

It is an honour to be invited to present to you my ideas of *How Humans Learn to Think Mathematically*. I thank Masami Isoda for inviting me here today and acknowledge our common interests. In particular, when we first met, he spoke about the way in which students built on their experience of the tangent in circle geometry and how the idea that it touches the circle at one point without passing through caused conflict with the more general concept of tangent in the calculus. Our common interests and differences and subsequent collaboration on Lesson Study have been a constant encouragement in the development of the theory I present today.

X



This presentation celebrates the publication of the Japanese translation of

How Humans Learn to Think Mathematically
in Three Worlds of Mathematics
based on perception, operation & reason.

This presentation offers an overview and focuses on the evolution of ideas in the Calculus (chapter 11, also chapter 13).

The plan is to encourage the student to make sense of mathematics at each stage, to develop powerful techniques and to learn to reason in appropriate ways.

It extends Lesson Study throughout the full range of mathematics including historical and individual evolution of mathematical ideas.

2

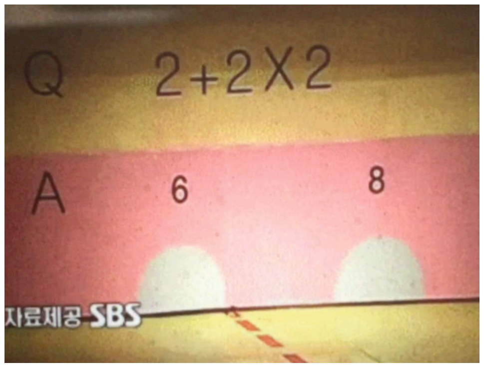
This presentation celebrates the publication of the Japanese translation of *How Humans Learn to Think Mathematically*. in a framework of **Three worlds of mathematics based on perception, operation & reason.** **X**

@ It offers an overview and focuses on the evolution of ideas in the Calculus (chapter 11, also chapter 13). **X**

@ The plan is to encourage the student to make sense of mathematics at each stage, to develop powerful techniques and to learn to reason in appropriate ways. **X**

@ The framework extends Lesson Study throughout the full range of mathematics including historical and individual evolution of mathematical ideas. **X**

Making Sense of Mathematics



3

Let us begin with how a student may make sense of an operation in arithmetic. **X**

@ (run video)

Making Sense of Mathematics

In mathematics, the symbol 2+2x2 means

2+4 which is 6

because multiplication takes precedence over addition.

The student is using the experience of reading symbols from left to right where

2+2 is 4 and 4x2 is 8.

4

In mathematics, the symbol 2+2x2 means
@ 2+4 which is 6
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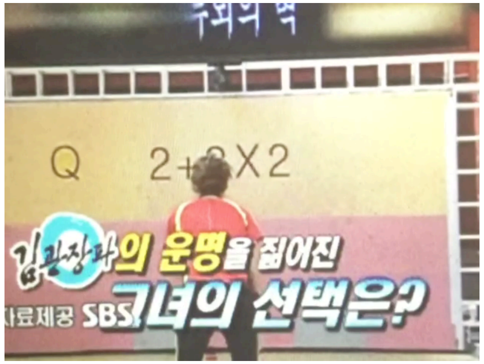
@ 2+2 is 4 and 4x2 is 8. **X**

Making Sense of Mathematics



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[run video, no comment necessary]

Making Sense of Mathematics

6

@ Making sense builds on previous experience.
@ Sometimes previous experience is appropriate and helps to make sense in a new situation, but sometimes it is problematic.
@ In this case, reading from left to right works in reading but requires a change in meaning in mathematics. **X**
@ Sense-making involves changes in meaning as mathematics becomes more sophisticated.
@ This happens surprisingly often in long-term learning and if sense is not made, later learning may become learning by rote without making sense.
@ This presentation considers the long term evolution of mathematical sense-making based on the increasing sophistication of human perception, action and reason. **X**

The Three Worlds of Mathematics

A framework for cognitive development, where each world is based on human perception, action & reason.

Actions performed for a specific reason are called operations, including constructions in geometry and symbolic operations in arithmetic & algebra.

***Embodied**: **properties of physical & mental objects** formulating verbal definitions,reasoning about relationships*

Symbolic: mathematical operations performed initially on real world objects that may be symbolised and then manipulated as mental objects

Formal: based on verbal/logical definition of properties with other properties deduced by mathematical proof.

7

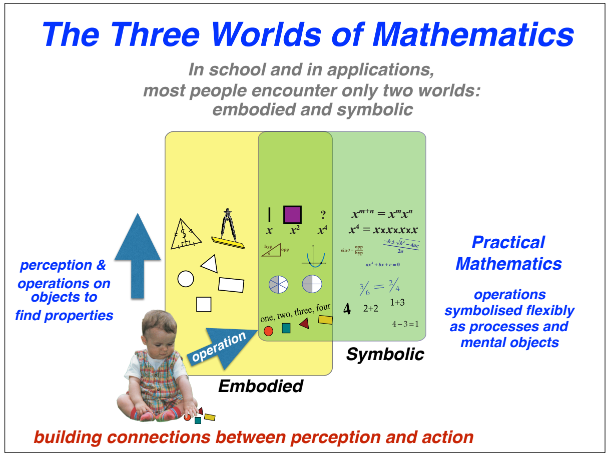
@ The three worlds of mathematics is **a framework for the *cognitive development*** of mathematical thinking, where **each worlds is based on human perception, action & reason.**

@ Actions performed with a specific purpose are called *operations* including constructions in geometry and symbolic operations in arithmetic and algebra. X

@ The **embodied world** explores the perceptual **properties of physical and mental objects**, formulating verbal definitions reasoning about relationships. **X**

@ The **symbolic world** develops out of **mathematical operations** performed initially on real world objects, that are **symbolised** and **manipulated as mental objects**. **X**

@ The **formal world** is **based on verbal/logical definition with other properties deduced by mathematical proof**. **X**



8

In school and in applications, most people encounter only two worlds: the embodied and symbolic **X**

@ The young child is born with a brain still maturing and spends the first months

@ **building connections between perception and action. X**

@ Using **perception and operations on objects to find properties** in what I term the

@ **Embodied** world. **X**

@ **Operation** on objects, **one, two, three, four** leads to the number concept **4.**

@ Operations can be performed on numbers and **different operations** such as 2 + 2 or 1 + 3 or 4 - 3 refer to **the same mental object**.

@ This leads to what I term the **Symbolic** world where

@ operations are symbolised flexibly as both processes and mental objects. **X**

@ Taking a fraction of an object, such as three sixths or two quarters, can give the same quantity, leading to the concept of equivalence of fractions. **X**

@ Equivalent fractions now refer flexibly to a single rational number. **X**

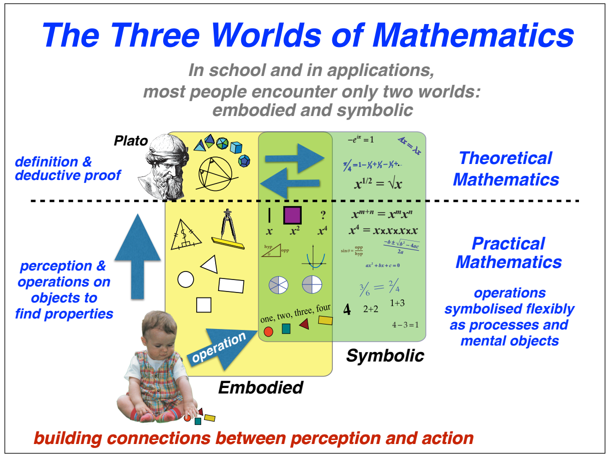
@ Successive embodied operations with triangles and graphs correspond to symbolic operations in trigonometry and algebra. **X**

@ Embodiment and symbolism do not always have a simple relationship. If *x* is a length, *x*^2 is an area, but x^4 has no obvious meaning. **X**

@ However, symbolically, x to the fourth is just x times x times x times x, which has the meaning of four xs multiplied together. **X**

@ This generalises to the power law for whole numbers m, n, where x to the power m + n equals x to the m times x to the n. **X**

I refer to all of this as @ **Practical Mathematics**. **X**



9

@ This shifts to a more sophisticated level that I term **Theoretical mathematics**

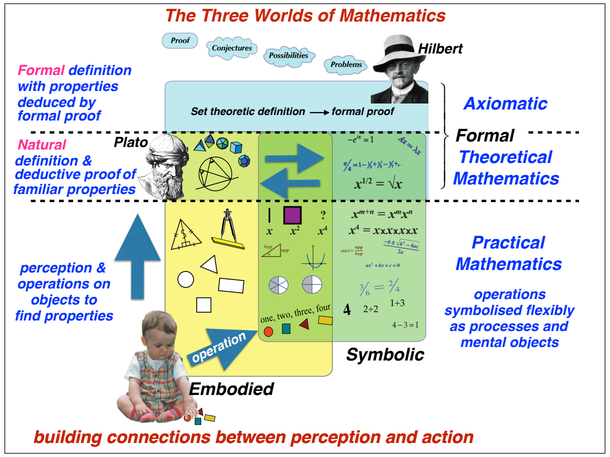
@ where the power law used as a definition gives x to the half a new meaning as the square root of x

@ This gives a new level based on definition and deductive proof.

@ represented by **Plato** in Euclidean Geometry

@ with corresponding definition and deduction in symbolism

@ with a continuing interchange between embodiment and symbolism. **X**



10 At a more sophisticated level, pure mathematicians shift to **set-theoretic definition** and **formal** proof introduced by **Hilbert**.

@ which extends the Formal world to a new level of

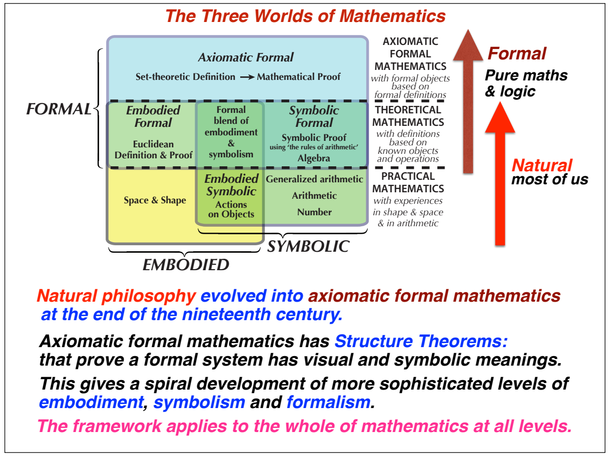
@ **Formal definition with properties deduced** only from the definition **using formal proof**

@ as distinct from the **Natural** form of **definition and deductive proof** based on **familiar properties**.

This extends Formal proof from Theoretical mathematics to @ **Axiomatic Formal. X**

From here, the research mathematician can develop new theories arising from

@ **problems** by considering **possibilities**, suggesting **conjectures** and seeking **Proof**. **X**



11 @ For **most of us**, including advanced applications, mathematics is a **natural** blending of embodiment and symbolism,

@ but for **pure mathematicians and logicians** the *axiomatic* ***formal*** world of mathematics builds properties based only on the formal definitions without any dependence on specific embodiments. **X**

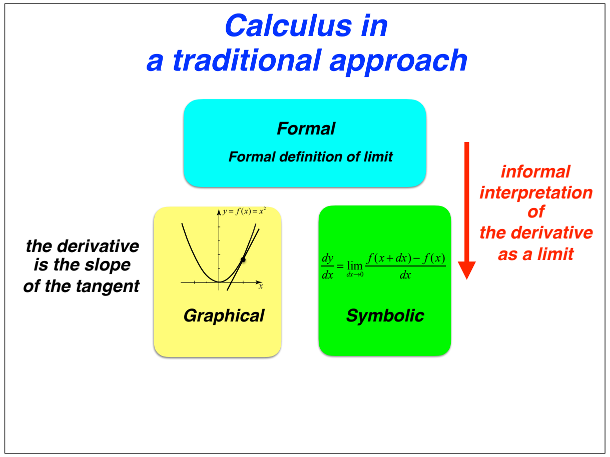
@ **In history, 'natural philosophy' evolved into axiomatic formal mathematics at the end of the nineteenth century.** The terms 'natural' and 'formal' have a historical as well as a cognitive meaning. **X**

@ Axiomatic formal mathematics has *Structure Theorems*: that prove a formal structure has visual and symbolic meanings.

@ This gives a spiral development of more sophisticated levels of embodiment, symbolism and formalism.

X

The framework applies to the whole of mathematics at all levels. X



12 Let us begin by considering a traditional approach to the calculus which may be considered in terms of graphical, symbolic and formal development.

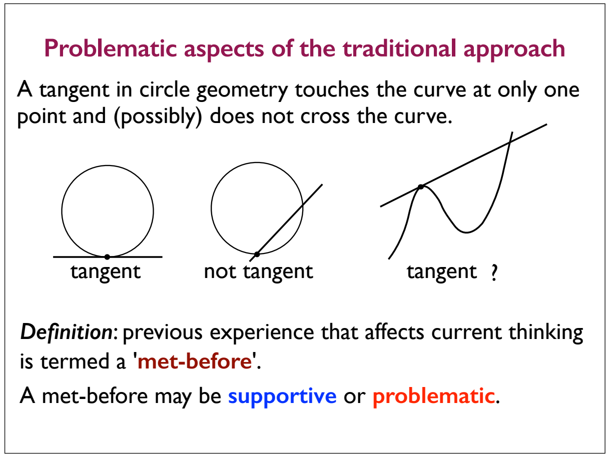
@ this begins by considering a graph, say y equals x squared.

@ the derivative is the slope of the tangent. **X**

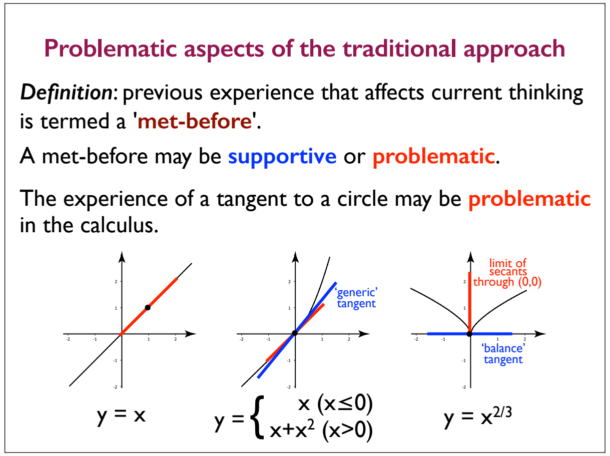
@ The formal definition of the limit is used to give

@ an informal interpretation of the derivative in which

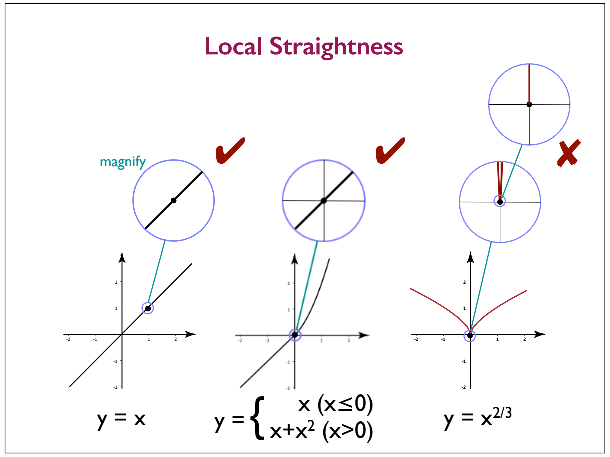
@ dy/dx is a limit but not a quotient dy divided by dx **X**



13 A tangent in circle geometry touches the curve at only one point and (possibly) does not cross the curve.
@ This is a tangent (shows tangent)
@ This is not a tangent (shows not tangent)
@ This is a tangent (shows third picture)
@ or is it? (extends tangent to meet curve again) **X**
@ Definition: previous experience that affects current thinking is termed a '**met-before**'.
@ A met-before may be supportive or problematic. **X**



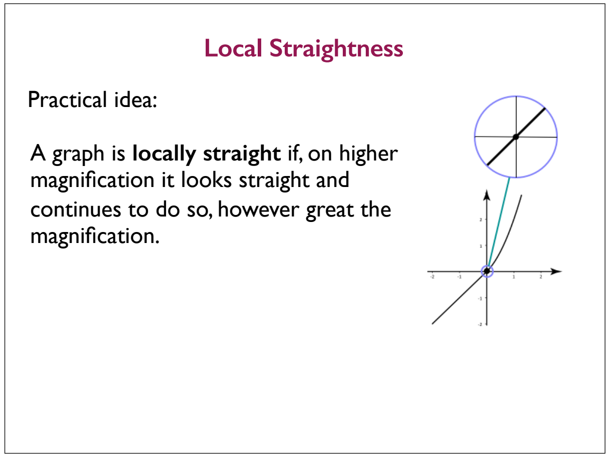
14 @ The experience of a tangent to a circle may be problematic in the calculus.
@ y = x has a tangent that coincides with the whole graph rather than touch at a single point. **X**
@ The second picture has y = x as a tangent at the origin, which does not touch at a single point.
so many students mark what I call a 'generic tangent' drawn to touch at a single point. **X**
@ The third picture is even more problematic.
@ The limit of the secants through the origin is a vertical half-line.
@ Other students imagine the tangent to touch at the origin as a 'balance tangent'. **X**



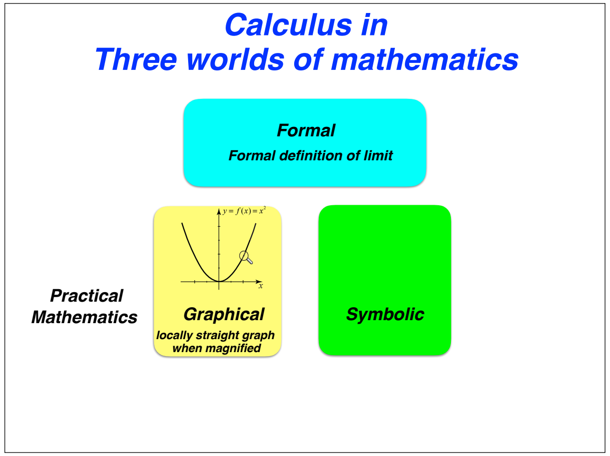
15 @ If we magnify the first graph
@ we see that a small portion looks straight. **X**

@ If we magnify the second graph at the origin
@ this also looks straight. **X**

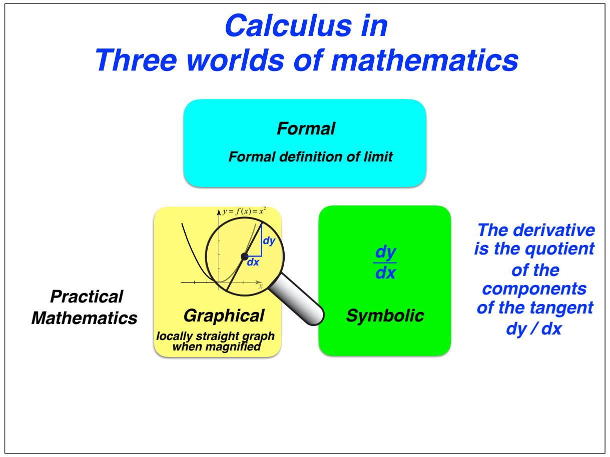
@ but if we magnify the third graph
@ eventually it looks like a half-line
@ so this is not a full locally straight line segment. **X**



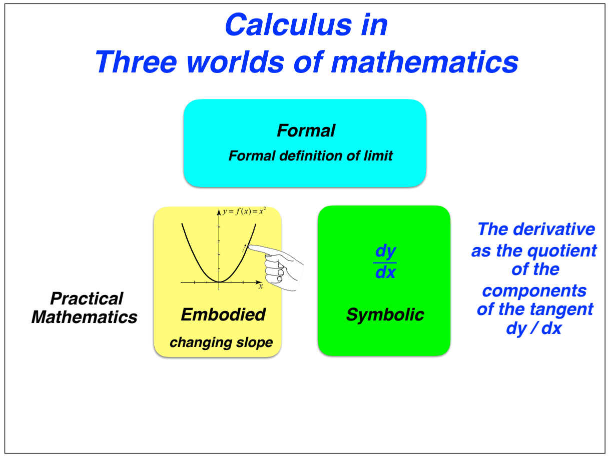
16 This leads to the practical idea that
@ A graph is locally straight if, on higher magnification it looks straight and continues to do so, however great the magnification. **X**



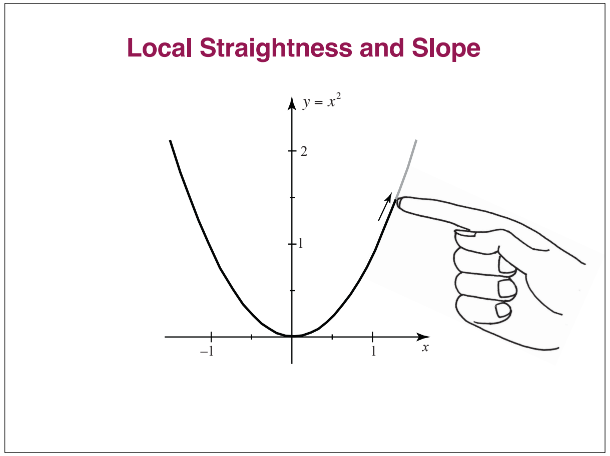
17 We now focus on the special case of calculus using the three worlds of mathematics.
@ It begins in practical mathematics with the graph of the function,
@ which looks straight when @ highly magnified. **X**



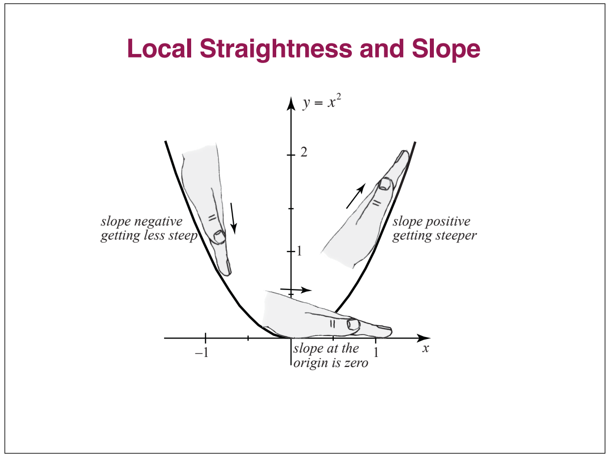
18 @ The components of the tangent are dx and dy.
@ [shows **dy/dx**]
@ The derivative is the quotient of the components of the tangent dy/dx. **X**



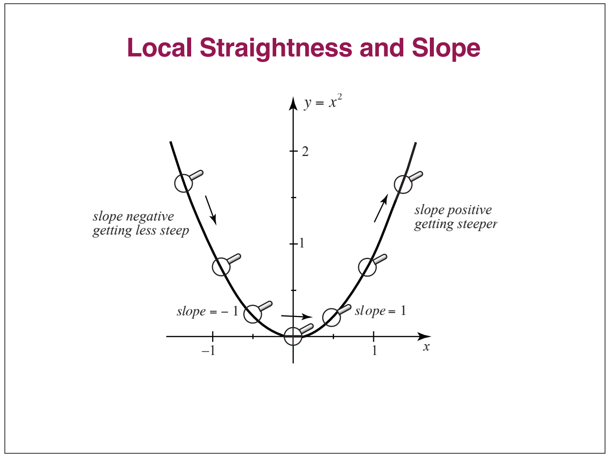
19 @ Our next operation is to move along the curve
@ in an **embodied** action that coordinates perception and operation
@ to focus on the **changing slope. X**



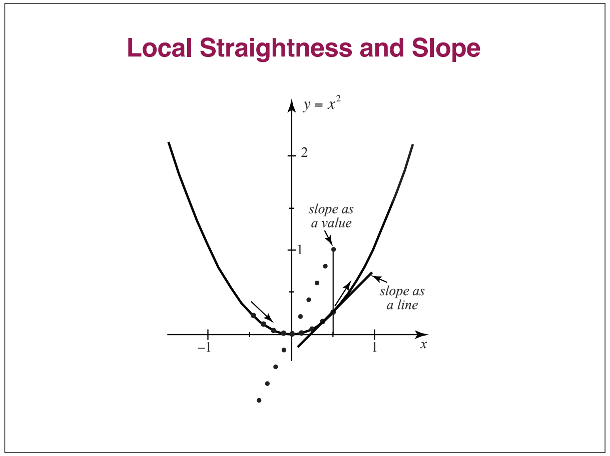
20 [no comment]



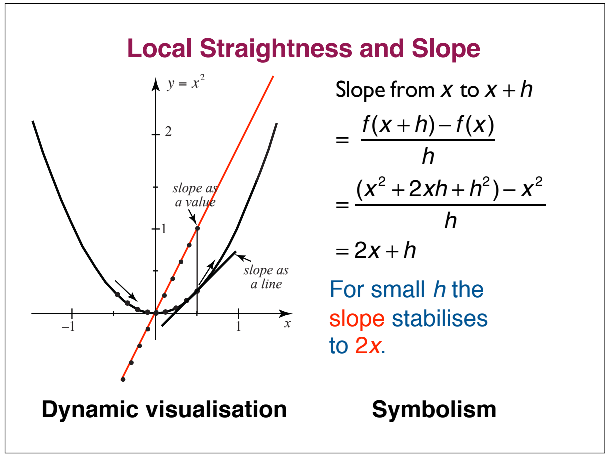
21 [illustrate with physical hand movements.]
As we move a hand along the curve, the slope is first negative getting less steep.
The slope at the origin is zero
and the slope then becomes positive, getting steeper. **X**



22 We note the numerical value of the slope.
At x =-1/2 the slope looks around –1,
at x = 1/2 the slope is around +1. **X**

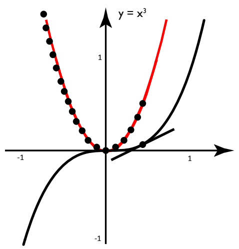


23 Looking at the changing value of the slope as a line, we can plot the slope as a point.
The slope as a value builds up a sequence of points which lie on a line. **X**



24 This gives a **dynamic visualisation of the changing slope**
@ which lies on a line
@ This has a corresponding **symbolism**
@ for the slope from x to x + h
@ which simplifies to 2x + h. **X**
@ For small h, the slope stabilises to 2x. **X**

Local Straightness and Slope



Slope from x to $x+h$

$$= \frac{f(x+h)-f(x)}{h}$$

$$= \frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$$

$$= 3x^2+3xh$$

For small h the slope stabilises to $3x^2$.

Dynamic visualisation **Symbolism**

25

For y = x cubed,
@ the slope from *x* to *x+h* simplifies to 3*x* squared plus 3*x* h
@ [draw red graph]
@ and for small *h* the slope stabilises to 3*x* squared. **X**

Local Straightness and Slope

More generally, use the binomial theorem

$$(x+h)^n = x^n + nx^{n-1}h + n(n-1)x^{n-1}h^2 + \ldots + h^n$$

to show symbolically that the slope function for x^n is nx^{n-1}.

Investigate the calculations on a visual display for other values of n, even negative and fractional.

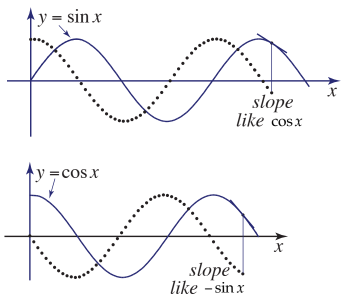
To be able to ***imagine*** the changing slope of a locally straight graph.

26

@ More generally, use the binomial theorem
@
@ to show symbolically that the slope function for x^n is nx^(n-1). **X**
@ Investigate the calculations on a visual display for other values of n, even negative and fractional.
@ To be able to imagine the changing slope of a locally straight graph. **X**

Local Straightness and Slope

Visualising the slope functions of sin*x* and cos*x*.



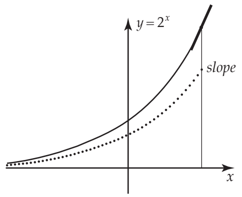
27

Visualising the slope functions of sin*x* and cos*x* for angles in radians.
@ For y = sin*x*, tracing along the graph, its slope is like cos*x*. **X**
@ For y = cos*x*, the slope is like the graph of sinx upside down, so it is like –sin*x*. **X**

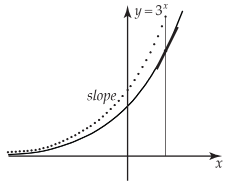
Local Straightness and Slope

Visualising the slope functions of 2^x, 3^x.

Both slope functions have steadily increasing shape.



Slope function below graph



Slope function above graph

Seek e where $2 < e < 3$ and slope of e^x is again e^x.

28

@ [show graphs] Visualising the slope functions of 2^x, 3^x,
@ Both slope functions have steadily increasing shape. **X**

@ The graph y = 2^x, has its slope function **below** the original graph,
@ The graph y = 3^x, has its slope function **above** the original graph. **X**

@ seek a value of e where e is between 2 and 3 and the slope of e^x is again e^x. **X**

Local Straightness and Slope

Seek e where $2 < e < 3$ and slope of e^x is again e^x.

Suppose that e^x can be approximated by a polynomial:

$$e^x = A + Bx + Cx^2 + Dx^3 + \dots$$

Putting $x = 0$ gives $e^0 = A = 1$.

The slope function of e^x is

$$e^x = B + 2Cx + 3Dx^2 + \dots$$

Comparing coefficients gives

$B = A = 1$, $2C = B$, so $C = ½$,
C =3D, so D=⅓!, etc

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

29

@ Suppose that e^x can be approximated by a polynomial.
@ [show polynomial] **X**

@ Putting x =0 gives e^0 is A, which is 1.
@ The slope function of e^x is
@ e to the x is B + 2Cx + 3Dx^2+ …
@ Comparing coefficients gives
@ B = A = 1, 2C = B, so C = ½,
@ C =3D, so D=⅓!, etc
@ so e to the x is 1 + x + x^2 over 2 factorial + … + x^n over factorial n + …
X

Local Straightness and Slope

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

and putting $x = 1$ allows e to be calculated

$$e^1 = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots$$

First term is	1.0000000000
divide by 1	1.1000000000
divide by 2	0.5000000000
divide by 3	0.1666666666…
divide by 4	0.0416666666…

etc and add up enough terms to get

e = 2.718…

It is easy to calculate, say, 10 decimal places.

30

@ putting x = 1 allows to be calculated by calculating successive terms @ **X**

@ the first term is 1.000… to as many places as desired
@ divide this by 1 to get 1.100 etc
@ divide the result by 2 to get 0.500 etc
@ divide the result by 3 to get 0.166 recurring
@ divide the result by 4 to get 0.4166 recurring
@ and add enough terms to get e to any desired accuracy,
@ for instance e=2.718
@ It is easy to calculate e to, say, 10 decimal places. **X**

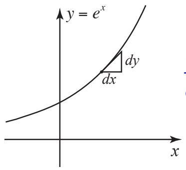
Local Straightness and Slope

A locally straight approach builds from the slope of straight lines and investigates calculating the slope of curves.

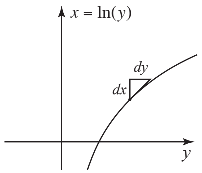
This works for locally straight curves, including

x, x^2, … x^n, sinx, cosx, e^x.

ln(x) follows from the inverse relationship between $y = e^x$ and $x = \ln(y)$



$\frac{dy}{dx} = y$



$\frac{dx}{dy} = \frac{1}{y}$

31

A locally straight approach builds from the slope of straight lines and investigates calculating the slope of curves.
@ This works for locally straight curves, including
x, x^2, … x^n, sinx, cosx, e^x. **X**
@ ln(x) follows from the inverse relationship between y = e^x and x = ln(y)
@ [graph] For e^x,
@ dy/dx = y
@ [next graph] for x = ln(x), dx/dy = 1/y. **X**

Local Straightness and Slope

If $y = f(x)$ then for dx ≠ 0, we define

$$f'(x) = \frac{dy}{dx}$$

The equation

$$dy = f'(x)dx$$

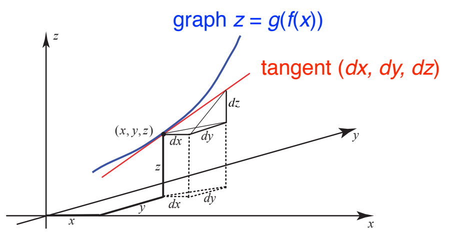
is then also true for $dx = 0$, in which case $dy = 0$ also.

32

@ If y = f(x) then for dx ≠ 0, we define
@ f'(x) = dy/dx
@ The equation
@ dy = f'(x) dx
@ is then also true for dx = 0, in which case dy = 0 also.
X

Local Straightness and Slope

$y = f(x)$, $z = g(y)$



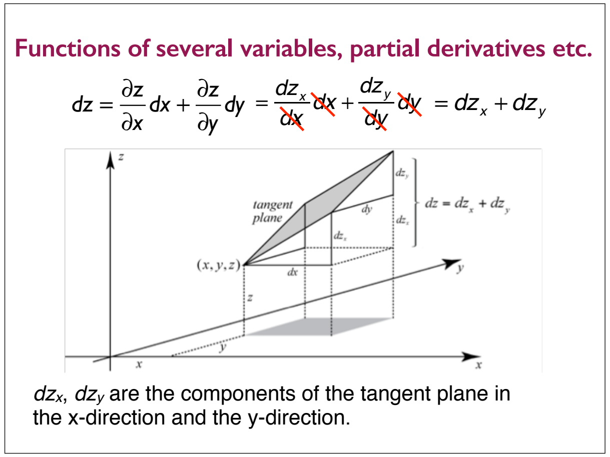
$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} \quad \text{where } dx \neq 0.$$

If $dy = 0$, then $dz = g'(y)dy = 0$, so $dz/dx = 0$.

33

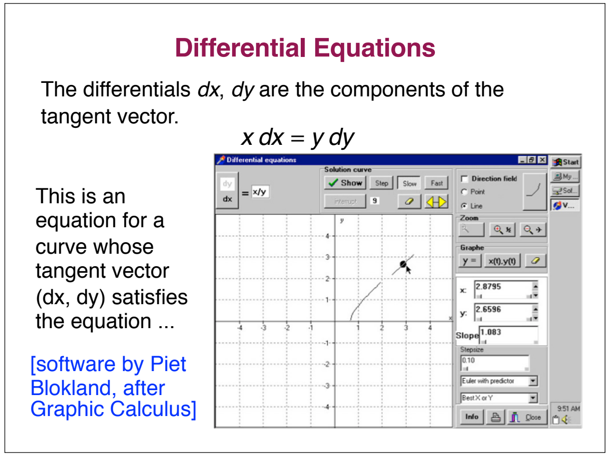
If y = f(x) and z = g(y), the picture shows the graph of z = g(f(x))
@ and the components of the tangent as *dx*, *dy*, *dz*
@ Because these are all lengths we get the standard equation
dz/dx = *dz/dy* times *dy/dx* where *dx* ≠ 0. **X**

@ The only problem here is that it may happen that *dy* = 0,
@ [show red line]
@ but then dz =*g'(y) dy* =0 and, for any value of *dx* ≠ 0, we must also have *dz*/*dx* = 0. **X**



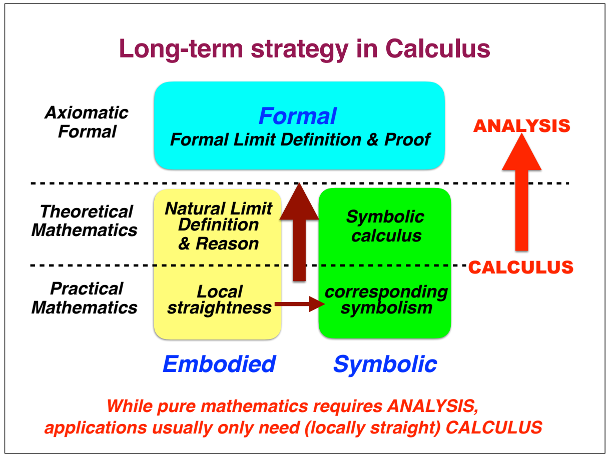
34 The same theoretical framework gives new insights for functions of several variables, partial derivatives etc.

@ For example, if z is a function of two variables, then we have the partial derivative equation where partial dz by dx is certainly not a quotient. Or is it? **X**
@ [reveal image]
@ let dz suffix x and dy suffix y be the components of the tangent plane in the x direction and the y direction,
@ then we can write dz as dzx over dx times dx plus dzy over dy times dy and
@ cancel to get the total differential dz equals the sum of dzx plus dzy. **X**



35 @ **The differentials dx, dy are the components of the tangent vector**.
@ for example **x dx = y dy**
@ **This is an equation for a curve whose tangent vector (dx, dy) satisfies the equation**.
@ [image appears] **X**

@ As the pointer is moved the line segment changes direction as given by the differential equation. Clicking drops the line segment, allowing the user to build up a solution whose direction follows the direction given by the differential equation.
@ Software by Piet Blokland for windows computers based on original Graphic Calculus by David Tall. **X**

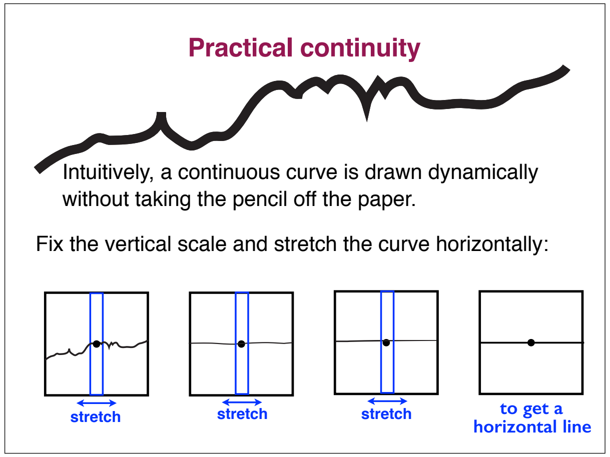


36 The long-term strategy in the calculus starts with
@ **practical mathematics**
@ **local straightness** and its **corresponding symbolism X**

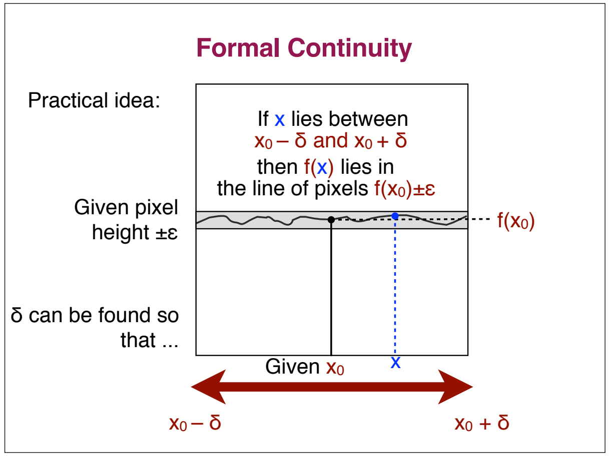
@ then we move up a level to **theoretical mathematics**
@ based on the **natural limit definition using natural reason** and the corresponding @ **symbolic calculus**
@ This gives the locally straight approach to the **CALCULUS X**

@ At a higher level we have the **axiomatic formal** world and the @ **formal limit definition and formal proof**
@ which leads to **mathematical ANALYSIS X**

@ I suggest that while pure mathematics requires ANALYSIS, applications usually only need (locally straight) CALCULUS. X



37 @ Intuitively, a continuous curve is drawn dynamically without taking the pencil off the paper. **X**
@ Fix the vertical scale and stretch the curve horizontally
@ [the curve is successively stretched horizontally] @ @ @ @ @ …
@ to get a horizontal line. **X**



38 In a practical sense, suppose we are given a value of x0 and a pixel height ±epsilon.
@ so that the value of f(x0) is in the middle of the pixel height. **X**
@ then a delta can be found so that **X**
@ If x lies between x0 - delta and x0 + delta **X**
@ then f(x0) lies in the line of pixels f(x0)±epsilon **X**
@ [draws curve]

Formal Continuity

Formal definition

Given $\varepsilon > 0$, $\delta > 0$ can be found so that ...

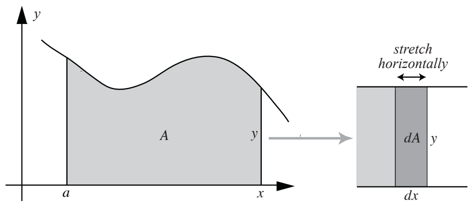
If x lies between $x_0 - \delta$ and $x_0 + \delta$

then $f(x)$ lies between $f(x_0)\pm\varepsilon$

39 This gives the formal definition
@ Given ε > 0,
@ δ> 0 can be found so that
@ If x lies between x0 – δ and x0 + δ
@ then f(x) lies between f(x0)±ε. **X**

Integral as area function

What is the area under a *continuous* function?



Practically $dA = y \times dx$ so $\frac{dA}{dx} = y$

Illustrating the Fundamental Theorem of the Calculus

40

@ What is the area A under a continuous function?
@ [draw picture] **X**
@ If we take a thin strip width dx and stretch it horizontally **X**
@ practically
@ dA = y x dA
@ so dA/dx = y
@ Illustrating the Fundamental Theorem of the Calculus **X**

A three-world approach to the calculus

A sequence of development from

Practical Mathematics
based on local straightness and practical continuity

to

Theoretical Mathematics
based on natural definition and coherent reason

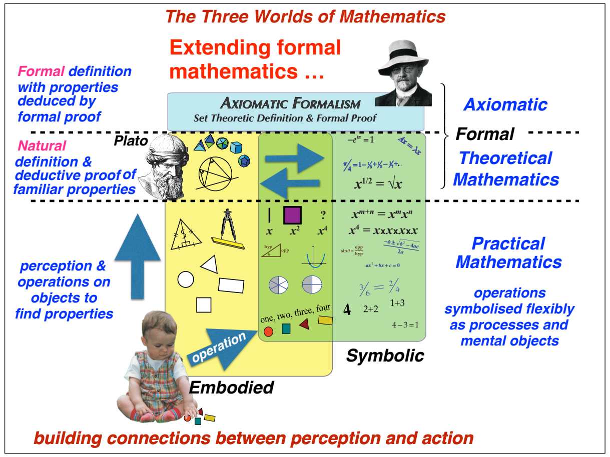
and later, if required, to

Formal Axiomatic Mathematics
based on formal definition and formal proof

The three world model takes us beyond familiar mathematical analysis …

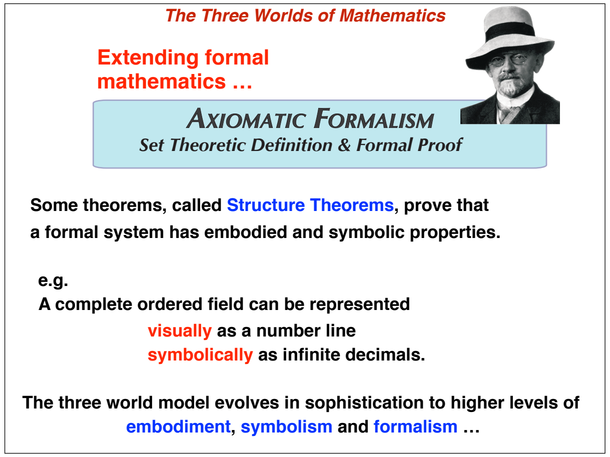
41

@ A sequence of development from
@ Practical Mathematics based on local straightness and practical continuity **X**
@ to Theoretical Mathematics based on natural definition and coherent reason **X**
@ and later, if required, to
@ Formal Axiomatic Mathematics based on formal definition and formal proof **X**
@ The three world model takes us beyond familiar mathematical analysis … **X**



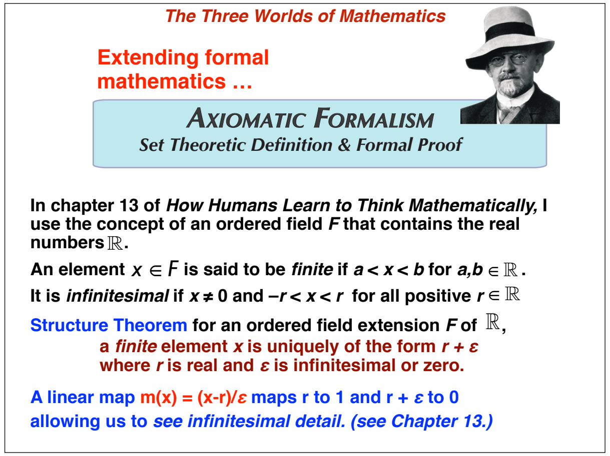
42

This is the framework for three worlds of mathematics. However, it continues to evolve.
Formal mathematics takes us to new levels of sophistication which give new forms of embodiment and symbolism. **X**

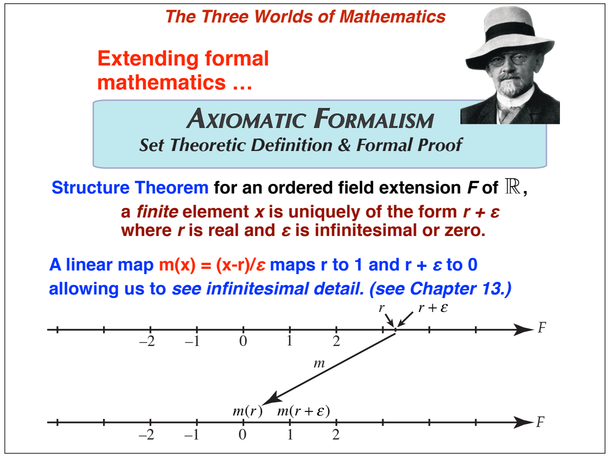


43 @ Some theorems, called Structure Theorems, prove that a formal system has embodied and symbolic properties.
@ e.g. A complete ordered field can be represented as
@ visually as a number line
@ symbolically as infinite decimals. **X**

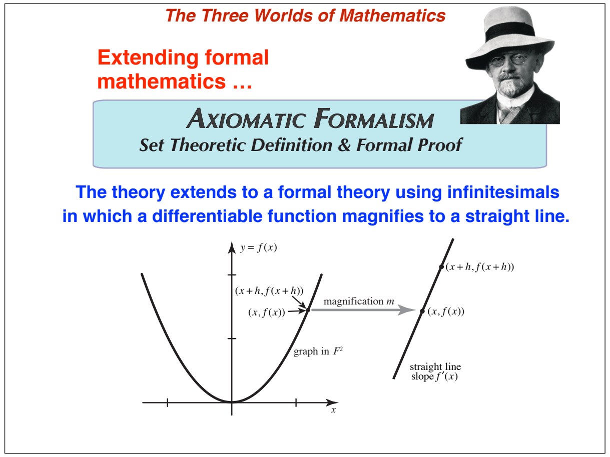
@ The three world model evolves in sophistication to higher levels of embodiment, symbolism and formalism … **X**



44 @ In chapter 13 of How Humans Learn to Think Mathematically, I use the concept of an ordered field F that contains the real numbers R. **X**
@ An element x in F is said to be finite if $a < x < b$ for a, b in R.
@ It is infinitesimal if $x \neq 0$ and $-r < x < r$ for all positive r in R. **X**
@ The **Structure Theorem** for an ordered field extension F of the real numbers,
@ a finite element x is uniquely of the form $r + \varepsilon$ where r is real and ε is infinitesimal or zero. **X**
@ A linear map $m(x) = (x-r)/\varepsilon$ maps x to 1 and $x+\varepsilon$ to 0 allowing us to magnify infinitesimal detail. (see Chapter 13.) **X**

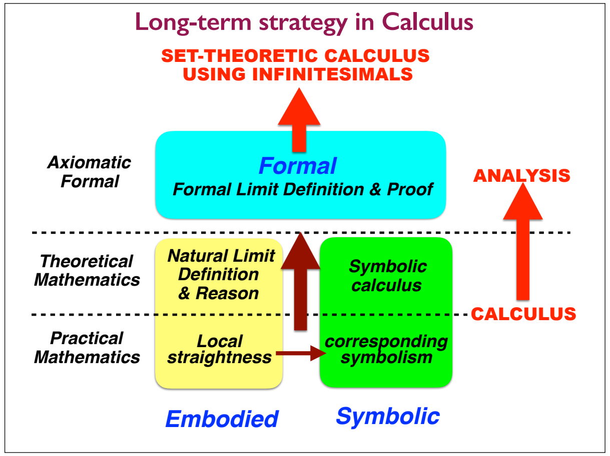


45 The function m maps points r and r+epsilon which differ by an infinitesimal to points a finite distance apart. **X**



46 It is possible to define a formal theory using infinitesimals in which a differentiable function magnifies an infinitesimal part of a locally straight graph to a straight line. **X**

This offers a formal justification of infinitesimals using modern set theory. I have taught an undergraduate course using these ideas *after* a standard mathematical analysis course and it was very well received. I would not use such an approach until students have made sense of the axiomatic formal world. For a first course in the calculus, I believe it is more sensible to begin with a locally straight practical approach. **X**

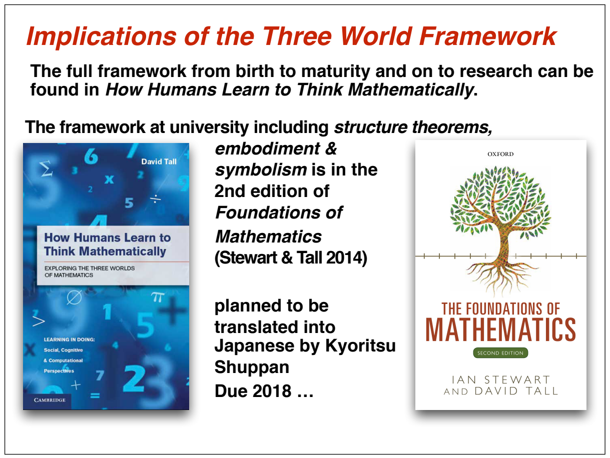


47 The framework for practical and theoretical calculus extending to axiomatic analysis therefore extends further to

@ set-theoretic calculus using infinitesimals.

I have taught this several times to undergraduates who have already taken a first year analysis course and it was very well received.

For a first introduction to calculus, I now always use a locally straight approach as it builds naturally on the students' experience. **X**



48 **@ The full framework from birth to maturity is given in *How Humans Learn to Think Mathematically.***

@ [picture appears] X

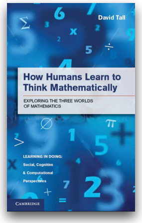
@ The framework at university including structure theorems, embodiment & symbolism is in the 2nd edition of

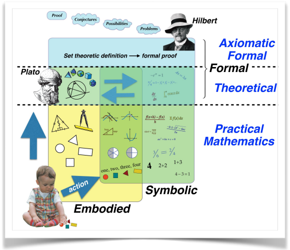
***Foundations of Mathematics* (Stewart & Tall 2014)**

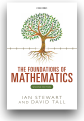
@ [picture appears] X

@ This is **currently planned to be translated into Japanese by Kyoritsu Shuppan. Due 2018 … X**

The Three Worlds of Mathematics







Thank you for listening

http://homepages.warwick.ac.uk/staff/David.Tall/

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Thank you for listening.