How Humans Learn to Think Mathematically

Exploring the Three Worlds of Mathematics and long-term international consequences

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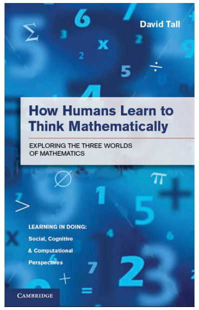
It is an honour to be invited to present to you the framework I have formulated for the long-term development of *How Humans Learn to Think Mathematically*. In particular I acknowledge the contribution arising from an early encounter with Masami Isoda where he spoke about the tangent in circle geometry touching the circle at one point and not passing through it in a manner that conflicts with the more general concept of tangent in the calculus. Our common interests and differences and subsequent collaboration on Lesson Study have been a constant encouragement in the development of the theory I present today.

I also declare that this framework is the distillation of many contributions from the literature and from my colleagues and research students. At a time when new technology is causing civilisation to evolve seemingly faster than ever before, it is time to reflect on how humans learn to think mathematically and build a new theory to encourage different communities to interact and plan together for the future. **X**



This presentation celebrates the publication of the Japanese translation of

How Humans Learn to Think Mathematically (HHLTTM)

Long-term development:

- ***Three Worlds of Mathematics***
- ***Biological development of mathematics****
- ***Emotional aspects** such as anxiety*
- ***Social aspects** in different communities*
- ***International comparisons & implications****

(Items marked * extend materials in the book HHLTTM.)

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This presentation celebrates the publication of the Japanese translation of *How Humans Learn to Think Mathematically*. which I will refer to as

@ **HHLTTM.** It offers an evolutionary framework of

@ **long term development** that should be read reflectively over several weeks. **X**

Here I present an outline of the main ideas beginning with the overall framework of

@ **Three worlds of mathematics X**

@ the underlying **Biological development of mathematics X**

@ its accompanying **Emotional aspects, particularly anxiety X**

@ and the very different **Social aspects in different communities X**

@ I will include observations about **international comparisons and implications X**

The Three Worlds of Mathematics

A framework for cognitive development, each world is based on human perception, action & reason.
Actions performed with a specific purpose are called operations including constructions in geometry and symbolic operations in arithmetic and algebra.

Embodied: properties of physical & mental objects

Symbolic: mathematical operations that may be symbolised and manipulated as mental objects

Formal: based on verbal/logical definition with properties deduced by mathematical proof.

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@ The three worlds of mathematics is **a framework for the *cognitive development*** of mathematical thinking, where **each world is based on human perception, action & reason. Actions performed with a specific purpose are called *operations* including constructions in geometry and symbolic operations in arithmetic and algebra. X**

@ The **embodied world** explores the perceptual **properties of physical and mental objects**, formulating verbal definitions used at a more sophisticated level to reason about relationships. **X**

@ The **symbolic world** develops out of **mathematical operations** performed initially on real world objects, where **the operations are symbolised** and the symbols themselves are **manipulated as mental objects**. These symbols can then be operated upon at successively higher levels in arithmetic, algebra, symbolic calculus, vector algebra and so on. **X**

@ The **formal world** is **based on verbal/logical definition with properties deduced by mathematical proof**. **X**

The Three Worlds of Mathematics

In each world there is a long-term development in sophistication.

***Objects** develop sophisticated structure*

***Operations** are symbolised and the symbols may be conceived as objects with structure*

***Properties** are later formulated verbally to define formal concepts whose other properties are deduced by formal proof.*

*In the long-term, powerful mathematical thinking builds what I term **crystalline concepts**.*

These have structures that are determined as a consequence of their context.

I will also suggest that failure to develop these flexible structures can lead to limited procedural learning.

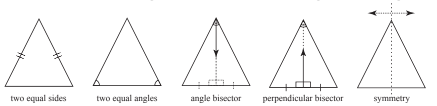
4

In each world there is a long-term development in sophistication.
@ Objects develop sophisticated structure
@ Operations are symbolised and the symbols may then be conceived as mental objects with structure.
@ Properties are later formulated verbally to define formal concepts whose other properties are deduced by formal proof. X
@ In the long-term, *powerful* mathematical thinking builds what I term *crystalline concepts*.
@ These have structures that are determined as a consequence of their context.
@ I will also suggest that failure to develop these flexible structures can lead to limited procedural learning.
X

The Three Worlds of Mathematics

Examples of crystalline concepts:

An isosceles triangle in Euclidean geometry



Different properties can specify the same mathematical object

A number in arithmetic, e.g. 6, 4+2, 2+4, 5+1, 3x2, 2x3
Different symbols, same mathematical object

A formal axiomatic system, e.g. the set of integers, a group, a complete ordered field.
Here, each system specifies a structure with properties that can be proved.
The same system may also be specified by a different list of axioms.

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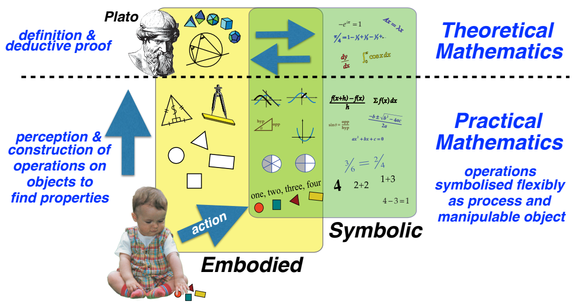
Examples of crystalline concepts include:
@ An isosceles triangle in Euclidean geometry
@ *It can be a triangle with* ***two equal sides****, with* ***two equal angles*** *or with various other properties*
@ **Different properties can specify the same mathematical object. X**
@ *A crystalline concept may be* **a number in arithmetic, for example, 6,** which is also **4+2, 2+4, 5+1, 3x2, 2x3**
Here **Different symbols give the same mathematical object. X**
@ *It can be* **a formal axiomatic system, for example the set of integers, a group, a complete ordered field. Here, each system specifies a structure with properties that can be proved.**
@ The same system may also be specified by a different list of axioms. X

For the vast majority, in school, in practice & in application mathematical thinking is a combination of embodiment and symbolism based on experience



building connections between perception and action

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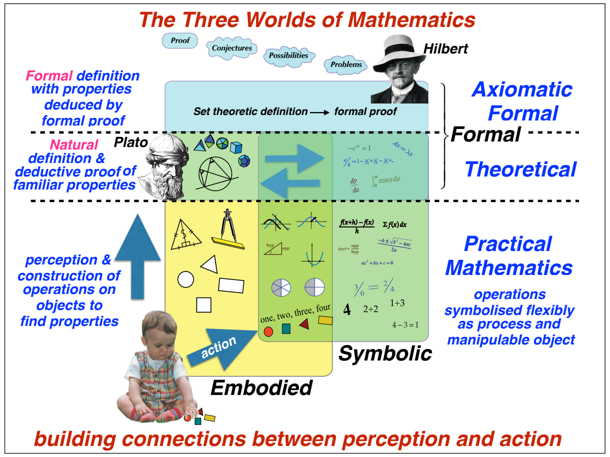
@ For the vast majority, in school, in practice & in application, mathematical thinking is a combination of embodiment and symbolism based on experience. **X**
@ The young child is born with a brain still maturing and spends the first months
@ building connections between perception and action. X
@ Using **perception and construction of operations on objects to find properties** in what I term the @ **Embodied** world. **X**
@ **Action** on objects, **one, two, three, four** leads to the number concept **4** where **operations** can be **symbolised flexibly as process and manipulable object** in what I term the @ **Symbolic** world. **X**
@ more sophisticated operations translated into symbolism build up @ **Practical Mathematics**. **X**
@ This may develop into a more sophisticated level of **Definition and Deductive Proof**
@ represented by **Plato** in Euclidean Geometry
@ with corresponding levels of definition and deduction in symbolism
@ as part of a higher level of **Theoretical Mathematics**
@ with a continuing interchange between embodiment and symbolism. **X**



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At a more sophisticated level, pure mathematicians shift to **set-theoretic definition** and **formal** proof introduced by **Hilbert**.

@ which extends the Formal world to a new level of

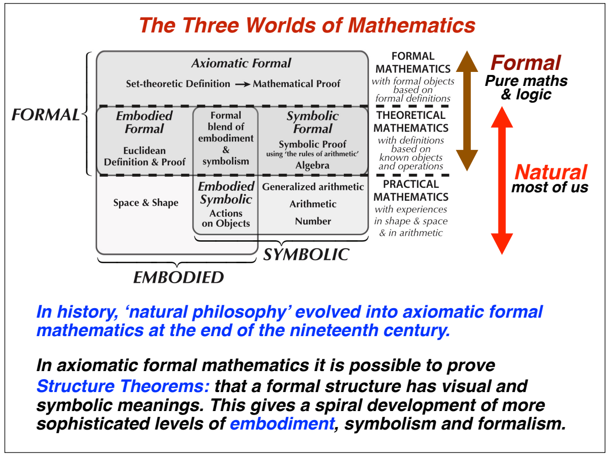
@ **Formal definition with properties deduced** only from the definition **using formal proof**

@ as distinct from the **Natural** form of **definition and deductive proof** based on **familiar properties**.

This extends Formal proof from @ **Theoretical** proof to @ **Axiomatic Formal. X**

From here, the research mathematician can develop new theories arising from

@ **problems** by considering **possibilities**, suggesting **conjectures** and seeking **Proof**. **X**



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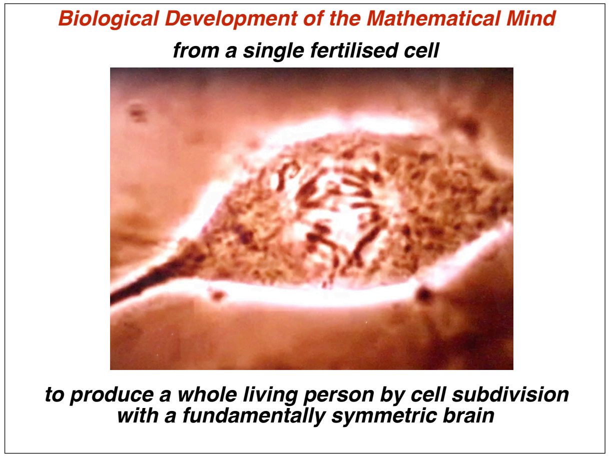
@ For **most of us**, including advanced applications, mathematics is a **natural** blending of embodiment and symbolism,

@ but for **pure mathematicians and logicians** a more fundamental *axiomatic* ***formal*** world of mathematics builds properties based only on the formal definitions without any dependence on specific embodiments. **X**

@ **In history, 'natural philosophy' evolved into axiomatic formal mathematics at the end of the nineteenth century.** The terms 'natural' and 'formal' have a historical as well as a cognitive meaning. **X**

In axiomatic formal mathematics it is possible to prove *Structure Theorems*: that a formal structure has visual and symbolic meanings. This gives a spiral development of more sophisticated levels of embodiment, symbolism and formalism. X



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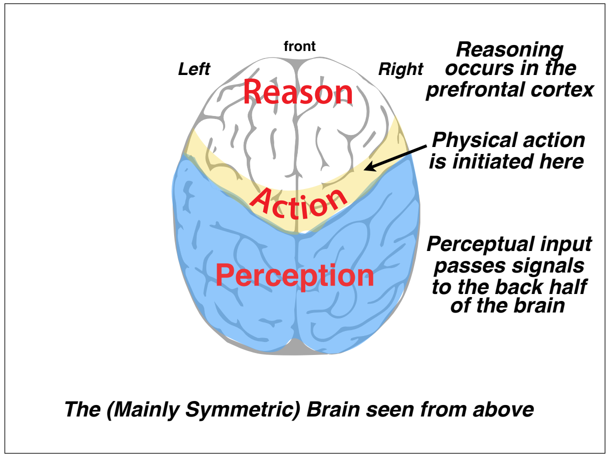
The biological development of the mathematical mind begins with

@ **a single fertilised cell. X**

@ (video) …..

@ to produce a whole living person by cell division

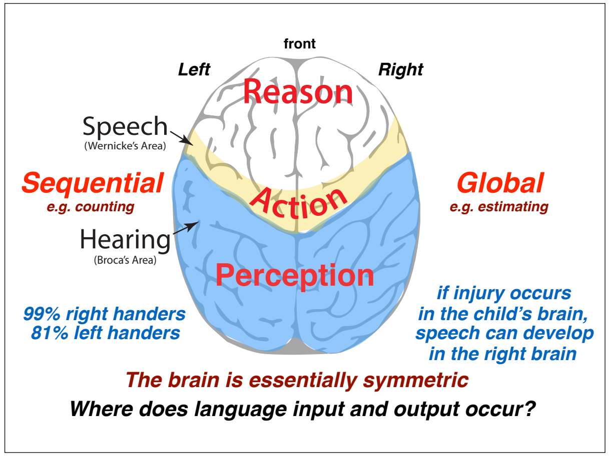
@ with a fundamentally symmetric brain X



The (Mainly Symmetric) Brain seen from above

10

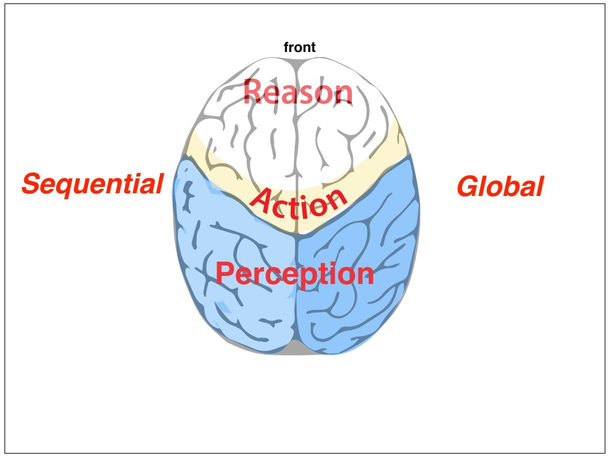
Here is a view of **the (mainly symmetric) brain seen from above.**
@ The **Left**
@ and the **Right** are essentially symmetric. **X**
@ **Perception** occurs as **perceptual input passes signals to the back half of the brain. X**
@ **Physical action** occurs at the rear of the forebrain. **X**
@ **Reason** occurs through links with the front of the brain. **X**.



Where does language input and output occur?

11

Where does language input and output occur? X
@ For **99% of right handers** and **81% of left handers**
@ **Hearing** is in **Broca's area** in the left rear brain.
@ **Speech** is in **Wernicke's area** in the front left brain. **X**
@ However, **if injury occurs in the child's brain, speech can develop in the right brain**
@ So **the brain is essentially symmetric** and speech resides on one side (usually the left)
@ **Sequential** operations, such as counting occur mainly in the left.
@ **Global** estimation occurs mainly in the right. **X**



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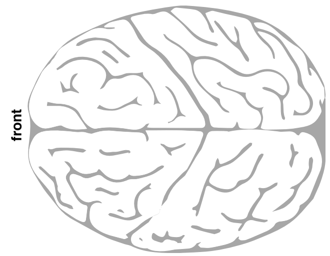
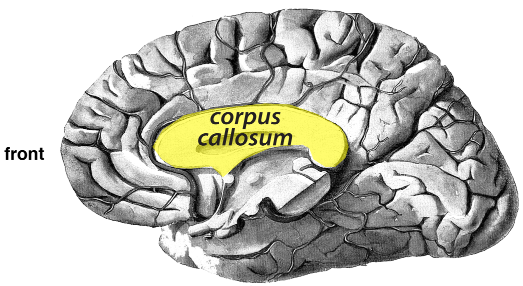
@ move to next slide



13 Taking a cross-section down the middle of the brain **X**

@



interconnections between the two halves of the brain

14 We now see a cross-section between the two halves.

The interconnections between the two halves of the brain are connected through

@ the **corpus collosum** **X**

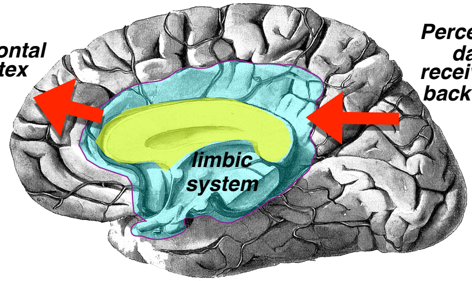
http://www.nature.com/naturejobs/science/articles/10.1038/nj7474-153a

interconnections between different parts of the brain

15 Here we see the long **interconnections between different parts of the brain** displayed by functional magnetic resonance imaging. Thinking occurs through multiple connections across the whole brain. **X**

@

Limbic system : several functions including making links to long term memories and emotional response to perceptual input



Unconscious emotional 'fight or flight' reaction occurs before logical reason

See, for example Kahnemann 'Thinking Fast, Thinking Slow'

16

@ In the centre of the brain is the **limbic system**.

@ The **Limbic system** has **several functions, including making links to long term memories and emotional response to perceptual input.** **X**

@ **perceptual data received in the back brain p**asses through the **limbic system** to the **prefrontal cortex.** **X**

Unconscious emotional 'fight or flight' reaction occurs before logical reason.

@ **See, for example, Kahnemann's book on 'Thinking Fast, Thinking Slow'** **X**

Emotional aspects of long-term learning

Unconscious emotional 'fight or flight' reaction occurs before logical reason

Mathematical thinking is affected subconsciously by ideas that fit together or that cause conflict.

Philosophers speak of thinking in 'metaphors'.

To relate new ideas to previous experience, I introduced the term 'met-before'.

A supportive met-before refers to previous experience that is now consistent with new learning.
A problematic met-before is in conflict with new learning

What is supportive in one context may become problematic in another.

17

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@ **To relate new ideas to previous experience, I introduced the term 'met-before'.** **X**

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Examples of problematic changes:

counting is supportive for simple arithmetic but problematic as the only strategy for larger numbers.

Procedures occur in time and flexible thinking involves seeing different symbols can represent the same thing.

4+2, 2+4, 5+1, 12/2, 3x2, 2x3 all represent the number 6.
***Procept*: different *pro*cesses, same con*cept*.**

As mathematics becomes more sophisticated, equivalent concepts are later considered as the same:
$\frac{3}{6}, \frac{2}{4}$ **are equivalent as fractions, the same as rational numbers.**

Different procedures but they represent a single flexible crystalline concept.

18

@ **Examples of problematic changes:**

@ **counting is supportive for simple arithmetic but problematic as the only strategy for larger numbers.** **X**

@ **Procedures occur in time and flexible thinking involves seeing different symbols can represent the same thing.**

@ **4+2, 2+4, 5+1, 12/2, 3x2, 2x3 all represent the number 6.** **X**

@ I use the term **Procept**: where symbols represent **different processes,** but are the **same concept.** **X**

@ **As mathematics becomes more sophisticated, equivalent concepts are later considered as the same: 3/6, 2/4 are equivalent as fractions, the same as rational numbers.**

@ They are **different procedures but they represent a single flexible crystalline concept.** **X**

Emotional aspects of long-term learning

Most current curriculum frameworks specify positive goals to be attained and assessed e.g. SIMSS, PISA.

Negative aspects, e.g. mathematical anxiety are usually researched separately.

The three-world framework integrates both supportive and problematic aspects of mathematics.

Historically, mathematics advanced from (unsigned) quantities to signed numbers.

Cognitively, our understanding of long-term mathematical thinking may be advanced by shifting from focusing only on positive aspects to including both positive and negative effects of mathematical thinking.

19

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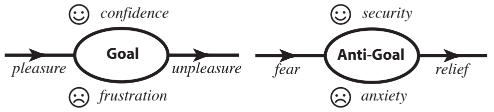
X

Emotional aspects of long-term learning

Skemp's theory (1986, 1989) focuses on both
- ***goals (to be achieved)***
- ***and anti-goals (to be avoided)***



(as pictured in HHLTTM Chapter 5)

Here we focus on one main aspect:
the teacher and learners' goals in learning mathematics:
- **long-term sense-making appropriate to the individual**
  or
- **short-term success passing the test.**

20

Skemp's theory (1986, 1989) focuses on both
goals (to be achieved) and anti-goals (to be avoided)
@ This is the full Picture …

@ For more information, see **How Humans Learn To Think Mathematically Chapter 5**. **X**

This has profound implications for making sense of mathematics. In this presentation there is only time to
@ Focus on one main aspect:
@ the teacher and learners' goals in learning mathematics: is it
@ long-term sense-making appropriate to the individual or

@ short-term success passing the test. X

Short & Long Term Goals & Anti-Goals

Is the goal of teacher or learner:
(1) long-term sense-making appropriate to the individual
or
(2) short-term success to pass the test

(1) is the more desirable goal, but if (1) fails at any stage, the goal may change to (2).

Problematic transitions can cause a change to learn to pass the test.

***The new goal of passing the test also can have a pleasurable outcome*. So many of us gain pleasure through repeating (2) which may then fail to attain (1).**

My belief is that this happens in many (most?) communities.

in particular where standards are specified to be attained.

21

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@ My belief is that this is happening in many, perhaps most, communities.

@ in particular where standards are specified to be attained. X

Social Aspects in Differing Communities

There are differing communities in (and within)

- Mathematics
- Mathematics Education
- Teaching
- Government
- Philosophy
- Science
- Other Applications

in addition to:

- differing international communities
- differing communities within a given country

22

@ There are differing communities in (and within)

@ Mathematics @ Mathematics Education @ Teaching @ Government **X**

@ Philosophy @ Science @ Other Applications **X**

@ in addition to:

@ differing international communities

@ differing communities within a given country **X**

Social Aspects in Differing Communities

Communities of advanced mathematicians have different beliefs and goals

e.g.
Pure mathematics – formal axiomatic proof with various foundations
Engineering – natural proof based on modelling problems visually and symbolically

Differing communities of math educators, teachers, researchers, curriculum designers, administrators etc. have very different perspectives

The three world framework offers an overall view to compare, contrast and evolve new understandings to improve long-term mathematical thinking.

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@ Communities of advanced mathematicians have different beliefs and goals

@ For example, Pure mathematics – formal axiomatic proof with various foundations

@ Engineering – natural proof based on modelling problems visually and symbolically **X**

@ Differing communities of math educators, teachers, researchers, curriculum designers, administrators etc. have very different perspectives **X**

@ The three world framework offers an overall view to compare, contrast and evolve new understandings to improve long-term mathematical thinking. **X**

International Comparisons & Implications

UK: continuing difficulties in 'raising standards'
USA: ongoing debate over Core standards
Netherlands: a disconnect between realistic math and later development of skills
Japan: earlier concern that children had good skills but hated maths is now focused on performance in PISA

+ many other instances
+ with some exceptions (Shanghai, Singapore, Finland?)

So what does the Three World Framework suggest?

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@ The **UK** has **continuing difficulties in 'raising standards'.**

@ In the **USA:** There is an **ongoing debate over Core standards.** **X**

@ In the **Netherlands:** There is **a disconnect between realistic math and later development of skills**

@ In **Japan:** There was an **earlier concern that children had good skills but hated maths**. The concern **is now focused on performance in PISA.**

@ There are **many other instances**

@ **with some exceptions (Shanghai, Singapore, Finland?)** **X**

@ **So what does the Three World Framework suggest?** **X**

Implications of the Three World Framework

Structurally, each world of mathematics increases in sophistication.

Operationally, there is a distinction between learning procedures and having a flexible (proceptual) meaning for symbols.

Formally, there is a difference in school between *practical mathematics* and *theoretical mathematics* and in more advanced mathematics in the transition to *axiomatic formal*.

'Making sense' changes subtly as mathematics becomes more sophisticated, with powerful supportive links and also problematic transitions that impede sense making.

25

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@ Operationally, there is a distinction between learning procedures and having a flexible (proceptual) meaning for symbols. X

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@ 'Making sense' changes subtly as mathematics becomes more sophisticated, with powerful supportive links and also problematic transitions that impede sense making. X

Implications of the Three World Framework

Structurally, making sense involves successive levels, particularly from practical to theoretical and from theoretical to axiomatic formal.

Practical properties are simultaneous,
Theoretical properties can be proved one from another.

Theoretical properties may be based on imagery,
Formal properties are based only on formal definition.

It is important for the teacher *and learner* to be aware of subconscious problematic aspects that impede learning, to make sense of the new context.

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Structurally, making sense involves successive levels, particularly from practical to theoretical and from theoretical to axiomatic formal. X

@ Practical properties are simultaneous, Theoretical properties can be proved one from another.

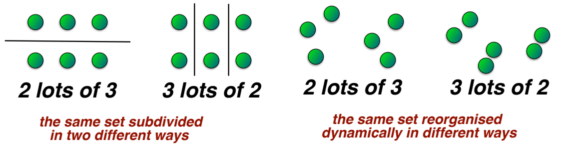
@Theoretical properties may be based on imagery, Formal properties are based only on formal definition. X

@ It is important for the teacher and learner to be aware of subconscious problematic aspects that impede learning, to make sense of the new context. X

Implications of the Three World Framework

Operationally, long-term sophistication requires the recognition of crystalline structures and the construction of crystalline concepts.

For instance, as procedures 2x3 and 3x2 are different but as crystalline concepts they are the same object.



Math Educators know that students sense the difference but in the long-term would it not be better to focus on the crystalline structure?

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Operationally, long-term sophistication requires the recognition of crystalline structures and the construction of crystalline concepts. X

@ For instance, as procedures 2x3 and 3x2 are different but as crystalline concepts they are the same object. X

@ The same set may be subdivided in two different ways,

@ or the same set may be reorganised dynamically in different ways. X

@ Math Educators know that students sense the difference but in the long-term would it not be better to focus on the crystalline structure? X

Implications of the Three World Framework

Conjecture: The best long-term solution in the symbolic world involves building flexible crystalline concepts.

This involves making sense of the ideas in context, to link ideas together coherently and to perform lengthy procedures efficiently (as in the Hakase principle).

Most curricula focus on the *positive* and fail to consider the implications of aspects that become problematic.

The consequence is that many seek the pleasure of being able to perform the required skill in context but not be aware of later problematic met-befores that require a new kind of sense-making.

I conjecture that this is the reason why so many skill-based curricula round the world fail to improve.

28

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@ I conjecture that this is the reason why so many skill-based curricula round the world fail to improve. X

Implications of the Three World Framework

In advanced mathematics, there is a major difference between

- natural proof based on thought experiment and imagery
- axiomatic formal proof based on only set-theoretic definition.

Axiomatic formal proof includes structure theorems that prove an axiomatic structure has embodied & symbolic structures.

e.g. the Peano Postulates give a unique crystalline structure: the natural numbers as successive points on a number line.

A complete ordered field has a unique crystalline structure visually as points on a line and numerically as decimals.

A group has a unique structure as permutations of a set and finite groups can be classified as generators and relations.

A finite dimensional vector space over *F* has structure as F^n. with both visual and symbolic interpretations.

29

In advanced mathematics, there is a major difference between

@ natural proof based on thought experiment and imagery

@ axiomatic formal proof based on only set-theoretic definition. X

@ Axiomatic formal proof includes structure theorems that prove an axiomatic structure has embodied & symbolic structures. X

@ For example the Peano Postulates give a unique crystalline structure: the natural numbers *visually* as successive points on a number line *with the familiar symbolic properties of arithmetic*. X

@ A complete ordered field has a unique crystalline structure visually as points on a line and numerically as decimals. X

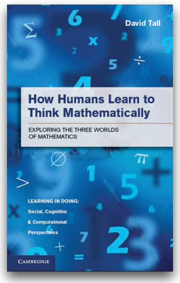
@ A group has a unique embodied structure as permutations of a set and finite groups can be classified *symbolically* in terms of generators and relations. X

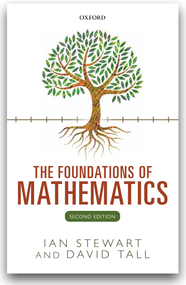
@ A finite dimensional vector space over F has structure as F^n with both visual and symbolic interpretations.

Implications of the Three World Framework

The full framework from birth to maturity is given in *How Humans Learn to Think Mathematically.*

The framework at university including *structure theorems, embodiment & symbolism* is in the 2nd edition of *Foundations of Mathematics* (Stewart & Tall 2014) planned to be translated into Japanese by Kyoritsu Shuppan Due 2018 …





30

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@ [picture appears] X

@ The framework at university including structure theorems, embodiment & symbolism is in the 2nd edition of

Foundations of Mathematics (Stewart & Tall 2014)

@ [picture appears] X

@ This is currently planned to be translated into Japanese by Kyoritsu Shuppan. Due 2018 … X

Implications of the Three World Framework

The three world framework offers a coherent framework for long-term mathematical thinking
with successive higher levels of
embodied, symbolic and formal sophistication.
It takes account of the
biological evolution of human thought
cognitively and emotionally
and the
natural and formal development of mathematics
in both the
development of the individual
and the
historical evolution of mathematics.

31 The three world framework offers a coherent framework for long-term mathematical thinking

@ with successive higher levels of embodied, symbolic and formal sophistication. X

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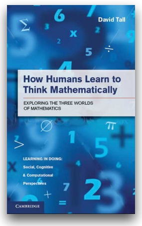
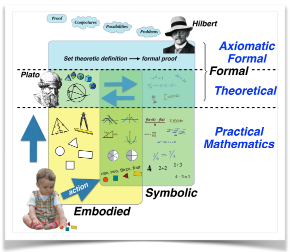
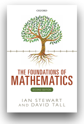
@ in both the development of the individual and the historical evolution of mathematics. X

The Three Worlds of Mathematics







Thank you for listening

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