SETTING LESSON STUDY WITHIN A LONG-TERM FRAMEWOK FOR LEARNING†

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Lesson Study is a format to build and analyze classroom teaching where teachers and researchers combine to design lessons, predict how the lessons might be expected to develop, then carry out the lessons with a group of observers bringing multiple perspectives on what actually happened during the lesson. This article considers how a lesson, or group of lessons, observed as part of a lesson study may be placed in a long-term framework of learning, focusing on the essential objective of improving the long-term learning of every individual in classroom teaching.

INTRODUCTION

This paper began as a result of a participation in a lesson study conference (Tokyo & Sapporo, December 2006) in which four lessons were studied as part of an APEC (Asian and Pacific Economic Community) study to share ideas in teaching and learning mathematics to improve the learning of mathematics throughout the communities. It included the observation of four classes (here given in order of grade, rather than order of presentation):

Placing Plates (Grade 2) – taught by Takao Seiyama
December 2nd 2006, University of Tsukuba Elementary School

Multiplication Algorithm (Grade 3) – Hideyuki Muramoto
December 5th 2006, Sapporo City Maruyama Elementary School

Area of a Circle (Grade 5) – Yasuhiro Hosomizu
December 2nd 2006, University of Tsukuba Elementary School

Thinking Systematically (Grade 6) – Atsutomo Morii
December 6th 2006, Sapporo City Hokuto Elementary School

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† Based on a plenary presentation given at the APEC–Tsukuba International Conference, December 3–7, 2006, extended as a chapter for a book of papers on Lesson Study.

The objective of this paper is to set these classes within a framework of long-term development outlined in Tokyo at the conference (Tall, 2006), which sets the growth of individual children within a broader framework of mathematical development. This long-term development of individual children depends not only on the experiences of the lesson, but in the experiences of the children prior to the lesson and how experiences 'met-before' have been integrated into their current knowledge framework.

In general, it is clear that lesson study makes a genuine attempt:

- to design a sequence of lessons according to well-considered objectives; to predict what may happen in a lesson;
- during lesson development, to have a group of observers bring multiple perspectives to what happened, without prejudice;
- to develop principles and curriculum materials to improve the teaching of mathematics for all involved.

Lesson study is based on a wide range of communal sharing of objectives. At the meeting in Tokyo, I was impressed by one essential fact voiced by Patsy Wang-Iverson:

> The top eight countries in the most recent TIMSS studies shared a single characteristic, that they had a smaller number of topics studied each year.

Success comes from focusing on the most generative ideas, not from covering detail again and again. This suggests to me that we need to seek the generative ideas that are at the root of more powerful learning.

For many individuals, mathematics is *complicated* and it gets more complicated as new ideas are encountered. For a few others, who seem to grasp the essence of the ideas, the *complexity* of mathematics is fitted together in a way that makes it essentially *simple* way. My head of department at Warwick University in the sixties, Sir Christopher Zeeman noted perceptively:

> " Technical skill is a mastery of complexity, while creativity is a mastery of simplicity." (Zeeman, 1977)

This leads to the fundamental question:

> How can we help *each and every child* find this simplicity, in a way that works, *for them*?

Lesson study focuses on the *whole class activity*. Yet within any class each child brings differing levels of knowledge into that class, related not only to what they have experienced before, but how they have made connections between the ideas and how they have found their own level of simplicity in being able to think about what they know.

To see simplicity in the complication of detail requires the making of connections between ideas and focusing on essentials in such a way that these simple essentials become generating principles for the whole structure.

In my APEC presentation in Tokyo (Tall 2006), I sought this simplicity in the way that we humans naturally develop mathematical ideas supported by the shared experiences of previous generations. I presented a framework with three distinct worlds of mathematical development, two of which dominate development in school and the third evolves to be the formal framework of mathematical research. The two encountered in school are based on (conceptual) embodiment and (proceptual) symbolism. I described these technical terms in Tall (2006) and they have been developed further in more recent publications, including *How Humans Learn to Think Mathematically* (Tall, forthcoming).

Essentially, conceptual embodiment is based on human perception and reflection. It is a way of interacting with the physical world and perceiving the properties of objects and, through thought experiments, to see the essence of these properties and begin to verbalize them and organize them into coherently related systems such as Euclidean geometry.

Proceptual symbolism arises first from our *actions* on objects (such as counting, combining, taking away etc) that are symbolized as concepts (such as number) and developed into symbolic structures of calculation and symbolic manipulation through various stages of arithmetic, algebra, symbolic calculus, and so on. This desirable form of flexible symbolism contrasts with procedural symbolism that involves only routine calculations.

Symbols such as 4+3, $x^2 + 2x + 1$, $\int \sin x\,dx$ all dually represent processes to be carried out (addition, evaluation, integration, etc) and the related concepts that are constructed (sum, expression, integral, etc). Such symbols also may be represented in different ways, for instance 4+3 is the same as 3+4 or even 'one less than 4+4' which is 'one less than 8' which is 7. This flexible use of symbols to represent different ***pro***cesses for giving the same underlying con***cept*** is called a ***procept***.

These two worlds of (conceptual) embodiment and (proceptual) symbolism develop in parallel throughout school mathematics and provide a long-term framework for the development of mathematical ideas throughout school and on to university, where the focus changes to the formal world of set-theoretic definition and formal proof.

In figure 1, we see an outline of the huge *complication* of school mathematics. On the left is the development of conceptual embodiment from practical mathematics of physical shapes to the platonic methods of Euclidean geometry. In parallel, there is a development of symbolic mathematics through arithmetic, algebra, and so on, with the two blending as embodiment is symbolized or symbolism is embodied.

The long-term development begins with the child's perceptions and actions on the physical world. In figure 1, the child is playing with a collection of objects: a circle, a triangle, a square, and a rectangle. The child has two distinct options, one is to focus on his or her *perception* of each object, seeing and sensing their individual properties, the other is through *action* on the objects, say by counting them: one, two, three, four.

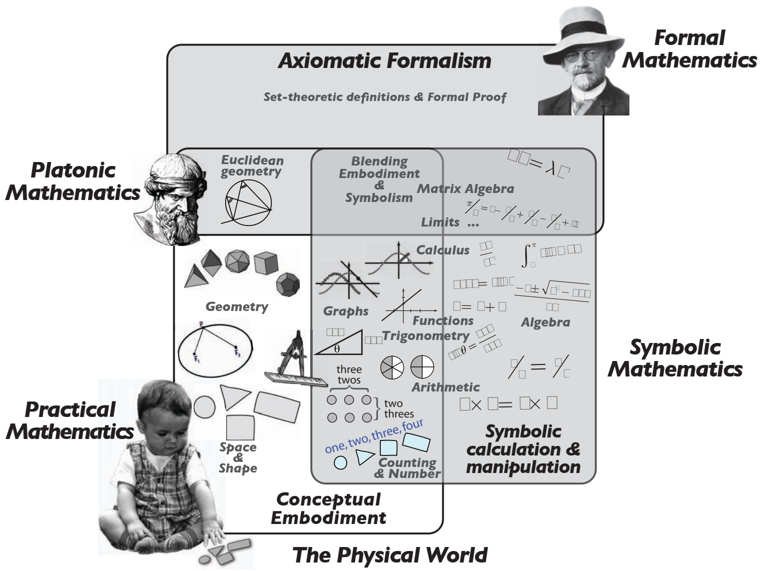

Figure 1. The three mental worlds of (conceptual) embodiment, (proceptual) symbolism and (axiomatic) formalism

The focus on perception, with vision assisted by touch and other senses to play with the objects to discover their properties, leads to a growing sense of space and shape, developing through the use of physical tools—ruler, compass, pinboards, elastic bands—to enable the child to explore geometric ideas in two and three dimensions, and on to the mental construction of a perfect platonic world of Euclidean geometry. The focus on the essential qualities of points having location but no size, straight lines having no width but arbitrary extensions and on to figures made up using these qualities leads the human mind to construct mental entities with these essential properties. Platonism is a natural long-term construction of the enquiring human mind.

Meanwhile, the focus on action, through counting, leads eventually to the concept of number and the properties of arithmetic that benefit from blending embodiment and symbolism, for example, 'seeing' that $2\times 3 = 3\times 2$ by visualizing 2 rows of 3 objects being the same as 3 columns of 2 objects. Long-term, there is a development of successive number systems, fractions, rationals, decimals, infinite decimals, signed numbers, real numbers, complex numbers. What seems to the experienced mathematician as a steady extension of number systems is, for the growing child, a succession of changes of meaning which need to be addressed in teaching. We will return to this shortly.

The symbolic world develops through increasingly sophisticated number systems which are given an embodied meaning through the number-line. These are extended further into the plane through cartesian coordinates, graphs relating symbolism to embodied visualization, with subjects such as trigonometry being a blend of geometric embodiment and operational symbolism. In the latter stages of secondary schooling, the learner will meet more sophisticated concepts, such as symbolic matrix algebra and the introduction of the limit concept, again represented in both embodied and symbolic form.

The fundamental change to the formal mathematics of Hilbert leads to an axiomatic formalism based on set-theoretic definitions and formal proof, including axiomatic geometry, axiomatic algebra, analysis, topology, etc.

Cognitive development works in different ways in embodiment, symbolism and formalism (Figure 2). In the embodied world, the child is relating and operating with perceived objects (both specific and generic), verbalizing properties and shifting from practical mathematics to the platonic mathematics of axioms, definitions and proofs.

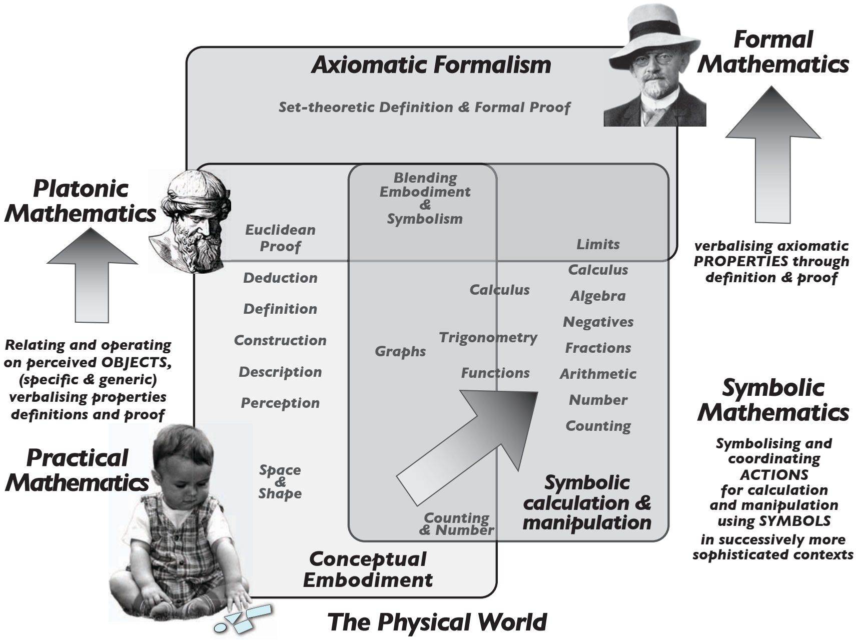

Figure 2: long-term developments in the three worlds

In the symbolic world, development begins with actions that are symbolized and coordinated for calculation and manipulation in successively more sophisticated contexts. The shift to the axiomatic formal world is signified by the switch from concepts that arise from perceptions of, and actions on, objects in the physical world to the verbalizing of axiomatic properties to define formal structures whose further properties are deduced through mathematical proof.

Focusing on the framework appropriate to school mathematics, we find the main structure consists of two parallel tracks, in embodiment and symbolism, each building on previous experience (met-befores), with

> *embodiment* developing through perception, description, construction, definition, deduction and Euclidean proof after the broad style suggested by van Hiele;
>
> *symbolism* developing through increasingly sophisticated compression of procedures into procepts as thinkable contexts operating in successively broader contexts.

These two developments are fundamentally different. Embodiment gives a global overall picture of a situation. Symbolism begins with coordinating and practicing sequences of actions to build up a procedure, perhaps refining it to give different procedures that are more efficient or more effective, using symbolism to record the actions as thinkable concepts. Many different procedures can, for some, seem highly complicated and so the teacher has the problem of reducing the complexity, perhaps by concentrating on a single procedure to show the pupils what to do. Procedures, however, occur *in time* and become routinized so that the learner can *perform* them, but is less able to *think about* them. (Figure 3.)

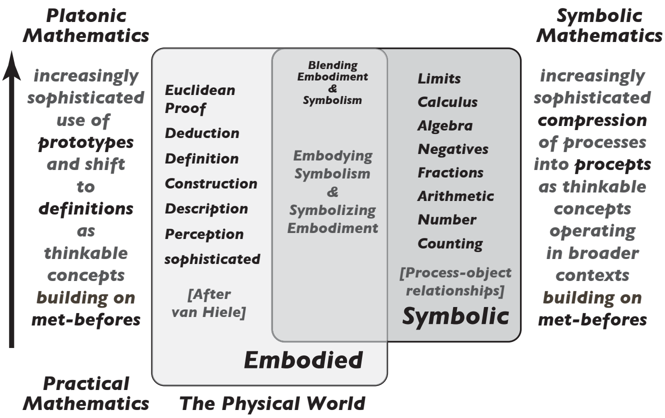

Figure 3: Developmental framework through embodiment and symbolism

An Example of Lesson Study in action

An example from the lessons observed involved the teaching of multi-digi multiplication. First children need to learn their tables for single digit multiplication from 0×0 to 9×9. They also need to have insight into place value and decimal notation.

The method used by Hideyuki Muramoto in one of the study lessons discussed later can be analyzed in terms of an initial embodiment representing 3 rows of 23. Here the learner can *see* the full set of counters: the problem is how to *calculate* the total. The embodiment can be broken down in various ways, separating each row into subsets appropriate to be able to compute the total. In the previous lesson the students had already considered 3 rows of 20 and had broken this into various sub-combinations, subdividing each row into 10+10 or 5+5+5+5, or even 9+9+2, or 9+2+9.

Now the problem related to breaking 23 into sub-combinations, results in the children proposing various possibilities including 10+10+3 and 9+5+9 (but not 5+5+5+5). Three lots of 10+10+3 gives 30+30+9, which easily gives 60+9, which is 69. Three lots of 9+5+9 is more difficult requiring the sum 27+15+27. Here we have two different procedures giving the same result, 69, and the most productive way forward is to break the number 23 into tens and units and multiplying each separately by 3.

In this analysis, the embodiment gives the *meaning* of the calculation of a single digit times a double digit number, while the various distinct sub-combinations give different ways of *calculation,* from which the sub-combination as tens and units is clearly the simplest and the most efficient.

The approach has a general format:

1. *Embody the problem* (here the product 3×23);
2. *Find several different ways of calculation* (here 3×23 is three lots of 10+10+3 or three lots of 9+9+5, etc) *where the embodiment gives meaning to symbolism*;
3. See *flexibility,* that all of these are the same;
4. See that the *standard algorithm is the most efficient.*

The embodiment gives meaning while the symbolism enables compression to an efficient symbolic algorithm that links flexibly to the embodied meaning.

It is not expected that all the children will be able to cope with every procedure (for instance, the suggestion 9+5+9 is likely to come from a more able child and the computation is likely to be too difficult for many of the others). The more successful may see the different ways of computing the result as different procedures with the same effect, and meaningfully see that the standard algorithm is just one of many that is chosen because it is efficient and simple. However, others may find it too complicated to calculate the product as 3 times 9+9+5 and not even desire to carry it through. Even so, some of these may still grasp the principle that different procedures can give the same result. Meanwhile, those who are less fluent in their tables and feel insecure with the more complicated procedures may seek use the standard method because it is less complicated. Focusing on a single procedure may have its attractions, showing *how to do it,* without the complication of *why it works.* However, such a procedural approach may have short-term success yet fail to produce long-term flexibility.

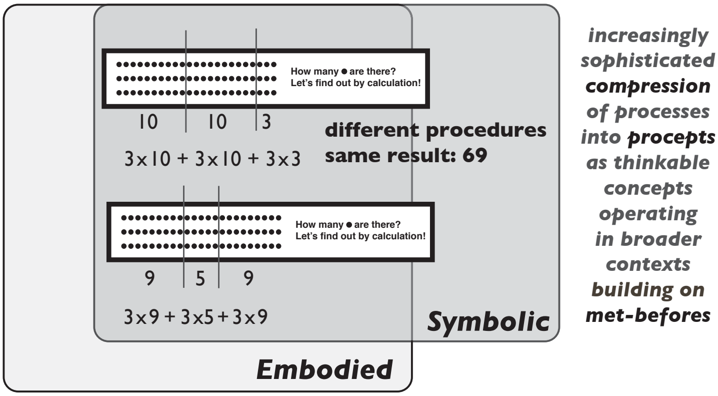

Figure 4: multi-digit arithmetic from embodiment to symbolism

Gray & Tall (1994) observed a growing divergence between those who succeed by developing flexible methods of operation and those who remain fixed in step-by-step procedures using rote-learnt rules that become increasingly fragile as the problems become more complicated.

The lesson is designed to encourage the child to build meaningfully on ideas that have been met before. However, different children build on their experiences in different ways. Sometimes the experiences met-before are *supportive* in a new context and sometimes they are *problematic*. Flexible use of number properties may be supportive for some, as is the development of efficient use of algorithms, but the fixation on procedural learning without meaning can become problematic.

BLENDING KNOWLEDGE STRUCTURES IN THE BRAIN

In addition to the combination of embodiment and symbolism to give meaning to number concepts and operations, there are subtle features of successive number systems that can become problematic. A mathematician may see successive numbers systems:

Whole Numbers
Fractions
Rational Numbers
Positive and Negative Numbers
Real Numbers consisting of rationals and irrationals.

They can all be marked on an (embodied) number line and the child should be able to *see* how each one is extended to the next. However, for the learner, each extension has subtle aspects that can become problematic. For example, there are subtle difficulties between counting and measuring:

> Counting 1, 2, 3, ... has successive numbers, each with a next number and no numbers in between. Multiplying these numbers gives a bigger result.
>
> Measuring numbers are continuous without a 'next' number and have fractions between. Multiplying can give a smaller result. Fractions involve new ideas of equivalence and new algorithms for addition and multiplication.

Not only must the learner deal with new number concepts and new procedures, they also encounter experiences that may be sensed as being problematic.

The examples we meet in the four lessons considered in this chapter focus on the supportive elements of prior knowledge, but in the overall picture, we should be aware of the problematic met-befores that occur as children encounter successive number systems. The majority of teachers and learners around the world seem to end up learning mainly procedural rules to pass tests rather than seeking flexibility that supports long-term understanding.

USING A LONG-TERM FRAMEWORK OF EMBODIMENT AND SYMBOLISM IN LESSON STUDY

Putting together the ideas of growth in elementary mathematics discussed here and in the earlier paper (Tall, 2006), we find that the parallel development of embodiment and symbolism suggests:

> Embodiment gives human meaning as prototypes, developing verbal description, definition, deduction.
>
> Symbolism is based initially on human action, leading to symbol use, either through procedural learning or through conceptual compression to flexible procept.
>
> Experiences build met-befores in the individual mind that are used later to interpret new situations.

Tall (2006) also observed:

> Embodiments may work well in one context but become increasingly complex; flexible symbolism may extend more easily.

This means that the flexible use of symbolism in the long-term can lead to ideas that are not only more powerful, they may also be more simple to use.

In our earlier discussions in Tokyo, great emphasis was made not only on meaningful learning of mathematical concepts and techniques, but also on *problem solving* in new contexts. Learning new concepts can be approached in a problem-solving manner. My own view is that learners must take responsibility for their own learning, once they have the maturity to do so, which includes developing their own methods for solving problems. I also believe that teachers have a duty, as mentors, to help focus students on methods that are more powerful and have more essential long-term value.

In lesson study we therefore require objectives to consider. There are so many theories in the literature: from Piaget's theory of successive stages of development, Bruner's (1966) analysis into enactive iconic and symbolic, Skemp's (1976) insight into instrumental and relational understanding and his (1979) modes of building and testing concepts, van Hiele's (1986) ideas of structure and insight in geometry, Fischbein's (1987) categorization of thinking into intuitive, algorithmic and formal, the unistructural–multistructural–relational–extended abstract modes of Biggs and Collis (1982), the process-object theories of Dubinsky (Asiala *et al.*, 1996) and Sfard (1991), the Pirie-Kieren theory (1994) with its ideas of 'making' and 'having' images and successive levels of operation, RBC theory (Recognizing, Building-with, Consolidating) formulated by Hershkowitz *et al.* (2001), theories of problem-solving (Polya 1945, Schoenfeld 1985, Mason *et al.* 1982) and so on.

With such a wealth of ideas to choose from and build on (or build with), to make sense in the classroom, we need to focus on a few simple yet profound ideas that are fundamentally helpful. You may choose different ones, but in the long run, it is essential for those involved in Lesson Study to have principles that offer a usable framework for any sequence of lessons. For instance, a long-term development may focus on three aspects:

> **Using** knowledge structures in routine and problem situations (where 'routine' includes practicing for fluency);
>
> **Building** thinkable concepts in (meaningful) knowledge structures;
>
> **Reasoning** about relationships (as appropriate for a given context).

I see these aspects as operating interactively rather than as a hierarchy and would see them being applied *before, during* and *after* each lesson, as follows:

> **BEFORE:** What is the purpose of the lesson?
>
> (**Using** known routines or problem-solving techniques, **Building** new constructs, **Reasoning** (to justify relationships), and what experience may the learners have to make sense of the lesson? (*met-befores, routines, problem solving techniques, reasoning*);
>
> **DURING:** How do learners use their knowledge structures during the lesson to make sense of it? (*met-befores, routines, problem solving techniques, reasoning*);
>
> **AFTER:** What knowledge structures are developing that may be of value in the future? (*met-befores, routines, problem solving techniques, reasoning*).

LESSON STUDIES

Four Lessons were studied in Japan in December 2006;

> **Placing Plates (Grade 2)** – Takao Seiyama
> December 2nd 2006, University of Tsukuba Elementary School
>
> **Multiplication Algorithm (Grade 3)** – Hideyuki Muramoto
> December 5th 2006, Sapporo City Maruyama Elementary School
>
> **Area of a Circle (Grade 5)** – Yasuhiro Hosomizu
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> **Thinking Systematically (Grade 6)** – Atsutomo Morii
> December 6th 2006, Sapporo City Hokuto Elementary School

My purpose is to focus on the role of these lessons in long-term learning, and to consider how the long-term development of each and every student may be affected by the lesson within the given framework.

There is already a great deal of evidence of the use of broad principles in the planning of the four lessons. Taking a few quotes at random from the plans, we find:

> The goal of the Mathematics Group at Maruyama is to develop students ability to use what they learned before to solve problems in the new learning situations by making connections.
>
> In addition, we want to provide 3rd grade students with experiences in mathematics that enable them to use what they learned before to solve problems in new learning situations by making connections.

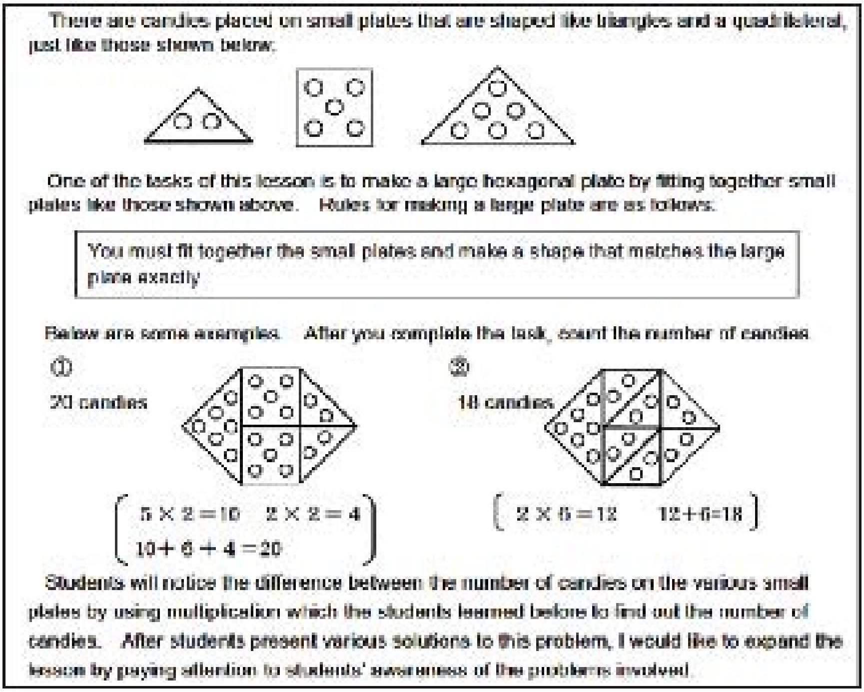
> Through teaching mathematics, I would like my students to develop a 'secure ability' for finding problems on their own, studying by themselves, thinking, making decisions, and executing those decisions. Moreover, I would like to help my students to like mathematics as well as enjoying thinking.
>
> In order for students to find better ideas to solve a problem, it is important for the students to have an opportunity to feel that they really want to do so.
>
> Starting in April (the beginning of the school year), I taught the students to look at something from a particular point of view such as 'faster, easier, and accurate' when they think about something or when they compare something.
>
> If you think about the method that uses the table from this point of view, students might notice that "it is accurate but it takes a long time to figure out" or "it is accurate but it is complicated."
>
> In order to solve a problem in a short time and with less complexity, it is important for the students to notice that calculation using a math sentence is necessary.

Each of these shows a genuine desire for students to make connections, to rely on themselves for making decisions and to seek more powerful ways of thinking with less complication. The videos of the classes themselves show high interaction between the students and with the teacher, as the teacher carefully guides the lesson to bring out essential ideas.

We now briefly look at each lesson in turn, to see how it fits with the framework of long-term development blending embodiment and symbolism, considering aspects of Using, Building and Reasoning that arise as an explicit focus of attention, before, during, and after the lesson. In particular, we consider how children respond to the lesson in ways that may be appropriate for long-term development of powerful mathematical thinking.

In the pages that follow, I use photographs that I took during each of the lessons to illustrate the overall plan of building ideas from a blend of embodiment and symbolism to using and reasoning about powerful mathematical concepts. This is, in no way, intended to be a once-and-for-all analysis. It is offered as a preliminary analysis to promote the use of lesson study as an approach to develop good curriculum materials that can be used widely by teachers to encourage learners to make sense of mathematics.

LESSON 1 Placing Plates (Grade 2) – Takao Seiyama
December 2nd 2006, University of Tsukuba Elementary School


There are candies placed on small plates that are shaped like triangles and a quadrilateral, just like those shown below.

One of the tasks of this lesson is to make a large hexagonal plate by fitting together small plates like those shown above. Rules for making a large plate are as follows:

You must fit together the small plates and make a shape that matches the large plate exactly

Below are some examples. After you complete the task, count the number of candies.

① 20 candies

(5 × 2 = 10 2 × 2 = 4
10 + 6 + 4 = 20)

② 18 candies

(2 × 6 = 12 12+6=18)

Students will notice the difference between the number of candies on the various small plates by using multiplication which the students learned before to find out the number of candies. After students present various solutions to this problem, I would like to expand the lesson by paying attention to students' awareness of the problems involved.

Figure 5: The problem: Placing Plates

The teacher’s notes included the following statement:

> There are two objectives in this lesson. The first is to foster students’ geometrical sense through composition of geometric shapes and the second is to foster students’ ability to think logically and understand mathematical expressions by asking them to think about the composition of geometric shapes and their corresponding mathematical expressions.

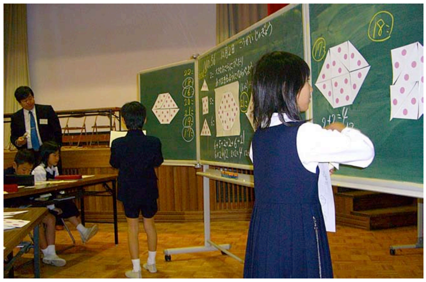
Instruction Plan

Phase 1: Meaning of triangles and quadrilaterals (2 periods).

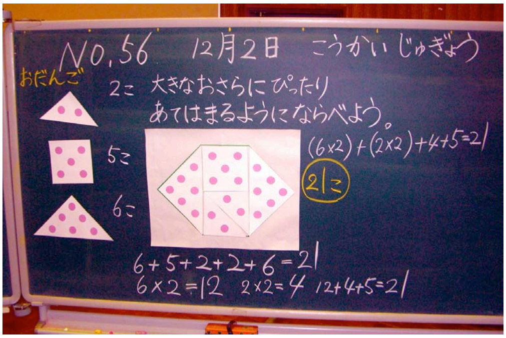
Phase 2: Composition and construction of triangles and quadrilaterals (2 periods).

Phase 3: Summary and practice – 1 period.

Experimenting with the problem



Sharing Data



Organizing Data

The lessons proved to be an enjoyable well-planned activity allowing a wide range of levels of performance. Elements involved included:

Using ideas in a non-routine *problem-solving* activity;

Reasoning by physical embodied experiment;

Met-before: shapes, simple arithmetic;

Activity: how to think flexibly in a specific problem situation;

Long-term: flexible thinking with specified rules, encouraging a problem-solving attitude in an idiosyncratic problem.

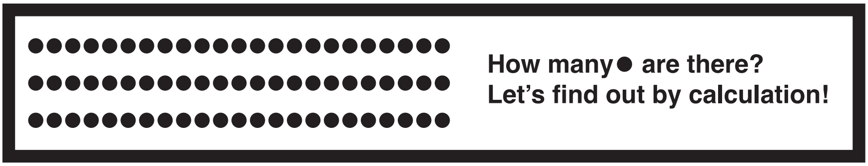
The activities included using arithmetic, problem solving (e.g. finding all possible combinations), with some idiosyncratic elements e.g. squares can have 5 or 6 candies on them. Questions arising in the discussion included:

What is the *important long-term role* of this lesson that the children should focus on?

What do individual children learn from this experience that are valuable in the long term?

LESSON 2: Multiplication Algorithm (Grade 3)

Hideyuki Muramoto
December 5th 2006, Sapporo City Maruyama Elementary School

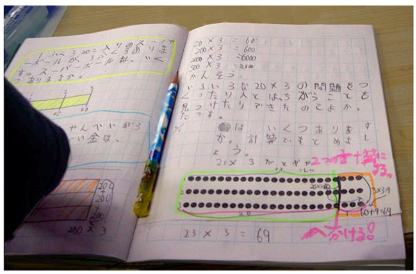


Goals of the unit proposed by the teacher

- Lessons that enable students to consciously think about the connection between what they learned before and what they are learning now;
- Lessons in which students learn from each other and that help them consciously think about their own solution processes;
- An evaluation method that helps foster students' logical thinking abilities;
- unit plan;
- This lesson (goals, process of lesson).

The teacher introduced the problem and the students worked on it together.

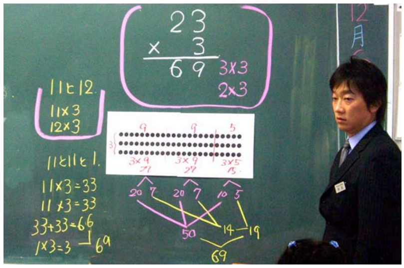

Introducing the problem


Trying out ideas


Sharing insights


Explaining to the class

Displaying different solutions



Comparing solutions

At the end of the lesson, the teacher had organized the material placed on the board by himself and his pupils, starting with simple pictures on the left, with a range of different approaches across to the right, culminating in the blending of the visual array and the symbolic addition using place value. (A more detailed analysis is given in Tall, 2008.)

Aspects that arose during the lesson included:

Building ideas in a flexible manner;

Met-befores: single-digit multiplication, subdividing a problem into smaller problems;

Activity: constructing different ways of calculating 3 times 23;

Long-term: flexible thinking about multiplication, revealing the standard algorithm as the most efficient.

LESSON 3: Area of a Circle (Grade 5) – Yasuhiro Hosomizu

December 2nd 2006, University of Tsukuba Elementary School

Plan of the unit: Area of circle, 10 lessons

1. Circle and regular polygons (2 lessons);
2. Length of circumference (3 lessons);
3. Area of circle (3 lessons, with this lesson the second of the three);
4. Summary and applications (2 lessons).

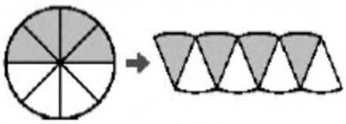
Goal of this lesson

> Students will be able to come up with ways to find the area of a circle by rearranging the shape of the circle so that they can use previously learned formulas for rectangles, parallelograms, triangles, to derive the formula for the area of a circle.

The plan is to present the problem as follows:

1. Present the problem

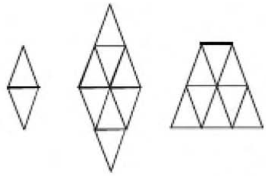
Come up with ways to find the area of the circle by using the sectors that are made by segmenting the circle into eight

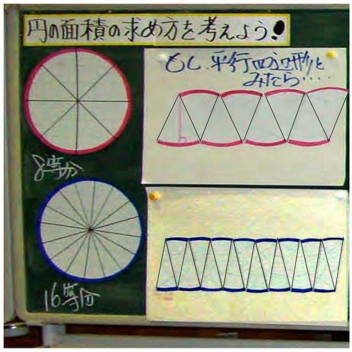


The area of parallelogram=base×height

2. Think about different ways to rearrange the shape so that other formulas for finding the areas of basic shapes can be used

Rearrange the shape and find different formulas to find the area





dividing into 8 and 16 pieces

The children worked through the problems and shared the results:

Making up Solutions

Explaining

Summarizing

This was again a well-organized lesson in a sequence designed to give a flexible insight into various ways of seeing the area of a circle.

> **Building** ideas of the area of rectangles, triangles, parallelograms; cutting a circle into 8 or 16 parts which approximate to triangles that can be rearranged into a shape looking much like a parallelogram.
>
> **Met-befores**: counting squares to calculate the area of a rectangle; experiences of adding and taking away areas.
>
> **Activity**: Cutting a rectangle in half to find the area of a right-angled triangle; generalizing to other cases such as a parallelogram, cutting off a triangle and shifting it to give a rectangle, cutting up a circle into approximately triangle areas and re-assembling into a near parallelogram.
>
> **Long-term**: Giving meaning to the area of a circle. Questions remain about the curved edges in the area Visibly, as the number of pieces increases the curved sides of the area approximate to a straight line.

The observers considered the understandings of different children and the long-term development of ideas such as approximating areas.

LESSON 4: Thinking Systematically (Grade 6)

Atsutomo Morii, December 6th 2006, Sapporo City Hokuto Elementary School

The purpose of this lesson is to introduce a problem that can be solved using tables, seeing patterns and producing a variety of solutions.

4. Process of the lesson

Students' activities and thinking process	Teacher's support
We bought pencils and ballpoint pens and the total number of items were 10 and the price was 460 yen. The price of each pencil was 40 yen and the ballpoint pen was 70 yen. How many pencils and how many ballpoint pens did we buy? If we calculate it... If we make a table...	○ Listening to the students' muttering (or voices) and pick up the idea to use a table to solve this problem. Then ask the students to fill in the table on the worksheet.

# of pencils	0	1	2	3	4	5	6	7	8	9	10
# of ballpoint pens	10	9	8	7	6	5	4	3	2	1	0
Total price (yen)	700	670	640	610	580	550	520	490	460	430	400

The table in the textbook shows the number of pencils and ballpoint pens from 1 to 9, but in this lesson I decided to use the number from 0 to 10. This is decision relates to my hope for a certain kind of mathematical thinking that I want my students to acquire.

The teacher's plan

In the 4th grade, students investigated changing quantities and expressing a relationship between two quantities with tables and math sentences.

In the 5th grade, students learned to solve problems by finding the relationships between two quantities and their regularity using tables.

The aim of this lesson is to use knowledge from prior grades to solve problems using tables that have more items. The lesson is in the textbook, as an individual lesson before a unit on "proportional relationships." In the unit, students will construct tables, finding patterns, and express the relationship using math sentences. I believe this lesson is included here to help students prepare to deal with proportional relationships.

In the lesson, I anticipate that students might solve the problem by coming up with an appropriate value and then calculating, or by constructing a table.

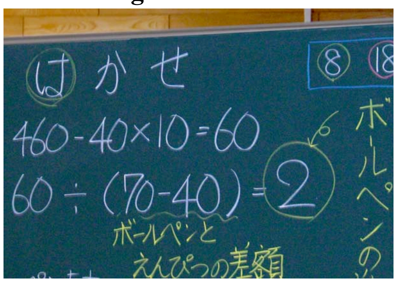
I would like to focus on a kind of mathematical thinking, i.e. hypothetical thinking. Something like "If it is … then …."

By changing the quantities of the items in the problem on their own, the students can come up with better solution methods. In order to do that, I think it is important for the students to see an extreme case in the table such as "I bought 10 items of one kind and 0 items of the other kind."

Starting a table with zero

Building data in columns

Organizing the complete table



A more sophisticated solution

Aspects that arose in this lesson included:

Building ideas relating 2 variables using tables.

Using problem-solving to use the data systematically

Met-before: Previous experience of relationships & tables.

Activity: more subtle solutions possible, but main focus on tables.

Long-term: to realise tables are systematic, but tedious, to create a need for a more powerful way to express and solve the problem.

Children may find that tables work but are not efficient, hence encouraging the later development of algebra in a more focused manner.

Questions:

What is the *important long-term role* of this lesson that the children should focus on?

What do individual children learn from this in the long-term?

Reflections

Around the whole world, there are concerns on how children learn, or fail to learn, mathematics. In Britain, attention is focused on the needs of 'pupils at risk' who need extra support and of the 'gifted and talented', who need extra challenges as successive governments attempt to 'raise standards'.

Mathematical learning is not a linear race, with some 'falling behind' and others 'racing ahead'. It is also a question of different kinds of learning based on different interpretations of previous experience and different ways of coping or making sense.

This focuses our attention on the need to improve the long term learning of every child. Lesson Study offers such an approach and this is enhanced by a long-term framework that focuses not only on what needs to be learnt and how, but also to take account of the supportive and problematic aspects of learning based on how the child builds on what has been met-before so that the mathematical thinking of every child can be enhanced.

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