The roles of visualization and symbolism in the potential and actual infinity of the limit process

Ivy Kidron,
Department of Applied Mathematics,
Centre for Educational Studies,
Jerusalem College of Technology,
Israel
ivy@jct.ac.il

David Tall,
Centre for Education Studies,
University of Warwick,
Coventry,
United Kingdom
david.tall@warwick.ac.uk

Abstract

A teaching experiment—using Mathematica to investigate the convergence of sequence of functions visually as a sequence of objects (graphs) converging onto a fixed object (the graph of the limit function)—is here used to analyse how the approach can support the dynamic blending of visual and symbolic representations that has the potential to lead to the formal definition of the concept of limit. The study is placed in a broad context that links the historical development with cognitive development and has implications in the use of technology to blend dynamic perception and symbolic operation as a natural basis for formal mathematical reasoning. The approach offered in this study stimulated explicit discussion not only of the relationship between the potential infinity of the process and the actual infinity of the limit, but also of the transition from the Taylor polynomials as approximations to a desired accuracy towards the formal definition of limit. At the end of the study, a wide spectrum of conceptions remained. Some students only allowed finite computations as approximations and denied actual infinity but for half of the students involved in the study the infinite sum of functions was perceived as a legitimate "object" and was not perceived as a dynamic "process" that passes through a potentially infinite number of terms. For some students the legitimate object was vague or generic but we also observed other students developing a sense of the formal limit concept.

Keywords: Actual infinity; potential infinity; limit process; limit concept; dynamic visualization; embodiment; symbolism.

1. INTRODUCTION

Over recent years there has been extensive research concerning the difficulties that students experience with the limit concept. Instead of conceptualizing a limit as a mental object in its own right, there is a strong tendency to see it as a potentially infinite process that goes on forever and never quite reaches its desired goal. This has been an ongoing fault line in mathematics since the Greeks debated the distinction between potential and actual infinity over two thousand years ago.

The problem becomes more complicated when shifting from limits of sequences of numbers to limits of sequences of functions $(f_n(x))$. The traditional technique is to calculate the limit for a fixed value of *x* and then allow *x* to vary to give the limit function. However, this means that the learner is building on difficulties experienced with the limits of sequences of numbers and moving to functions at an increasing level of complexity.

The invention of interactive computer graphics allows an alternative approach. Instead of focusing initially on the symbolic process of a sequence of numbers tending to a limit, then moving on to a sequence of functions tending to a limit pointwise, it becomes possible to focus on the graphs of successive functions $(f_n(x))$ as whole *objects* that change dynamically as *n* increases to stabilize visually on the limit function and then to relate this to the symbolic limit as an object that can be approximated as closely as is desired. For example, in figure 1, $P_n(x)$ is the polynomial approximation to the function sin(*x*) given by

$$P_1(x) = x,\ P_2(x) = x - x^3/3!,\ P_3(x) = x - x^3/3! + x^5/5!,\ ....$$

The successive graphs not only get *close* to the limit function sin(*x*), they become *visually indistinguishable* from the graph of sin(*x*) to within the accuracy of the picture.

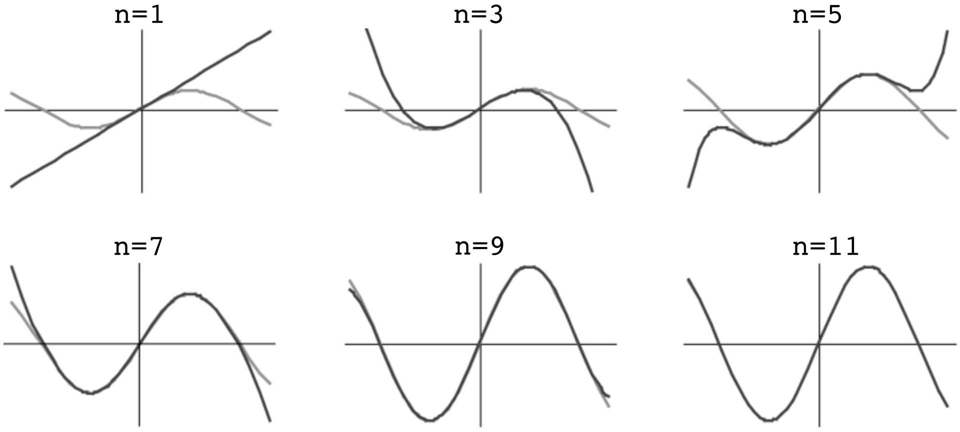

Figure 1: Taylor polynomials of degree 1, 3, 5, 7, 9, 11, superimposed on the graph of sin(*x*)

Kidron (2002) used this observation to advantage to compare two distinct approaches to computing the rational function $f(x) = 1/(1-x)$. One followed the historical symbolic approach of Euler, writing $f(x) = Q(x)/R(x) = a_0 + a_1x + a_2x^2 + ...$ to represent the function as an algebraic calculation dividing polynomials on the left and as an infinite sum on the right. In this representation, the error term can be written as $f(x) - P_n(x)$ where $P_n(x) = a_0 + a_1x + ... + a_nx^n$ and, for a numerical value x_0, the error may be calculated numerically as $Q(x_0)/R(x_0) - P_n(x_0)$. The other used a dynamic graphical approach, drawing successive polynomial approximations $P_n(x)$ with *Mathematica* to see that, as the error term becomes small as n increases, the graph of successive polynomials soon look virtually the same as the graph of $f(x)$.

In the latter approach the infiniteness of the process of 'getting close' is replaced by the practical notion of being 'good enough' for the graphs to look the same on a computer screen. The animation that illustrates how the different Taylor polynomials approach the graph of the function blends together the process of "getting close" and the visual appearance of "looking the same". It then becomes possible to perform a thought experiment to imagine drawing the graph on a computer screen of far higher resolution, to begin to think about the conception: 'given a required level of accuracy, can we find a suitably large n such that $P_n(x)$ and $f(x)$ are indistinguishable from each other to this required level' We might even reason that (over a given interval $[a,b]$), given a specific numerical error $\varepsilon > 0$, we may seek an N such that, for $n \geq N$, the graphs of $P_n(x)$ and $f(x)$ differ by less than ε.

Such an approach blends together the visual perception of the changing function with the corresponding numerical calculations to lead to the formal definition. This precisely fits the 'three worlds' framework of Tall (2004, 2013), which characterizes the development of mathematical thinking through human perception, symbolic operation and mathematical reasoning. According to this framework, school mathematics involves a natural blending of two distinct mental worlds of thought: *conceptual embodiment* coordinating perceptions and thought experiments and *operational symbolism* where operations are performed by numeric and symbolic computations. Reflections on the properties that arise in these two worlds lead to embodied forms of definition and proof in geometry and symbolic forms of definition and proof in arithmetic and algebra. The *axiomatic formal* world of mathematics arises through the introduction of quantified set-theoretic definitions with properties deduced using formal proof.

2. HISTORICAL EVOLUTION OF IDEAS

Historically, the mathematics of change and growth emerged in Greek times with the conflict between the potential infinity of performing a repeating sequence of actions as long as desired and the actual infinity of completing such a process. Initially geometry and number were seen as separate ideas. In geometry, a length had magnitude, so a product of two lengths was an area and of three was a volume. By the simple expedient of choosing a unit length, Descartes (1637) used the theory of proportions to interpret the product of lengths as a length and introduced Cartesian coordinates to link geometry to algebra. Early developments of the calculus in the seventeenth century involved mathematicians such as Kepler (1604) and others who imagined a smooth curve as a polygon with an infinite number of infinitesimally small straight sides, with the tangent at a point simply being the prolongation of one of these sides. Barrow (1670) used this idea to formulate geometrical principles that Newton (1671) and Leibniz (1684) developed into symbolic methods to compute rates of change and growth. In the eighteenth century, Euler (1748) focused on the symbolism and performed operations on infinite series as if they behaved according to the same rules as finite arithmetic. In the nineteenth century, Cauchy (1821) and others continued to imagine the number line visually in the manner of the Greeks as an entity in itself on which numbers could be marked. Bolzano (1817) realized the need to introduce formal definitions of the real numbers, which led to the formal definition of limit introduced by Weierstrass that became the foundation of modern analysis. Subsequently Hilbert (1900) proposed the formalist approach based on formal set-theoretic definition and deduction.

Before Hilbert, mathematics had always been linked to *natural* phenomena, observed perceptually, imagined mentally within the mind and calculated symbolically. This is exemplified by the name 'natural philosophy' to describe the kind of mathematics used by Newton to imagine mathematical models to linking the falling of an apple to the motion of the planets. After Hilbert, pure mathematics is given a *formal* foundation that is 'future-proofed', in the sense that any theorem proved in a system satisfying explicit set-theoretic axioms and definitions would hold in any other context where those axioms and definitions are satisfied, so that formal mathematics has greater generality than natural mathematics.

The shift from the natural mathematics in school to formal mathematics in university continues to present formidable obstacles to the learner whose experience is based on symbolic computation in arithmetic and algebra and visual perception in geometry, blended together in a natural approach to the calculus.

In the twenty-first century, new tools are being introduced that enable us to visualize thought experiments and calculate powerfully and accurately. This offers a new approach to mathematical analysis through a natural approach blending dynamic embodiment and computational symbolism leading naturally to formal definitions and proof (Mejia - Ramos & Tall, 2004; Tall, 2009, 2013).

This study considers an approach by Kidron (2002, 2003) to use *Mathematica* to enable her students to investigate and debate the historical development of the calculus, contrasting Euler's symbolic approach with power series as infinite polynomials and a dynamic visual approach as the sequence of graphs of polynomials $P_n(x)$ stabilizes on the limit function $f(x)$. The strategy is to consider the error term $f(x)-P_n(x)$ as the difference between a known function $f(x)$ and a polynomial, both of which are computable, and so the error function can be calculated. It transpires that, in this approach, some students imagine the function $f(x)$ as an infinite sum $a_1(x)+a_2(x)+...+a_n(x)+...$ with a polynomial approximation

$$P_n(x)=a_1(x)+a_2(x)+...+a_n(x),$$

where the error term is conceived as an 'infinite tail' in the form

$$a_{n+1}(x)+a_{n+2}(x)+...$$

The various student experiences with *Mathematica* are used to stimulate a class discussion to encourage students to formulate and debate their individual views of the potential and actual aspects of the limit concept, to seek a resolution that may lead to the formal limit concept.

3. RECENT RESEARCH ON STUDENT VIEWS OF INFINITE PROCESSES AND CONCEPTS

As individuals we only live for a finite time and can only carry out actions a finite number of times. Since Greek times, the natural conception of infinity is the concept of *potential* infinity, including the unlimited possibility of counting or the possibility of dividing an interval into successively smaller parts. However, when considering a sequence of numerical values given by a formula such as 1, $1/2$, $1/3$,…the successive terms follow a pattern that may be conceived as a variable quantity $s_n = 1/n$ that 'becomes arbitrarily small' as *n* increases (Cornu, 1981, 1991). From such a viewpoint, the sequence (s_n) where

$$s_n = 0.\underbrace{999...9}_{n \text{ places}} = 1 - (1/10^n)$$

tends to a quantity that is less than one by an arbitrarily small quantity. Fischbein, Tirosh and Hess (1979) reported a case where a student insisted that $1 + 1/2 + 1/4 + 1/8 + ...$ is $2 - 1/\infty$. This is consistent with the idea that the limit object will have the same properties as the objects tending to the limit. For instance, the sequence, 0.9, 0.99, … is a sequence whose *n*th term is $1 - (1/10^n)$, which is always less than one, so the infinite decimal 0.999…, is also less than one. Tall (1986) called such a limit a 'generic limit' that is conceived as having the same properties that are common to all the terms in the sequence.

Lakoff and Núñez (2000, p. 258) appeal to the same fundamental idea in their 'metaphor of infinity' in which:

> We hypothesize that all cases of infinity—infinite sets, points at infinity, limits of infinite series, infinite intersections, least upper bounds—are special cases of a single conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result.

Such a principle was essentially formulated by Leibniz in his 'principle of continuity', that 'In any supposed [continuous] transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.' (Child, 1920, p.147.)[1] It is also directly related to the symbolic approach of Euler to deal with infinite power series as if they obey the same rules as finite polynomials.

The transition from practical arithmetic involving finite operations and finite decimals is known to be problematic.

[1] The translation given by Child omitted the term 'continuous' that was in the original Latin.

Kidron & Vinner (1983), Vinner & Kidron (1985) observed that the infinite decimal is perceived as one of its finite approximations: "*three digits after the decimal point are sufficient, otherwise it is not practical",* or as a dynamic creature which is in an unending process. Monaghan (1986) found that students often viewed both repeating and non-repeating infinite decimals as '*improper numbers which go on for ever.'* An expression such as $\sqrt{2} = 1.414...$ is not read as '$\sqrt{2}$ can be computed exactly as the limit of a decimal expansion', but more usually as saying that '$\sqrt{2}$ can be described to any required accuracy by approximating to an appropriate number of decimal places.' Li & Tall (1993) found students who agreed that 1/9=0.1+0.01+… yet did not accept the equation written in the other direction as 0.1+0.01+… =1/9. Their reasoning was that dividing 1 by 9 successively gave 0.1, then 0.11, and so on, while the terms 0.1+0.01+… could not be added together to give 1/9, because the process is potentially infinite and can never be completed in finite time.

However, by using a dynamic approach visualizing whole graphs instead of first dealing with limits at a point, it is evident that students can conjecture the slope function for specific functions such as x^2, x^3, x^n, sin(x), cos(x), by drawing the graph of $(f(x+h)-f(x))/h$ and investigating the visual picture for smaller values of h (Tall, 1986). This offers a long-term approach to the calculus, building on a perceptual foundation, leading long-term to symbolic, then formal theory. (See Tall (2013), chapter 11.)

Research in the transition from visual and symbolic modes of operation to formal set-theoretic definitions in analysis reveal that students differ in the way that they attempt to make sense of new ideas. Some students attempt to build on their visual imagery, some attempt to build from the formal definitions and others simply learn proofs by rote (Pinto & Tall, 2002, Weber, 2004). Alcock & Simpson (2004, 2005) found that some students use images to provide access to the subtleties of formal analytical concepts while others focus almost entirely on the symbolism and rarely use visual ideas. The effectiveness of dynamic images in learning the concept of limit has been discussed and studied by Kidron and Zehavi (2002), Kidron (2003, 2008), Tall and Vinner (1981), Williams (1991), and others.

4. THE EMPIRICAL STUDY

This study uses the symbolic and graphical facilities provided by *Mathematica* to enable the students to follow the historical development of expressing functions in terms of power series, contrasting the symbolic approach of Euler and a visual approach that pictures Lagrange's remainder formula for $R_n(x) = f(x) - P_n(x)$ to give an upper estimate for the difference between the function and its Taylor approximation. In both cases, visual pictures produced by *Mathematica* are used to explore how successive Taylor approximations $P_n(x)$ soon become indistinguishable from the function $f(x)$ (as in figure 1).

For instance, the polynomial approximation to sin(x) around x = 0 of degree 5 is $P_5(x) = x - x^3/3! + x^5/5!$, ... The error $f(x) - P_5(x)$ given by the Lagrange remainder is $\frac{f^{(6)}(c)\, x^6}{6!}$ for some c between 0 and x. The absolute value of the error (as a function of x and c, with $-\pi \le x \le \pi$, $-\pi \le c \le \pi$) can then be plotted. The c value in $-\pi \le c \le \pi$ that corresponds to the exact error is unknown, so the students were requested to look at all pairs (x,c) such that $-\pi \le x \le \pi$, $-\pi \le c \le \pi$. The 3-dimensional plot in figure 2 represents the error (in fact, an upper estimate of the absolute value of the error) as a function of the two variables x and c. In this specific plot the upper estimate of the error is obtained, for example for $x = \pi$ and $c = \frac{\pi}{2}$.

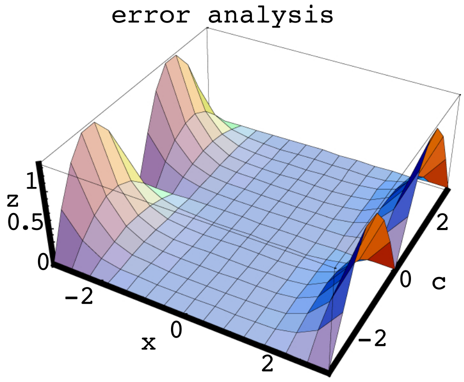

Figure 2: The error as a function of x and c

In our research study, we analyze how students progress from the embodied and symbolic worlds to the formal definition of limit. The participants were high school students learning at the highest level (grade 11, N = 63). They learned mathematics six hours a week during the entire year. Two hours in the PC lab were devoted to the topics "Approximation" and "Interpolation". The other four hours were devoted to standard subjects in Calculus, Algebra, and Trigonometry. The students were introduced to the notion of derivative as a limit intuitively with no formal approach. Therefore the learning experience in the PC lab was their first encounter with convergence phenomena with the aim of enhancing their ability to pass from a visual interpretation of the limit concept to formal reasoning. The laboratory consisted of 20 PCs, each equipped with *Mathematica*. The lab was also equipped with a system that transmits the content of the screen of each computer to all the computers in the classroom. Kidron (2003) analyzed the role played by *Mathematica* in enabling the students to "walk the same paths" as the founders of mathematical theory from the period of Euler in 1748 to the period of Runge in 1901.

They were introduced to two approaches to approximate functions by Taylor polynomials. One involved the students following the original text of Euler (1988), aided by the *Mathematica* package in order to do the "continued division procedure" to calculate $1/(1-x)$ as described by Euler. The students were requested to translate

Euler's algorithmic thinking into *Mathematica* commands, which required them to be very explicit to instruct *Mathematica* to carry out his ideas and to gain insight concerning Euler's "development of functions in infinite series".

In the second approach, students used the notion of order of contact, finding a polynomial of given degree that has the highest possible order of contact with a given function. In this approach, the teacher presented dynamic graphical animations and invited the students to analyze the process of convergence and to describe what they see in the dynamic pictures. They were asked to translate the dynamic pictures into analytical language and construct their own animations by changing parameters and choosing different functions, for example, to test the convergence of the Lagrange Remainder (displayed as a 3-D animation) for different functions.

Three types of data were collected for research purposes: the students' *Mathematica* files, a record of student questions and comments during the sessions, and written tests without *Mathematica*.

In this paper we analyze the students' evolution of ideas from a potentially infinite process to the limit as an object, in terms of Euler's symbolic view of a power series as an 'infinite polynomial', and the visual convergence of the finite polynomial approximations to the function itself. This was studied using the record of a discussion that illustrated the various conceptions that arose as the students debated the ideas between themselves and a written test that enabled the categorization of individual personal conceptions.

5. STUDENTS REACTIONS: FINDINGS AND DISCUSSION

5.1 The class discussion

The class discussion took place in a context where the function $f(x)$ and the Taylor polynomial $P_n(x) = a_0 + a_1x + ... + a_nx^n$ were both known, so the computation of the error $f(x) - P_n(x)$ is the numerical difference between two known functions. However, this error function is also the infinite tail of the power series which leads to:

Research Question 1. Were the students able to overcome their potential infinite process view? In particular, did they conceptualise the error term $f(x) - P_n(x)$ as a potentially infinite calculation or as a legitimate object?

This problem was voiced by one of the students:

Dina*: How could we speak about a graph that describes the error* $f(x) - P_n(x)$*? This difference is the 'infinite tail',* $a_{n+1}\,x^{n+1} + a_{n+2}\,x^{n+2} + ...$ *. How could this difference be well defined?*

A similar reaction is shown at the beginning of the following point-wise discussion, which took place as the teacher proved Taylor's theorem at $x = 0$. In one step of the proof, in order to calculate the derivative of a certain term, she mentioned that the error term

$d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$
is a constant when x_0 is a constant.

Student reactions to this idea demonstrated different phases in destabilizing the conception of an infinite sum as growing to be infinite.

Phase 1: The beginning of a conflict

[1]. Julia*: Why is d a constant?*

[2].Teacher*: $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$, therefore d is the difference between two constants.*

[3]. Julia*: But d is the 'infinite tail'... of the polynomial. How could it be a constant?*

Phase 2: The conflict

[4]. Tomer*: We compute the error for a given n.*

[5]. Julia*: But $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$. So, how could it be that d is a constant?*

Phase 3: A turning point. An infinite sum of numbers could be equal to a given number.

[6]. Tomer*: $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$ is an infinite sum that is equal to a given number!*

[7]. Ron*: Yes! For example, you have* 1/2+1/4+1/8+1/16+1/32+*... which equals* 1.

Phase 4: Back to the conflict phase. The infinite sum is growing all the time.

[8]. Julia*: Will it not be bigger than* 1 *when we continue to add terms?*

Phase 5: An attempt to resolve the problem.

[9]. Dan: *In the example* 1/2+1/4+1/8+1/16+1/32+*..., the infinite sum is a defined number but there are other examples in which the infinite sum is not a given number. It tends to ∞!*

[10]. Adi*: So how could we know if $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$ is an infinite sum which is equal to a given number?*

Phase 6: Focusing on the 'process itself' and not on the divergent process of adding terms to the sequence.

[11]. Yifat: *In the last lab we have seen animations, which demonstrate that when* $n \to \infty$*, the expression $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$ tends to* 0.
Therefore, the expression $d = a_{n+1}x_0^{n+1} + a_{n+2}x_0^{n+2} + ...$ is a given number and not an expression that tends to ∞!

As we reflect on the developing class discussion, we begin with the first research question to analyse to what extent the students involved were able to overcome their potential infinite process view and to conceive an infinite sum as something that is not necessarily growing to be infinite.

Dina and Julia, both high achievers in mathematics, are unwilling to give the error term the status of legitimate object. For Dina, the object spoken about is the graph of the difference between the given function and the Taylor polynomial and for Julia the object is a number – the numerical value of the error term for a given numerical value of *x*. We want to stress one interesting point: In spite of the fact that in the two cases, the teacher presented the error term as the difference between the function and the approximating polynomial, both Dina and Julia considered the error term in its other representation as an infinite sum of functions (Dina) or as an infinite sum of numbers (Julia).

The class discussion indicates that, at least for some students, a conflict exists. The teacher focused on the error term as the difference d between two constants, $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + \ldots.. + a_nx_0^n)$. Julia considered the representation of d as an infinite sum $d = a_{n+1}x_0^{n+1} + \ldots$. She agreed that the difference between two constants is a constant and yet she was not ready to accept the infinite tail as a constant. Julia looked at the infinite sum as an unending process, as something growing all the time. Her question: 'Will the sum 1/2+1/4+1/8+1/16+1/32+… not be bigger than 1 when we continue to add terms?' shows that her attention is drawn by the never-ending process of adding terms to the sequence.

A turning point takes place in phase 5 where some students differentiate between two types of infinite sums: those who tend to ∞ and infinite sums that give a finite number, despite the divergent process of adding terms to the sequence. The animation of the error term helped the students to reflect on the question to which of these two types of infinite sums the 'infinite tail' $d = a_{n+1}x_0^{n+1} + a_{n+2}x_0^{n+2} + \ldots.$ belongs. Phase 6 demonstrates how the dynamic visualization encourages a focus on the convergent process of the error term tending to zero rather than the potentially infinite process of adding terms together. Even though the computer was turned off during the discussion, the mental images may still remain. As a consequence, by substituting a value x_0 for *x*, it was possible to encourage the students to realize that the infinite sum of numbers $a_{n+1}x_0^{n+1} + a_{n+2}x_0^{n+2} + \ldots.$ is not growing to be infinite in this case and could equal a given number.

The class discussion illustrates the influence of blending embodied visualization and operational symbolism on students' potential infinite conceptions (Research question 1). The "infinite tail" is potentially infinite and problematic to compute in a finite time. However, it is also represented symbolically by the Lagrange remainder formula, which can be visualized dynamically to show how the upper estimate of the absolute value of the error tends to zero for large *n*.

In the class discussion, a spectrum of conceptions were voiced: some students were reluctant to give the infinite sum the status of a 'legitimate' object, seeing it instead as a potentially infinite process as reported in the literature. To address the second question relating to the further development from potentially infinite process towards the limit concept, we used a written test.

5.2 The written test

The written test sought to analyse the personal conceptions of individual students and to respond to the following:

Research Question 2. What progress did the students make in shifting towards the formal definition of the limit concept?

Did they conceive the infinite sum as a process of approximation or were they able to move towards the formal limit concept?

To investigate this question, the students were presented with the following problem from the textbook by Davis, Porta and Uhl (1994).[2]

> **Question**: When someone says $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$ for $-1 < x < 1$, the meaning of this statement is subject to a controversy that raged in mathematical circles for years and still raises its head every so often even now.
>
> One group, including Zeno, said: 'Infinite sums exist only in theory. In practice, there is no such thing as an infinite sum. Saying that
>
> $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ... \text{ for } -1 < x < 1$$
>
> really means that, by using enough of the expansion of $1/(1-x)$ in powers of x, you can get as close to $1/(1-x)$ as you like.'
>
> A second group, including Euler, said: 'This is an actual infinite sum.'
>
> The third group, including Bolzano and Weierstrass, said:
>
> 'The first two groups really have nothing to fight about. You just define the infinite sum $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$ to mean that given an x with $-1 < x < 1$ and a tolerance t then you can find a power m so that
>
> $$\left| \frac{1}{1-x} - (1 + x + x^2 + x^3 + ... + x^m) \right| < t.\text{'}$$
>
> Which group are you in, and why?

These three representations relate to the approach used in the teaching experiment. Some students respond as with Zeno that there is no actual limit, only a potential one. Some respond to the idea of actual infinity as represented by Euler and a third group resolved

---

[2] In the original question of Davis, Porta and Uhl, and in our study, the sequence of responses in the question was given with Euler first and Zeno second. Here we place Zeno first and Euler second, to follow the same order both in history and in cognitive development.

the conflict in a way that builds towards the formal definition. The lab sessions involved situations like: 'you give me a measure of error e > 0 and I will find a power *N* such that the absolute value of the error term is smaller than e', 'you give me another measure of error e, smaller than the previous one, and I will find a suitable power *N* such that … .' This was done in the context of the embodied visualizations of the upper estimate of the error (see figure 2). The students were encouraged to think of the process of finding a suitable *N* for each e in the sense that 'what I have done once, I can do again.' The concept of infinity involved in such situations is the natural concept of potential infinity but it also has the possibility to lead to the formal definition of limit.

We were interested in seeing how the experience in the lab may help the students move towards the formal definition, although we were also conscious that various difficulties could arise. For example, a range of conceptions observed in the literature arose in the study, including 'only the finite is meaningful,' 'there is no such thing as an infinite sum,' 'the infinite sum is just a finite approximation,' or 'the infinite sum is the process of approximation itself.' We were interested in how the students' experiences combining dynamic visualization and symbolic calculations may contribute to a natural approach to the limit definition.

The analysis of the students' arguments in justifying their answers enabled us to reflect on the second research question dealing with the students' ability to make a transition from the potentially infinite process view to the concept of infinite sum and on towards the formal definition of limit.

5.3 Categorising the written responses

We examined the students' arguments in justifying their answers and, apart from category 0 that offered no relevant information, we placed them in four categories:

Category 0 No relevant answer or no explanation.

Category I There is no such thing as an infinite sum.

Category II The infinite sum is perceived as a dynamic process that passes through a potentially infinite number of terms: $1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3, \ldots\ldots$

Category III The infinite sum is perceived as a legitimate object in one of the following ways:

i) The 'infinite polynomial' is recognized as a legitimate entity with a vague and unclear meaning;

ii) The infinite sum is perceived as a generic limit object, described as $\frac{1}{1-x}$ is $1+x+x^2+x^3+\ldots+x^\infty$.

Category IV The infinite sum is conceived as the limit of the infinite process of approximation.

The various responses were placed in different categories and percentages are given to indicate the approximate size of each category:

Category I There is no such thing as an infinite sum (13 %)

This includes responses indicating that a large number of terms is perceived to be very big but finite:

'I am in the first group. I agree with those who claim that in fact there are no infinite sums. Infinite sums exist only in theory. Nothing in the world is endless. There is an end to the approximation. ... The problem is how to get this finite approximation since the number of its elements is so big that there is yet no computer that is able to find it. The approximation exists but we are unable to compute it. The difference $\frac{1}{1-x}-(1+x+x^2+....+x^m)$ *equals* 0 *if we choose the appropriate m. We do not know how to compute this number because the number of computations that are required. [...] I know that the third group is right because that is the way that I learnt but I tend to agree with the first group since their arguments seem reasonable to me and fit my way of thinking.'*

Note the conflict in this response. The student 'knows that the third group is right' but tends to 'agree with the first group' because the arguments 'seem reasonable' and 'fit my way of thinking.' It proves often to be the case that assigning a response to a single category is a matter of degree rather than an absolute allegiance to a single view. However, viewing the overall spectrum of responses enables us to gain insight into the different ways in which students express the subtlety of their opinions.

In the majority of answers that illustrate Category I, the infinite sum is often seen to exist in theory, but not in practice:

'An infinite sum exists only in theory. It is not practical. In order to do something practical with an infinite sum, you have to stop it at some stage.'

In some answers, the student reports the strong influence that the computer has on their conceptions:

'In the laboratory, I learn that the expression 'infinite polynomial' means the possibility to approach $\frac{1}{1-x}$ *as much as we want. The computer also considers the expansion in power series as a theoretical concept and it permits the expansion only after we fixed a certain power.'*

'Euler, considering the infinite sum as an actual infinite sum, as an entity (something like π) argues that it will be easier to define (while working by hand) the infinite sum as an expansion in a power series. But, since there exist nowadays computers with suitable software , it is preferable to work in a way that enables us to feel that at the moment we decide what will be the measure of error, the deviation in the approximation will not be exaggerated.'

Category II The infinite sum is perceived as a dynamic process that passes through a potentially infinite number of terms: 1, $1 + x$, $1 + x + x^2$, $1 + x + x^2 + x^3$, … (35%)

Here the process is infinite but each stage is represented by an approximating polynomial with a finite number of terms.

The following answer illustrates Category II:

'The polynomial can always approach the function better and in the same way that when we add digits 3 to 0.3 then it approaches 1/3 – we can add digits 3 as much as we want and as a result we can approach better 1/3 (0.333 is nearer 1/3 than 0.33) – in the same way, the more powers there are in the expansion, the better the polynomial approaches the function.'

Sometimes, this perception of the infinite sum as a potential infinite process is accompanied with some reservation that expresses a conflict in the process of obtaining 100% accuracy:

'Infinity is a tendency and not a number. Thus, when we tend to an infinity of terms, the error will tend to 0. The first group is right: I can get the precision I request only if I use enough terms. But, after all, I have a problem with the definition of the first group: If I request a precision of 100% – a zero error – I cannot reach it except if I will be helped by the definition of the second group.'

Category III The infinite sum of functions is perceived as a legitimate object but not clearly as the formal limit definition (28%)

These were subdivided into two subcategories:

(i) The 'infinite polynomial' is recognized as a legitimate entity with a vague and unclear meaning. (19%)

As an example, in the following answer, the student views the infinite polynomial as a legitimate object but does not yet fully grasp the representation of the third group as a formal definition of the second group's actual infinite sum:

> *'To each power m we can compute the error between the function and the expansion and to each tolerance t small as we want, we can find m that meets the requirement* $\left|\frac{1}{1-x} - (1 + x + x^2 + x^3 + ... + x^m)\right| < t$*. If we choose t smaller and smaller, we will be able to find m bigger and bigger that meets the requirement but every time that we fix m and t we move from an infinite polynomial to a finite polynomial.'*

(ii) The infinite sum is perceived as a generic limit object. (9%)

> *'I am in the second group since the limit of the approximating polynomial of* $1 + x + x^2 + x^3 + ... + x^m$ *is* $1 + x + x^2 + x^3 + ... + x^\infty$*. We know this fact from the expansion in power series around 0 according to Euler. We can choose a polynomial with ∞ terms such that the coefficient of each term of the polynomial* $a + b\,x + c\,x^2 + O(x^3)$ *equals* 1.*'*

The term $O(x^3)$ in the student's answer represents the infinite tail as it is represented in the syntax of *Mathematica*. This symbol and the way the student wrote the infinite sum $1+x+x^2+x^3+...+x^\infty$ creates the impression of a symbolic manipulation of an infinite number of terms.

Students may also note that the quest to get an exact answer will always involve a non-zero error:

'I am in the second group since even if we will add more and more terms to the sequence $1+x+x^2+...+x^n+..$ yet a difference will remain (maybe a very small one) between the sum and $\frac{1}{1-x}$, i.e.

$$\frac{1}{1-x}-(1+x+x^2+....+x^k+....)=\frac{x^{(n+1)}f^{(n+1)}(c)}{(n+1)!}.$$

It means that only at infinity will we get a zero error. I do not think that we can take enough terms to get a zero error. We can get some precision but not a 100% precision.'

By using the Lagrange expression for the error, which may never equal zero, the student reinforces the impression that it is not possible to get 100% precision.

Category IV The infinite sum is conceived as the limit of the infinite process of approximation (22%)

Some answers demonstrate that the infinite sum is conceived as the limit of the sequence of partial sums:

'I am in the third group. I think that the argument

$$\lim_{n\to\infty}(1+x+x^2+....+x^n)=\frac{1}{1-x}$$

is the best description of the expansion of the function in power series.'

Other answers reveal that the infinite sum

$$f(x)=a_0+a_1x+a_2x^2+....+a_nx^n+...$$

is defined because $\lim_{n\to\infty}R_n(x)=0$ with

$$R_n(x)=f(x)-(a_0+a_1x+a_2x^2+...+a_nx^n).$$

Other answers placed in this category refer to the importance of the infinite polynomial 'tending to' the limit:

' I do not agree with the second group since this group pays no attention to the fact that the infinite polynomial $1+x+x^2+....+x^k+...$ tends to $\frac{1}{1-x}$.'

Some answers refer to upper bound ideas that are relevant in this particular example:

'$\frac{1}{1-x}$ is the bound of the sum of the infinite series.'

This was, for example, the answer of Julia, the student who played an important role in the previous class discussion.

While some students are happy to view an infinite sum as an actual infinity, others see it as a potentially infinite process. Others, as in the following response, may acknowledge both views as being possible:

> ‘*The infinite polynomial means that we can approach the function* $\frac{1}{1-x}$ *as much as we want. I do not agree with the second group since the infinite sum is a limit and I cannot consider it as a sum of an infinite number of terms at once. Nevertheless, there are some cases in which it is possible to work with the infinite sum. For example, we can look at Euler's way of finding the coefficients of the expansion into power series by his method of undetermined coefficients. Euler did not mention a specific power.*’

In this case, the student had used *Mathematica* in the computer lab to follow the original text written by Euler and seems influenced by the way that Euler manipulated the infinite number of terms in the power series.

Some students were able to rationalise the two views in a complementary manner as an ongoing process or a legitimate object. For instance Dror responded as follows:

> ‘*I agree with the third group. The first and the second group complement each other. The second group looks at the infinite expansion as an entity that is perceived at once. This is the right way of looking:* $\frac{10}{3} = 3.33\dot{3}$ *and, in spite of the fact that it is a periodic infinite decimal, one can perceive it as an entity at once.*
> *The first group considers the number from the mathematical theory and sees it in reality. In practice, there is no such thing as an infinite sum: neither human being nor computers can perform infinite computations; they can perform only approximations.*
> *The second group looks at the expansion* $\frac{1}{1-x} = 1 + x + x^2 + ...$ *from left to right and the first group looks at it from right to left (they start from the expansion and try to arrive to the quotient). For this purpose, they need a finite number of terms.*
> *The third group simply confirms that the equality could be read in both two directions.*’

Dror's answer demonstrates that the formal definition offers a resolution of the difference between Euler's approach from left to right, $\frac{1}{1-x} = 1 + x + x^2 + ...$, and the analytical limit from right to left, $1 + x + x^2 + ... = \frac{1}{1-x}$.

In the preceding discussion, it is clear that an individual may not respond precisely in a single category. The much-quoted definition of concept image (Tall & Vinner, 1981, Vinner & Hershkowitz, 1980) formulated the principle that:

> Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole.
>
> We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion (Tall & Vinner, 1981, p. 152).

In this sense, a student in conflict may say one thing at one time and something different at another. Thus the classifications and percentages involved may not be precise. Nevertheless, they are indicative of the variety of responses that occur.

The study offers the students the opportunity to express their own views in a context that covers a wide range of conceptions. Taking Categories I and II together, nearly half the students either reject the infinite sum (13%) or see it as a potentially infinite process (35%). Category III sense the limit as a mental object either relating it, often unclearly, to the infinite polynomial of Euler (19%) or to a generic limit (9%). Category IV, (22%) are beginning to see that the opposing views need not be so much in conflict, moving towards a view consistent with the formal definition.

6. REFLECTIONS

This study has used the power of *Mathematica*'s symbol manipulation and dynamic graphics to study the historical development of Taylor series using Euler's infinite polynomials and Lagrange's error function. The students were provided with an environment to manipulate the symbolism and represent the error function graphically to see how it tended to zero.

The introduction of Euler's use of infinite polynomials proved to be problematic for students familiar only with performing finite operations. However, it offered an explicit conflict in the relationship between the process of tending to the limit and the limit concept as a legitimate mental object. It stimulated explicit discussion not only of the relationship between the potential infinity of the process and the actual infinity of the limit, but also of the transition from the Taylor polynomials as approximations to a desired accuracy towards the formal definition of limit.

A wide spectrum of interpretations remained, from those who only allowed finite computations as approximations and denied actual infinity, through those who had a sense of the limit as a legitimate object that was vague or generic, to those developing a sense of the formal limit concept.

The part of the study focusing on the graphical approach alone (as distinct from Euler's work with infinite polynomials) offers a natural transition from embodiment and symbolism to the formal definition. While the limit of a numerical sequence involves a concept image where the terms may get close to the limit but never equal it, the visual picture of a sequence of functions tending to a limit can be illustrated in a visual way by seeing how the graphs of the approximations $P_n(x)$ become indistinguishable from the limit function $f(x)$. This experience may be used to suggest the definition 'given any error e > 0, it is possible to find an *N* such that, for $n \geq N$, the graphs of $P_n(x)$ and $f(x)$ differ by less than e.'

Paradoxically therefore, the introduction of the limit of a sequence of functions may be cognitively easier to comprehend in an embodied sense than the limits of a sequence of numbers. This is certainly true in the initial stages of the calculus where looking at the graph of the practical slope function

$$\frac{f(x+h)-f(x)}{h}$$

and allowing *h* to get small dynamically allows the student to *see* that, for $f(x)=x^2$, the practical slope function stabilizes to 2*x* visually and this can be translated directly into the symbolic calculation of the derivative. Such an approach can be used to *see* the derivatives of standard functions and also motivate the derivative as the slope of the tangent vector. Once the learner has mental imagery for the notion of the derivative as the slope function for several standard functions, it is natural to seek to compute the derivatives of sums, products, composite functions and only at this stage need one begin to introduce a more formal definition of limit. (See Tall, 2013, chapter 11.) Using dynamic imagery and symbol manipulation, alternative approaches to the calculus become possible that build *from* embodied perception and symbolic manipulation in the calculus *to* formal definitions and more sophisticated ideas in multi-dimensional calculus and analysis.

Different approaches will be appropriate in various contexts. Some may require calculus in applications to model physical situations mathematically to solve and predict consequences of the mathematical model, some may study analysis in pure mathematics or non-standard analysis in logic. The approach here offers a combination of historical development and modern technology to enable students in high school with very different views to participate in a reflective discussion of the issues involved in historical development. At the end of this study a spectrum of conceptions remain. However, instead of personal conceptions remaining implicit in the students' minds, the differences have been made explicit as part of an ongoing debate concerning the evolution of mathematical meaning from potentially infinite process to approximation within a desired error and on to the later formal definition of limit.

This research was supported by the Israel Science Foundation (grant No. 843/09).

REFERENCES

Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57*(1), 1–32.

Alcock, L., & Simpson, A. (2005). Convergence of sequences and series 2: Interactions between nonvisual reasoning and the learners beliefs about their own role. *Educational Studies in Mathematics, 58*(1), 77–100.

Barrow, I. (1670). *Lectiones Geometricae*, translated by Child (1916) *The Geometrical Lectures of Isaac Barrow*. Chicago and London: OpenCourt.

Bolzano, B. (1817). *Rein Analytischer Beweis des Lehrsatzes,dass zwischen je zwey Werthe, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* [Pure analytical proof of the theorem that between two values that give opposite signs, there is at least one real root of the equation], Prague, Haase. Facsimile, Berlin, Mayer & Mueller, 1894, 2nd edition, Leipzig, Engelmann, 1905.

Cauchy, A. L. (1821). *Cours d'analyse de l'Ecole Royale Polytechnique* [Analysis course of the Royal École Polytechnique]. Paris: Imprimérie Royale. See http://gallica.bnf.fr/ark:/12148/bpt6k90195m/f12

Child, J. M. (editor and translator) (1920). *The early mathematical manuscripts of Leibniz*. Chicago, IL.

Cornu, B. (1981). Apprentissage de la notion de limite: modèles spontanées et modèles propres, [Learning the notion of limit: spontaneous and formal models] *Actes du Cinquième Colloque du Groupe Internationale PME*, Grenoble, 322–326.

Cornu, B. (1991). Limits. In D. Tall, (Ed.) *Advanced Mathematical Thinking* (pp.153–166), Dordrecht: Kluwer Academic Publishers.

Davis, B., Porta, H. & Uhl, J. (1994). *Calculus & Mathematica: Derivatives, measuring growth*. Reading, MA: Addison-Wesley Publishing company.

Descartes, R. (1637). *La Géométrie*, translated by D. E. Smith & M. L. Latham (1954) as *The Geometry of René Descartes*, Dover Publications: New York.

Euler, L. (1748). *Introductio in analysin infinitorum*. [Introduction to the analysis of the infinite] Lausanne: Bousquet. Reprinted as *Opera mathematica*, Series I, Volume VIII.

Fischbein, E., Tirosh, D. & Hess, P. (1979). The intuition of infinity, *Educational Studies in Mathematics, 10*, 3–40.

Hilbert, D. (1900). Mathematische Probleme. *Göttinger Nachrichten* pp. 253–297, translated into English as Mathematical Problems, *Bulletin of the American Mathematical Society,* 8 (10), 437–479.

Kepler, J. (1604). *Astronomiae Pars Optica* [The Optical Part of Astronomy].

Kidron, I. (2002). Concept definition, concept image, and the notion of infinite sum in old and new environments. *Proceedings of the 26th International Conference for the Psychology of Mathematics Education,* (Vol.3, pp.209–216), Norwich, England.

Kidron, I. (2003). Polynomial approximation of functions: Historical perspective and new tools. *The International Journal of Computers for Mathematical Learning,* (IJCML) 8, 299–331.

Kidron, I. (2008). Abstraction and consolidation of the limit procept by means of instrumented schemes: the complementary role of three different frameworks. *Educational Studies in Mathematics 69*(3), 197–216.

Kidron, I. and Vinner, S.(1983). Rational numbers and decimals at the senior high level: density and comparison, *Proceedings of the 7th Annual Meeting for the Psychology of Mathematics Education*: Israel, pp.301–306.

Kidron, I., & Zehavi, N. (2002). The role of animation in teaching the limit concept. *The International Journal of Computer Algebra in Mathematics Education, 9*, 205–227.

Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Leibniz, G. (1684). *Nova methodus pro maximis et minimis* [New method for maximas and minima], translated by Struik, D. J., (1969): *A Source Book in Mathematics, 1200–1800*, Harvard University Press, pp. 271–281.

Li, L. & Tall, D. O. (1993). Constructing different concept images of sequences and limits by programming, *Proceedings of PME 17*, Japan, **2**, 41–48.

Mejia-Ramos, J. P. & Tall, D. O. (2004). Reflecting on post-calculus reform: Opening Plenary for *Topic Group 12: Calculus, International Congress of Mathematics Education*, Copenhagen, Denmark. http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot2004b-tall-meija-icme.pdf

Monaghan, J. (1986). *Adolescent's understanding of limits and infinity*. Unpublished Ph. D. thesis, Warwick University, U.K.

Newton, I. (1671) *Methodus Fluxionum et Serierum Infinitarum*. First published in 1736 in English translation as *Method of Fluxions.*

Pinto, M., & Tall, D. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the Learning of Mathematics*, 22, 2–10.

Tall, D. O. (1986). *Building and testing a cognitive approach to the calculus using interactive computer graphics*. Ph.D. Thesis in Education, University of Warwick.

Tall, D. O. (2004). The three worlds of mathematics. *For the Learning of Mathematics*, 23 (3). 29–33.

Tall, D. O. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM – The International Journal on Mathematics Education*, 41 (4), 481–492.

Tall, D. O. (2013). *How humans learn to think mathematically*. Cambridge University Press: New York.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151–169.

Vinner, S. & Kidron, I. (1985). The concept of repeating and non repeating decimals at the senior high level. *Proceedings of the 10th PME Conference*, The Netherlands: Noorwijkerhout. 357–361.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the 4th Annual Meeting for the Psychology of Mathematics Education* (pp. 177–184). Berkeley, CA: PME.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course, *Journal of Mathematical Behavior*, 23 115–133.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*, 219–236.