EVOLVING A THREE-WORLD FRAMEWORK FOR SOLVING ALGEBRAIC EQUATIONS IN THE LIGHT OF WHAT A STUDENT HAS MET BEFORE

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In this paper we consider data from a study in which students shift from linear to quadratic equations in ways that do not conform to established theoretical frameworks. In solving linear equations, the students did not exhibit the 'didactic cut' of Filloy & Rojano (1989) or the subtleties arising from conceiving an equation as a balance (Vlassis, 2002). Instead they used 'procedural embodiments', shifting terms around with added 'rules' to obtain the correct answer (Lima & Tall, 2008). Faced with quadratic equations, the students learn to apply the formula with little success. The interpretation of this data requires earlier theories to be seen within a more comprehensive framework that places them in an evolving context. We use the developing framework of three worlds of mathematics (Tall, 2004, 2013), based fundamentally on human perceptions and actions and their consequences, at each stage taking into account the experiences that students have 'met-before' (Lima & Tall, 2008; McGowen & Tall, 2010). These experiences may be supportive in new contexts, encouraging pleasurable generalization, or problematic, causing confusion and even mathematical anxiety. We consider how this framework explains and predicts the observed data, how it evolves from earlier theories, and how it gives insights that have both theoretical and practical consequences.

KEYWORDS: Theories of learning; solving equations; quadratic equations; procedural embodiment; three worlds of mathematics.

Empirical data and theoretical frameworks for the solution of linear equations

It is our view that theories of learning evolve over time as phenomena are noticed and formulated in coherent ways that later need to take new data into account. In this way initial ideas may be enriched and become part of a more comprehensive whole. In this paper, specific data in linear equations and the transition to quadratic equations will be placed in a broader framework for cognitive development that brings together several distinct strands of research within a single theory.

The research of Filloy & Rojano (1989) suggested that an equation such as $3x-1=5$ with an expression on the left and a number on the right is much easier to solve symbolically than an equation such as $3x+2=x+6$. This is because the first can be 'undone' arithmetically by reversing the operation 'multiply by 3 and subtract 1 to get 5' by 'adding 1 to 5 to get $3x=6$ and then dividing 6 by 3 to get the solution $x=2$. Meanwhile the equation $3x+2=x+6$ cannot be solved by arithmetic undoing and requires algebraic operations to be performed to simplify the equation to give a solution. This phenomenon is called 'the didactic cut'. It relates to the observation that many students see the 'equals' sign as an *operation*, arising out of experience in arithmetic where an equation of the form $3+4=7$ is seen as a dynamic operation to perform the calculation, 'three plus four *makes* 7', so that an equation such as $3x-1=5$ is seen as an operation which may possibly be solved by arithmetic 'undoing' rather than requiring algebraic manipulation (Kieran, 1981).

Lima & Healy (2010) classified an equation of the form 'expression = number' as an *evaluation* equation, because it involved the numerical evaluation of an algebraic expression where the input value of x could be found by numerical 'undoing', and more general linear equations as *manipulation* equations, because they required algebraic manipulation for their solution.

On the other hand, if the solution of linear equations is considered in terms of the conceptually embodied notion of a 'balance', the difficulty of the equations is reversed. The equation $3x+2=x+6$, can easily be solved as a balance by imagining the xs to be identical unknown objects of the same weight and representing the equation with 3 xs and 2 units on the left and one x and 6 units on the right. It is then possible to remove 2 units from either side to retain the balance as $3x=x+4$, and then remove an x from both sides to get $2x=4$, leading to $x=2$. In writing the prophetic paper entitled '*the balance model: hindrance or support for the solving of linear equations with one unknown*', Vlassis (2002) noted that, as soon as negative quantities or subtraction are involved, then the embodiment becomes more complicated and hinders understanding. For instance, the equation $3x-1=5$ cannot be represented directly as a balance because the left-hand side $3x-1$ cannot be imagined as $3x$ with 1 taken away if the value of x is not known.

This reveals that the didactic cut and the balance model give rise to very different orders of difficulty. In the didactic cut the equation $3x-1=5$ is easier to solve than the equation $3x+2=x+6$, but in the balance model the order of difficulty is reversed.

The data of Lima & Tall (2008) presented an analysis of Brazilian students' work with linear equations that did not fit either the didactic cut or the balance model. Their teachers had used an 'expert-novice' view of teaching and had introduced the students to the methodology that they, as experts, found appropriate for solving equations, using the general principle of 'doing the same thing to both sides' to simplify the equation and move towards a solution. However, when interviewed after the course, students rarely used the general principle. They did not treat the equation as a balance to 'do the same thing to both sides', nor did they show any evidence of the didactic cut.

Instead, they focused more on the specific actions that they performed to shift symbols around and 'move towards a solution' using two main tactics:

1) 'swop sides, swop signs'

in which an equation $3x-1=3+x$ is operated upon by shifting the 1 to the right and the x to the left and changing signs to get:

$$3x - x = 3 + 1$$
$$2x = 4.$$

2) ‘swop sides and place underneath’

in which the 2 associated with the expression $2x$ in the equation above is moved from one side of the equation to the other, then placed underneath to give

$$x = \frac{4}{2} = 2.$$

In an attempt to use such rules, some students made mistakes, such as changing $2x = 4$ to

(a) $x = 4 - 2$ (b) $x = \frac{4}{-2}$ (c) $x = \frac{2}{4}$.

In (a) the 2 is passed over the other side and its sign is changed; (b) correctly ‘shifts the 2 over and puts it underneath’ but also ‘swops the sign’; (c) shifts the 2 over and puts the 4 underneath. When questioned, *no* student mentioned the principle of ‘doing the same thing to both sides’, instead they developed what Lima and Tall called *procedural embodiments* which involved embodied actions on the symbols to ‘pick them up’ and ‘move them to the other side’ with an extra ‘magic’ principle such as ‘change signs’ or ‘put it underneath’ to ‘get the right answer’. Procedural embodiments worked for some students but they also proved to be fragile and misremembered by many others, leading to the wide range of errors that are well-known in the literature (Matz, 1980; Payne & Squibb, 1990).

Our purpose is not simply to find and catalogue errors. Instead we seek to evolve a single theoretical framework that covers all three aspects: the didactic cut, the balance model and the problem with ‘doing the same thing to both sides’. Such a theoretical framework should relate to both cognitive development and the emotional effects of the learning experience. To integrate these different aspects into a single framework, we begin with a theoretical construct that relates current learning to previous experience.

Supportive and problematic met-befores

The effect of previous experience on current learning may be studied using the notion of ‘met-before’, which has a working definition as ‘a structure we have in our brains *now* as a result of experiences we have met before’ (Lima & Tall, 2008, McGowen & Tall, 2010). The effect of previous experience has both cognitive and emotional aspects. In general, students encountering algebra for the first time already have experience of arithmetic, in which expressions such as $3\times5-2$ have answers. This acts as a met-before that causes problems in algebra where an expression such as $3x-2$ is a generalized operation that does not have an answer unless x is known.

The concern that algebraic expressions ‘do not have answers’ is often referred to as the ‘lack of closure’ (Collis, 1978) and is seen as an obstacle in the general solution of equations. However, when the full data is examined, we can see that it is not always an obstacle. If the value of x is known, then the expression $3x-2$ can be evaluated and so the expression can have a clear meaning as an operation of evaluation. This has positive implications when solving an equation such as $3x-2=7$ where the process of evaluation may be undone: start with the 7, add 2 to get 9, then divide 9 by 3 to get the solution $x=3$.

The notion of an expression as 'an operation to be evaluated' therefore acts in different ways with different types of equation: it is *problematic* for equations requiring algebraic manipulation but it is *supportive* in solving an equation that can be interpreted as an arithmetic evaluation.

Met-befores have both mathematical and emotional consequences. We conjecture that supportive met-befores operating in a new context allow old methods to be used in a pleasurable way to make generalizations of established techniques in new settings. Problematic met-befores impede generalization and cause confusion. A confident individual may be frustrated by such impediments and work to find new ways of thinking that conquer the problems. A less confident individual may feel alienation that grows over time as successive problematic aspects in new contexts cause anxiety and increase the desire to avoid the pain by attempting to learn 'what to do' to seek at least the pleasure of passing tests.

This offers a refined formulation of the original research into the didactic cut by Filloy & Rojano (1989), where many of the students were able to solve simple evaluation equations before being taught to solve equations using algebraic manipulation. The notion of an equation as a process of evaluation is supportive for solving evaluation equations but problematic for manipulation equations. Another observation made at the time is that the introduction of the algebraic technique in solving linear equations caused a loss in ability for some students to solve simple equations using arithmetic undoing. This loss in facility when faced with a new technique is common in mathematics learning. For instance, Gray (1991) noted that some children introduced to column subtraction may make errors that did not occur when they performed the same operation using simple mental arithmetic.

This is consistent with the absence of the didactic cut in the data of Lima & Tall (2008). The students had been presented with a new formal principle for solving equations by 'doing the same thing to both sides'. This new principle was not generally implemented as intended, instead the students focused on shifting symbols with additional rules as procedural embodiments that treated both evaluation and manipulation equations in the same way. Thus the students performed the same type of operation in both cases and made the same sort of error.

This suggests a need to encompass the earlier analyses involving the 'didactic cut', the 'help or hindrance' of the balance metaphor, or the reasoning of 'doing the same thing to both sides' within a single framework that sees the students' ideas evolve as they encounter new contexts where previous experiences may be supportive or problematic. It involves more than simply studying a single context, say quadratic equations, to see how it can be taught and learnt to best advantage. It requires a framework to make sense of the whole growth of mathematical knowledge of individuals, as they build personal ways of thinking over the long-term, based on fundamental human ways of thinking and the consequences of previous experiences.

The three worlds of mathematics

The framework of three worlds of mathematics (Tall, 2004, 2013) is an overall theory of cognitive and affective growth in mathematics that has evolved to build from the early development of ideas in the child, through the years of schooling and on to the boundaries of research in formal mathematics.

It is strongly related to a wide range of theoretical frameworks formulated by Piaget (1970), Dienes (1960), Bruner (1966), Van Hiele (1986), Skemp (1979), the SOLO taxonomy of Biggs & Collis (1982), the structural and operational mathematics of Sfard (1991), process-object theories (such as those of Sfard (1991), Dubinsky (e.g. Asiala et al., 1996), Gray & Tall (1994)), theories of advanced mathematical thinking (Tall (ed.), 1991), as well as theories from cognitive science such as the embodied theory of Lakoff and his colleagues (e.g. Lakoff & Núñez, 2000), the blending of cognitive structures formulated for example by Fauconnier and Turner (2002) and other aspects such as the role of various levels of consciousness (Donald, 2001). Detailed discussion of all these aspects can be found in Tall (2013). However, the main purpose of the theoretical framework is not to collate all these theories together with all their intricate details that differ in many ways, but to seek the fundamental essence of essential ideas that they have in common.

Following Skemp (1979), whose theoretical framework builds from perception (input) and action (output) and becomes increasingly sophisticated through reflection, the three-world framework builds on the tripartite structure of perception, operation and reason. All three of these aspects arise throughout mathematics. Van Hiele (1986) provides a growth of perception of geometric figures, where operations on figures produce geometric constructions and reasoning develops in sophistication through Euclidean definition and proof. Process-object theories build on actions that become mathematical operations, encapsulated as mental objects (procepts) that operate dually as processes (such as addition) and concepts (such as sum) (Gray & Tall, 1994). Overall, the learning of school mathematics requires that the student blends together, in the sense proposed by Fauconnier and Turner (2002), embodied perception and operation that lead to geometry on the one hand and arithmetic and algebra on the other. Both may be blended together, for instance, through representation of relationships in the cartesian plane, where perceptual ideas of dynamic change are related to operational techniques for computing change and growth in calculus.

At the higher levels of school mathematics, methods of reasoning lead to Euclidean proof in geometry and symbolic proof – based on the ‘rules of arithmetic’ – in arithmetic and algebra.

In university, applied mathematicians broadly build on their experience of natural phenomena to construct mathematical models that can be used to reason about situations and compute solutions. Pure mathematicians take natural ideas and translate them into formal objects specified set-theoretically and deducing their properties using mathematical proof.

Underlying this whole development is the nature of the species *Homo Sapiens* where the child builds on initial sensory perception and action and evolves increasingly sophisticated forms of mathematical thinking using language and symbolism.

The sensory side develops through exploring and interacting with the structures of objects, recognizing properties, using language to describe, define and deduce relationships in an increasingly sophisticated mental world of *conceptual embodiment* that includes geometry and other perceptual representations; it develops over the longer term from physical perception to increasingly subtle mental imagination using thought experiments. This may be described using the four van Hiele levels that may usefully be subdivided into two distinct forms of thinking: the *practical* ideas of shape and space developed through *recognition* and *description* and the *theoretical* ideas of Euclidean geometry developed through *definition* and *deduction* using Euclidean proof (see Tall, 2013).

The motor side of human action develops into a world of *operational symbolism* in which operations on objects such as counting, measuring, sharing, adding, multiplying, and so on, are symbolized as mathematical concepts such as number, fraction, sum, product, and operations are generalized as manipulable expressions in algebra. At every stage, operations are practiced and internalised as mental objects. Properties of the operations that have been recognised and described in practical mathematics may then be defined as 'rules of arithmetic' that become the basis for more technical aspects, such as the properties of prime numbers and the theory of factorization in arithmetic, and the formulation and solution of equations in algebra.

Conceptual embodiment and operational symbolism blend together in the calculus where embodied ideas of rates of change and growth are blended with numerical and algebraic processes to formulate the symbolic operations of differentiation and integration and their inverse relationship expressed in the fundamental theorem of calculus.

At a later stage in university pure mathematics, fundamental properties are formulated as axioms in a third world of *axiomatic formalism* where concepts are defined set-theoretically and further properties are proved as theorems using mathematical proof. Even here, the full evolution of formal mathematics essentially follows the same underlying van Hiele-like framework. The development of formal mathematical theory begins with the *recognition* and *description* of possible properties (in the form of conjectures) and the *definition* and *deduction* of formal theory using set-theoretic definition and formal proof.

The algebra of linear and quadratic equations studied in this paper occurs in the later stages of operational symbolism in school where arithmetic is generalized to algebra. It also has conceptual embodiments not only as graphical representations but also as physical and mental representations as a balance. The solution of equations introduces more general forms of reasoning such as the principle of 'doing the same thing to both sides' that has meaning in both embodied terms, as a balance, and symbolic terms, as an equation.

More formal techniques in algebra involve manipulating symbols to simplify expressions, factorizing expressions and performing operations such as multiplying out brackets. For example, the expression $2x+6+x$ may be rewritten more simply as $3x+6$ and factorized as $3(x+2)$. Here the operation may be imagined as changing one expression into another or as representing the same underlying conception written in two different ways. There is clearly a difference as *processes*: $3x+6$ multiplies 3 times x and adds 6, while $3(x+2)$ multiplies 3 times the sum of x and 2. However, the *results* of the operations are the same, and in algebraic manipulation, they are considered to be different ways of representing the same underlying object. This difference between a focus on carrying out various procedural rules of operation to change something into something else and the more flexible view of working with the same idea represented in different ways is fundamental in simplifying mathematical thinking. This leads to the introduction of a central simplifying idea.

Crystalline concepts

The curriculum is full of examples where mathematical concepts are represented in different ways that can also be considered as being essentially the same. For example, we speak of 'equivalent fractions' where the fractions $2/3$ and $4/6$ are 'equivalent' but different (as processes) but the rational numbers $2/3$ and $4/6$ are one and the same concept.

Tall (2011) formulated a working definition of a *crystalline concept* as 'a concept that has a structure of relationships that are a necessary consequence of its context'. Such a concept has strong internal bonds that hold it together so that it can be considered as a single entity. Just as Sfard (1991) spoke of 'condensing' a process from a sequence of distinct steps which we may interpret as a metaphor for transforming a gas that is diffuse to a liquid that can be poured in a single flow, we can think of 'crystallizing' as the transition that turns the flowing liquid into a solid object that can be manipulated in the hand, or, in mathematics, manipulated in the mind as an entity. This metaphor does not mean that a crystalline concept has uniform faces like a chemical crystal, but that it has strong internal bonds that cause it to have a predictable structure.

Crystalline concepts are found throughout mathematics in many guises. They arise throughout geometry where specific figures have interrelated properties as a consequence of their definitions and more general concepts such as congruent triangles and parallel lines have definitions that cause them to have predictable structure. For instance, a triangle with two equal sides must, as a consequence, have two equal angles, even though the definition specifies only the equality of the sides and does not mention angles. In operational symbolism, numbers, algebraic expressions and, more generally, procepts, are crystalline, where the same underlying concept may be symbolised and manipulated in various ways. For instance, the number 5 may also be written as 2 + 3 and if 3 is taken away from 5, the result must be 2. In axiomatic formalism, axiomatic systems and defined concepts within those systems all have necessary properties that can be deduced by mathematical proof (Tall, 2011).

Our interest in this paper focuses on the crystalline structure of equations and how they have necessary structures that can be seen to operate in flexible ways. For instance, if we begin with an equation and operate on it by 'doing the same thing to both sides' in a way that can be reversed (such as adding the same quantity to both sides, or multiplying throughout by a non-zero number), then the new equation has the same solutions, as do any further equations produced by a reversible operation. This offers an overall coherence to the solution of equations where the underlying solutions remain unchanged by the operations on the equations. However, students who use procedural embodiments remained focused more on the step-by-step sequence of actions to move towards a solution in which the equations are changed into new equations rather than grasping the overall principle of 'doing the same thing to both sides' which has the effect of maintaining the coherence of the solution throughout the whole activity.

An overall framework

Taken together, the ingredients of our framework suggest that the development of mathematical thinking involves three distinct ways of making sense of mathematics, each of which develops in sophistication:

> First, through making sense through our physical perceptions and actions developing into mental structures through thought experiments;
>
> second, through our actions which become organized mathematical operations that are symbolized and lead to increasingly sophisticated calculation and symbol manipulation;
>
> and third, through the increasingly subtle use of language and reason that begins with recognition and description of properties, then develops through definition and various forms of mathematical deduction.

Learning builds on previous experiences that may be supportive and encourage generalization of ideas in new contexts or problematic and impede understanding. Supportive met-befores give pleasure and problematic met-befores cause frustration. The student who succeeds in making sense of the new situations develops in confidence and responds to problematic met-befores by responding to the challenge to conquer the difficulties. In the longer term this may lead to increasingly rich knowledge structures and the vision of mathematical ideas as crystalline concepts.

The student who is unable to cope with new situations sees them as becoming increasingly complicated and may feel alienated and develop mathematical anxiety.

A student who uses a method that has problematic undertones may be able to 'do' a problem and get the correct answer, while feeling uncomfortable about its meaning. Obtaining the correct answer is only part of long-term learning. A student may succeed at one stage but a problematic met-before lying in the subconscious may impede future learning. Using the goal-oriented theory of Skemp (1979), this may drive the student away from the goal of understanding mathematics to the alternative goal of learning procedures to solve standard problems. Procedural learning may give initial success yet fail to provide a flexible basis for successful learning in new contexts, leading to increasingly complicated procedures rather than richly connected crystalline structures. For example, procedural embodiment may give some success in solving linear equations but may impede learning when solving quadratics.

The case of algebra and the shift from linear to quadratic equations

The specific case under discussion in this paper involves the long-term growth of mathematical thinking that at an earlier stage involved the generalization of arithmetic to algebra and here focuses on the shift from linear to quadratic equations. It occurs as students build on their previous experience in arithmetic, developing mainly symbolic methods of solving linear equations that do not link either to the symbolic didactic cut or to the embodied notion of a balance, and do not explicitly use the more general reasoning of 'doing the same thing to both sides'. Instead they shifted the symbols around in an embodied sense, 'collecting like terms together', 'moving terms to the other side', and using additional techniques such as 'change signs'. Our attention now turns to how these students develop as they encountered quadratic equations.

We first report and analyse the collected data, then we consider this data in relation to the wider literature and the overall theoretical framework outlined in previous sections. Our purpose is to evolve a practical theory that explains and predicts why students learn in a manner based on their previous experience that may be supportive or problematic in a new context. In particular we take note of the observation of McGowen and Tall (2010) that the effect of previous experience applies not only to the met-befores of students, but also of the theorists who build the theories. We therefore expect theoretical frameworks to evolve over time to take account of new ways of making sense of observed data.

THE RESEARCH STUDY

The data presented in this paper was collected in the doctoral study of Lima (2007), developed at The Pontifical Catholic University of São Paulo (Brazil) and the University of Warwick (UK). The research involved sharing ideas with a group of high-school teachers whose objective was to examine their current teaching practices to seek ways to improve their teaching. The researcher encouraged the teachers to carry out their own ideas and to share the design of research instruments and the collection of data. The data came from 80 high school students in three groups, one of 32 15-year-olds, one of 28 15-year-olds, both from a public school in the city of Guarulhos/SP; and one group of 20 16-year-olds from a private school in São Paulo/SP. All of them had already been taught how to solve linear equations at least two years before the research took place, followed by quadratic equations a year later. This research focuses on their long-term grasp of experiences that they had met before.

In the study, there were three data collections, each one administered by the class teacher in a lesson lasting 100 minutes. The first invited the students to construct a concept map of their knowledge of linear and quadratic equations, the second was a questionnaire and the third was an equation-solving task. After an initial analysis of data, twenty students were selected for interviews, conducted by the researcher, in the presence of an observer, and tape recorded for further analysis. Students who participated in interviews were chosen by the kind of work they presented – including either typical mistakes or correct answers. In the interviews, we wished to investigate why students performed as they did. In particular, they were asked to explain what kind of symbol manipulation they had performed and why they believed it was a proper way to proceed. In this paper, we focus specifically on the work students performed when they were asked to solve quadratic equations and relate this data to the overall framework of three worlds of mathematics. (Detailed analyses of other parts of the study can be found in Lima & Tall, 2006a; Lima & Tall, 2006b; Lima, 2007; Lima & Tall, 2008, Lima & Healy, 2010.)

Tasks with quadratic equations

The data used to investigate the students' conceptions of quadratic equations came from two instruments, an equation-solving task, with three linear equations and four quadratic equations:

$$3l^2 - l = 0\,,\ r^2 - r = 2\,,\ a^2 - 2a - 3 = 0\,,\ m^2 = 9\,,$$

together with a questionnaire that included two quadratic equations:

$$t^2 - 2t = 0\,,\ (y-3)(y-2) = 0\,.$$

The questionnaire also included a request to respond to the solution of the final quadratic equation as given by an imaginary student 'John':

> To solve the equation $(x-3)(x-2)=0$ for real numbers, John answered in a single line that:
>
> '$x = 3$ or $x = 2$.'
>
> Is his answer correct? Analyse and comment on John's answer.

Figure 1: John's Problem (question 8 of the questionnaire).

Interviews with selected students revealed additional personal information on how they interpreted the tasks and their thinking in seeking solutions.

DATA AND RESULTS

A total of 68 students gave their answers to the equation-solving task and 77 responded to the questionnaire, due to absences on each day. From an analysis of all the instruments, our findings are that the students mainly interpreted an equation as a calculation, building on their previous experience working with numbers. For instance, when asked, ‘What is an equation?’ in the questionnaire, 36 out of 77 students (47%) answered that *‘it is a calculation in mathematics’* or some equivalent response. Less than half the students mentioned the unknown. Instead, the responses often focused on the equals sign interpreted as a signal to perform a calculation (termed an *operational sign* by Kieran, 1981), consistent with their earlier experience of using an equals sign in calculations in arithmetic.

The responses to the six equations are given in Table 1, with the number using the formula in square brackets.

Equation	Correct	One root	Incorrect	Blank	Total
$a^2-2a-3=0$	4 [4]	0 [0]	41 [6]	23 [0]	68 [10]
$r^2-r=2$	3 [3]	9 [0]	31 [5]	25 [0]	68 [8]
$3l^2-l=0$	3 [3]	0 [0]	40 [3]	25 [0]	68 [6]
$m^2=9$	1 [1]	15 [0]	27 [2]	25 [0]	68 [3]
$t^2-2t=0$	6 [6]	0 [0]	62 [11]	9 [0]	77 [17]
$(y-3)\cdot(y-2)=0$	8 [8]	0 [0]	63 [7]	6 [0]	77 [15]
Total	25 [25]	24 [0]	264 [34]	113 [0]	426 [59]

Table 1: Solutions of equations [those using the formula in brackets]

Two correct roots using the formula

The first column reveals that in total, of the 426 responses, only 25 (6%) gave a correct response with two roots and *all* of these used the formula. A study of the individual solutions reveals that *not one student completed the square* to solve any of the quadratic equations and *not one student used factorization*, not even in the case of equations $t^2-2t=0$ or $3l^2-l=0$.

Even in the final equation $(y-2).(y-3)=0$, which is already factorized, *none* of the students used the given factorization. Only 8 out of 77 responses to this question (10%) gave a correct solution and all of them multiplied out the brackets and used the formula. Seven of the 63 incorrect solutions also used the formula but were unable to carry out the required manipulation.

One correct root using evaluation or procedural embodiment

The partially correct solutions with one root (column 3) either guessed a value that satisfied the equation or used procedural embodiment shifting the power to the other side and turning it into a root.

Of the 9 students finding one root for $r^2 - r = 2$, *all* solutions were found by arithmetic evaluation. Several students explained in interview that they thought of the equation $r^2 - r = 2$ as '*a squared number taking away the same number resulting in 2, is 2'* using the familiar fact that 2^2 is 4, so $2^2 - 2$ is 2.

The equation $m^2 = 9$ was solved correctly by just one student, who quoted the formula which therefore gave the two roots. Of the 68 students responding to this question, 15 (22%) found the solution as a single square root, either by square-rooting 9 or by using a procedural embodiment to shift the square root over the other side where it became a (positive) square root (Figure 2).

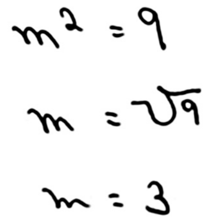


Figure 2: Passing the exponent to the other side as a square root

In interview, one of them explained, *'the power two passes to the other side as a square root.'* In this explanation, the student makes it clear that there is a movement of the exponent and a transformation from a square power to a square root.

Just as with linear equations, what seems to be happening in this student's explanation is a movement of symbols and an additional magic rule for changing something: the power is passed to the other side and is transformed into a square root. It is a new variation of familiar procedural embodiments such as 'swop sides, swop signs', and, perhaps for this reason, the students were satisfied to find just one value. Neither this student, nor any of the others interviewed, mentioned the possibility of another (negative) root.

No correct solutions

Out of a total of 426 solutions, 264 (62%) were incorrect and 113 (27%) were blank. Only two out of the 27 erroneous solutions of $m^2 = 9$ attempted to use the formula and both failed to use it correctly. All the others followed a general strategy of 'moving towards a solution' by 'simplifying' the quadratic equation in a mistaken procedural way to obtain a linear equation that they could then attempt to solve by procedural embodiment.

For example, one student rewrote m^2 as *m.m* and then interpreted this as 'two *m*s' to give a linear equation that led to an erroneous solution (Figure 3).

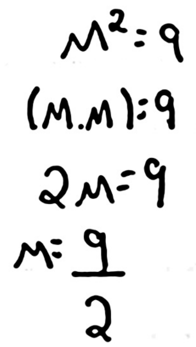


Figure 3: m^2 seen as $2m$

A common error made on various equations by nine students (13%) out of 68 on the equation solving task was simply to replace m^2, r^2 or a^2 respectively by m, r and a, and then solve the equation as if it were linear. Others used the exponent of the squared term to square its coefficient (Figure 4).

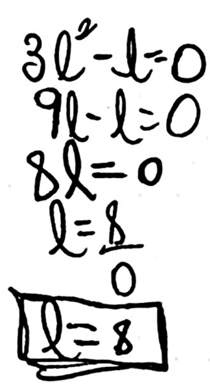


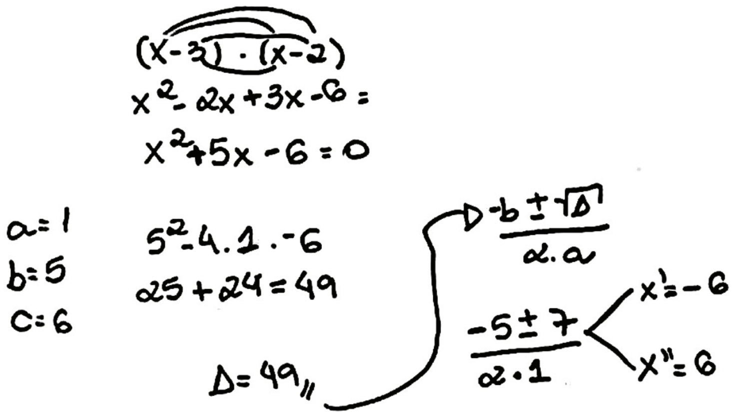
Figure 4: Using the power of the unknown in its coefficient

Here the switch from $3l^2$ to $9l$ may involve seeing the power applying to both terms and applying it only to the part that the student can actually calculate, namely to the numerical coefficient.

The effect of these faulty operations may be seen as an attempt to 'move towards a solution' by a procedural embodiment that transforms the quadratic equation into a more familiar linear problem, which then proceeds by procedural embodiment. In Figure 3, the final part correctly 'moves the 2 underneath'. In figure 4, after reducing the quadratic to the equation $8l = 0$, the student shifts the 8 over the other side, putting it on top with the zero below, then moves to the final 'solution' by ignoring the zero (perhaps because it may 'do nothing') to leave the solution as 8.

John's Problem

There is clear evidence that some of the students believe the formula to be the 'right' way to solve quadratic equations (despite the difficulties that they had in applying it). Evidence for this arises in the responses to 'John's Problem' (Figure 1). Thirty students out of 77 (39%) claimed that his solution was correct. Three (4%) mentioned the formula saying things like, *'He must have used the quadratic formula in his mind.'* Eleven students (14%) declared that *'John didn't solve the equation'* essentially *'because he did not use the formula.'* Four students (5%) used the formula to solve the equation and compared the result with John's solution. One of these used the formula incorrectly and obtained different values from John, insisting that John was wrong (Figure 5).



'Ah! I don't know,but I think that John is wrong and I think that my way is right; I said my way, not my results, ok?'

Figure 5: A student's use of the quadratic formula and his verbal comments

Most of the students who believe that they needed to use the formula to get the solutions for a quadratic equation lacked the flexibility to manipulate algebraic symbols. No one responded to say that John's answer is correct by referring to the principle that when a product is zero, one of the factors must be zero. Some responses referred to *the need to carry out the calculation*, to test whether the solution is correct:

> *'If he guesses that, as it equals zero, x should be 3 or 2, it is wrong. But maybe, he is very clever, calculated in his mind, and supposed that this is the answer.'*

or

> *'I don't know, but I think it is wrong because he didn't do the calculation, he just put the results that were by the side of x.'*

Such responses often involve a procedurally embodied form of evaluation by mentally 'putting' numerical values for the variable 'into' the equation. Four students (5%) (three in the questionnaire and one during interview) said that John is right *'because putting $x = 3$ or $x = 2$ gives the number zero'*, while two others substituted both values successively into the equation to check both solutions (Figure 6).

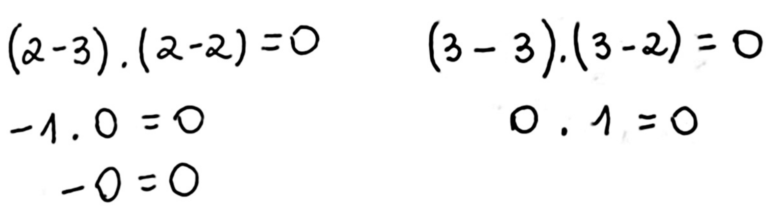


Figure 6: Replacing values for *x* in the equation.

One of those performing the substitution explained in interview:

> *Student: To see if the answer is right, I have put 3 here [in the place of x] to see*

what result I would get, and then another calculation with 2.

Interviewer: Why have you put 3 in the place of x, and then 2 in the place of x?

Student: Because here it says that x is equal to 3 so, if x is 3, then I replace the number to see what I get.

Interviewer: And what happens if the result is the same as the one in the equation?

Student: If it is zero, then x is 3.

The language here speaks of 'putting' or 'substituting' a numerical value into an equation to evaluate the expression to check the equation, combining both evaluation and procedural embodiment successfully in a manner reminiscent of experiences with linear equations. This operation is successful for those who use it and reflects not only the particular operation of evaluation but also a formal characteristic of the solution process: that the solution is a number that satisfies the equation when it is put in place of *x* and the evaluation is carried out.

DISCUSSION

What is evident from data collected in this study is that very few of these students use flexible algebraic symbol manipulation or formal principles such as 'do the same thing to both sides'. Having developed a technique of embodied procedural symbol shifting in linear equations, some used a similar technique to solve equations of the form $x^2 = k$, by shifting the power over the equal sign where it becomes a square root and gives only a single solution. No one completed the square or factorized equations to find the solution. A small number used the formula and many of these had difficulty if algebraic manipulation was required to get the equation into the right form to use the formula. Students who used procedural embodiments all failed to get both roots, either finding a single root by shifting a square on one side to the other where it became the (positive) square root or by making errors in symbol-shifting that gave erroneous results.

In summary, all correct results giving two roots (6%) used the formula, while all the results giving a single correct result (6%) either used a procedural embodiment shifting the power to a square root on the other side, or guessed a single correct solution. All other solutions were either blank (27%) or often used a form of procedural embodiment to give a wrong answer (62%).

Now we see that the attempts at solutions involve either fragile procedural embodiments (as in linear equations) or a minority use of the quadratic formula with little understanding. This could relate to the teachers desire to give the students a technique (the formula) that they knew could be used in *all* cases, in preference to the complications of completing the square or factorizing quadratics. However, the strategy had extremely limited success, especially in cases where it required symbolic manipulation to translate the equation into the needed form $ax^2 + bx + c = 0$, which most students in the study found difficult.

This data does not make for comfortable reading. The teachers as experts attempted to teach the students as novices to practice the procedures that they had found to be successful for their own solution of equations, but the students saw the operations in terms of their own experience and most did not grasp the general theory.

The three-world framework suggests the need to take into account three main aspects:

- *(i) conceptual embodiment and the transition to operational symbolism,*
- *(ii) the symbolic transition from arithmetic to algebra,*
- *(iii) the introduction of general formal principles, such as 'do the same thing to both sides'.*

We consider each of these in turn.

Conceptual embodiment and the transition to operational symbolism

Students' responses bring little evidence of attempts to make use of conceptual embodiments of equations. Indeed, if we look at previous research involving both linear and quadratic equations, we find that such embodiments tend to have limitations beyond the more simple cases. The work of Vlassis, for example, has already shown how the conceptual embodiment of a linear equation as a balance proves to be supportive in simple cases but is problematic where negative quantities are involved.

In relation to quadratic equations, an interesting visual approach arose from the time of the Babylonians, and extended in Arabic mathematics in terms of physically 'completing the square'. Based on this idea, Radford and Guérette (2000) designed 'a teaching sequence whose purpose is to lead the students to reinvent the formula that solves the general quadratic equation' (p.69). An example is given in Figure 7.

The length of a rectangle is 10 units. Its width is unknown. We place a square on one of the sides of the rectangle as shown in the figure. Together the two shapes have an area of 39 square units. What is the width of the rectangle?

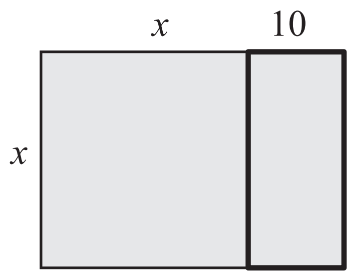


Figure 7: The Babylonian Geometric Model

The pieces were cut out of cardboard and the solution could be found by cutting the rectangle vertically in half (Figure 8a), rearranging the pieces to move one half rectangle round to the bottom (Figure 8b), then realizing that what is missing to 'complete the square' is the corner square with sides 5×5. Filling this in to get a total area of $39 + 25 = 64$ units (Figure 8c), we find the larger square has side 8 units and so, taking off the 5 units leaves $x = 3$.

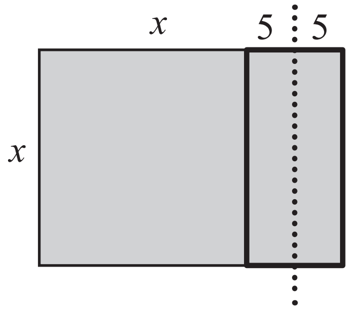


Figure 8a
Cut the $10\times x$ rectangle

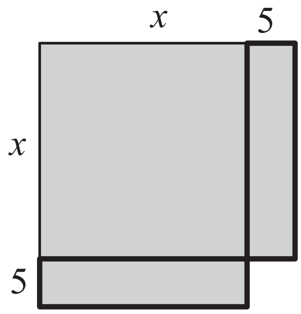


Figure 8b
Rearrange the pieces

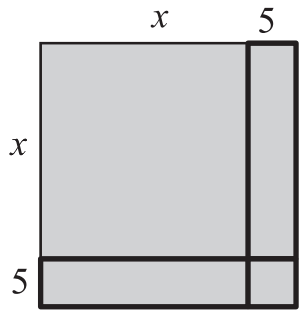


Figure 8c
Complete the square

Students were then encouraged to think of a number of similar examples and derive a symbolic solution to equations of the form $x^2+bx=c$ to find the general solution:

$$x=\sqrt{c+\left(\frac{b}{2}\right)^2}-\left(\frac{b}{2}\right).$$

They were shown how $ax^2+bx=c$ could be rewritten as $x^2+\frac{b}{a}x=\frac{c}{a}$, and substituting b/a for b and c/a for c gives the general solution of $ax^2+bx=c$ as

$$x=\sqrt{c+\left(\frac{b}{2a}\right)^2}-\left(\frac{b}{2a}\right)$$

The next step suggested is to replace c by $-c$ to obtain the solution of $ax^2+bx+c=0$ as

$$x=\sqrt{-c+\left(\frac{b}{2a}\right)^2}-\left(\frac{b}{2a}\right)$$

The paper continues (p.74) with the comment:

Of course, this formula is equivalent to the well-known formula

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

where, in order to obtain all the numerical solutions, one also needs to consider the negative square root of b^2-4ac. *This leads us to the formula:*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

The authors suggest that this is a good way to introduce the quadratic formula for students because it relates geometry and algebra, aiming 'to provide a useful context to help the students develop a meaning for symbols' (p.74). They note that many students were able to solve the initial tasks but 'need some time to abandon the geometrical context themselves to the numerical formulae', commenting on the complexity of the semiotic structures, without any explicit reasons for the difficulties.

The three-world framework clarifies the details. The representation of variables geometrically as lengths requires the quantities to be *positive*. If the same method is applied to an equation of the form $x^2 - bx = c$ such as $x^2 - 10x = 64$, instead of adding rectangles $5 \times x$, this involves cutting them away. Having cut off one rectangle from the right-hand side of the square, as in Figure 9, the lower right 5×5 square has already been removed, so it is no longer possible to cut away the full rectangle size $5 \times x$ along the bottom.

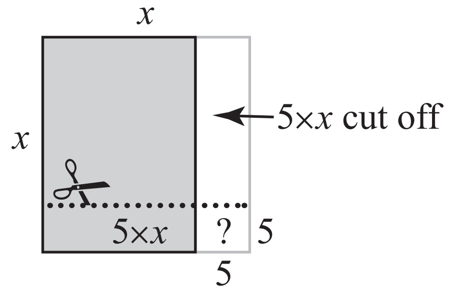


Figure 9: Attempting to cut off two rectangles of size 5×x

This reproduces the phenomenon observed in the linear case where an embodiment supportive for unsigned numbers becomes problematic when signed numbers are introduced.

The symbolic transition from arithmetic to algebra

We have already seen the 'didactic cut' in action for linear equations where the evaluation equation $ax + b = c$ can be 'undone' but the more general manipulation equation $ax + b = cx + d$ requires algebraic manipulation to find a solution.

The quadratic case is more complicated. Some simple equations have solutions that can be 'undone' by arithmetic operation, such as $x^2 = 9$ where the operation of squaring can be undone to give the square root $x = 3$. An equation that is already factorized such as $(x-2)(x-3) = 0$ may also be solved by evaluation, by substituting each of the values 2, 3 to see that they both satisfy the equation. However, although these solutions are self-evident for an expert, they prove to be problematic for the student who has learned to solve linear equations by procedurally embodied symbol shifting. In the first case, only the positive root is found, consistent with the experience in solving linear equations that have only a single solution. In the second case the students did not use the general principle that if a product of brackets is zero, then one of the brackets must be zero; instead all of those who sought to find a solution did so by attempting to multiply out the brackets and use the formula.

The 'didactic cut', which has proved to be a helpful theoretical construct in dealing with the symbolic solution of linear equations, is less relevant in the solution of quadratic equations. An equation 'quadratic expression = number' in general does not have a quadratic in a form of an operation that can be ‘undone’. Lima and Healy (2010) suggested that a quadratic of the particular form $a(x+b)^2+c=d$ may be ‘undone’ as an evaluation equation by starting with d, subtracting c, dividing by a, taking a square root and then subtracting b to find x. Such a generalized procedure does not occur in any of the data. It is not a procedure that the students have practiced and it does not offer a method of ‘undoing’ more general quadratic expressions. Instead, one needs to manipulate the symbols, either by factorization into two linear factors or by ‘completing the square’. Thus the ‘didactic cut’, while being a suitable theoretical construct for linear equations, does not readily extend to quadratic equations.

There are three main symbolic techniques for solving quadratic equations: factorization (if that is appropriate), completing the square, or using the formula arising from completing the square. Vaiyavutjamai & Clements (2006) analysed the written solutions of 231 Grade 9 Thai students after eleven lessons studying all three techniques and found data similar to the present study. Students did not use the principle that if the product of two brackets was zero, then one of them must be zero. They solved the already factorized equation $(x-3)(x-5)=0$ by multiplying out the factors and using the formula. To check if these solutions are correct, some replaced the x in the first bracket by 3 and in the second bracket by 5, as if the equation simultaneously had both solutions. In dealing with the equation $x^2=9$, some students responded by saying that ‘in that equation x appears only once, and therefore there is only one solution’ (p.72).

Thorpe (1989) reported that even when students could successfully find solutions for quadratic equations using the formula, the ‘±’ sign in $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ might not be meaningfully understood.

When Gray & Thomas (2001) used graphic calculators to combine symbolic manipulation with graphic representations, they encouraged their students to practice paper and pencil methods of solution and to plot the graphs of functions to solve quadratic equations in various ways. They found little progress in procedural skills to solve quadratic equations, that the students seemed not to understand the principle of performing the same operation in both sides of an equation, and that they used procedures without understanding why they worked. Students were able to perform a range of individual tasks yet lacked the flexibility to move easily from one representation to another, for example, to switch from a symbolic to a graphical representation to visualize the solutions of equations in terms of where the graphs meet the horizontal axis.

In all these studies, many students have difficulty making flexible sense of the solutions of quadratic equations.

The introduction of the formal principle 'do the same thing to both sides'

The introduction of the principle of 'doing the same thing to both sides' also made little impact on many students in the current study or in the research papers quoted above. This is consistent with the shift in meaning in the worlds of embodiment and symbolism from the practical aspects of school mathematics to the more formal aspects of embodied and symbolic reasoning. It does not involve the higher level of axiomatic systems and formal proof, but it does signal a shift to a more general level of operational symbolism, building not on specific operations, but on a general strategy. While there were some students that had a flexible view of operational symbolism who showed some appreciation of its meaning, most students in the current study either found incorrect solutions or left the solution blank.

REFLECTIONS

The data in this paper shows only 6% of the responses in this study correctly finding both solutions of a quadratic equation, all of whom use the formula, with 6% finding one solution, either by inspection or by procedural symbol shifting. All other solutions were either blank (27%) or gave an incorrect answer (62%). Our previous analyses of these students (Lima & Tall, 2008) indicated that they solved linear equations based on their previous experience of arithmetic operations, in which operations are carried out to obtain an answer, mainly by procedural symbol shifting to move towards a solution. With neither type of equation did the students use flexible manipulation of symbols or the general principle of 'doing the same thing to both sides'. The problems that students using procedural methods encountered with linear equations become even more severe with quadratic equations, a finding not limited to this study, but also in all the other research considered in the literature review.

Gray and Tall (1994) proposed the *proceptual divide* in which a spectrum of performance in arithmetic grows from those students who begin to use flexible relationships between numbers to make their task easier to those who continue to focus on procedures of counting where the difficulties grow even greater as the topics become more sophisticated.

This study reveals that the proceptual divide continues further into algebra. While some students may develop flexible methods to solve linear equations, most of those in this study solved linear equations by procedural symbol shifting that leads to even greater problems when attempting to solve quadratic equations. The learning that occurs at each stage affects subsequent stages and the bifurcation between those who make flexible use of symbolism to make sense of the mathematics and those who use procedural embodiments can only grow wider until, as here, those succeeding in solving quadratic equations are a small minority.

The development of algebra is part of the whole growth of mathematical thinking which is formulated as blending embodiment and symbolism in school mathematics, leading to embodied and symbolic forms of reasoning, which are later transformed into an axiomatic formal world of set-theoretic definition and proof in university pure mathematics.

The three-world framework formulates the cognitive and affective development of mathematical thinking over a lifetime from a newborn child to the full spectrum of adult mathematical thinking. It includes the effects of supportive met-befores that enable generalizations in new contexts and problematic met-befores that impede progress, with a growing awareness of the crystalline structure of mathematical concepts that enable them to be grasped and manipulated as mental entities with flexible meaningful links between them.

The particular study of the solution of linear and quadratic equations occurs in operational symbolism with some support from embodied representations. The forms of reasoning appropriate to school algebra involve more formal use of embodiment and symbolism without any reference to the third world of axiomatic formalism. The reasoning in the solution of algebraic equations builds symbolically on the operations of generalized arithmetic, shifting from evaluation equations to equations requiring more general symbolic manipulation that give rise to the problematic aspects of the didactic cut. This may be blended with various conceptual embodiments such as seeing the solution of equations as the intersection of graphs, imagining the equation as a physical balance or cutting up squares in the case of quadratic equations. Methods that work with physical quantities – such as the equation as a balance, or the representation of x^2 as a physical square – become problematic when negative quantities are introduced. The introduction of more general strategies, such as 'doing the same thing to both sides' prove to be problematic for students who interpret the generalities in terms of procedural symbol-shifting. The proceptual divide reveals a spectrum of performance between those who remain limited to learning step-by-step procedures and those with the flexibility of being able to grasp the crystalline structure of mathematical concepts.

To address these issues requires more than focusing on the particular context at a particular level of the curriculum. The problems encountered in quadratic equations lie not only in that topic, nor in what is carried forward from linear equations, but in the whole build up of mathematical structures over the student's lifetime. The bifurcation between success and failure is likely to become even wider as supportive and problematic met-befores affect successive learning in increasingly sophisticated mathematical contexts. This makes it incumbent on us as mathematical educators to evolve an approach to long-term learning in the light of what each student has met before.

References

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., Thomas, K. (1996), A framework for research and curriculum development in undergraduate mathematics education, *Research in Collegiate Mathematics Education II*, CBMS Issues in Mathematics Education, 6, 1–32.

Collis, K. F. (1978). Operational thinking in elementary mathematics. In *Cognitive development: Research based on a Neo-Piagetian approach*, edited by J. A. Keats, K. F. Collis and G. S. Halford. Chichester: John Wiley & Sons.

Biggs, J. & Collis, K. (1982). *Evaluating the Quality of Learning: the SOLO Taxonomy*. New York: Academic Press.

Bruner, J. S. (1966). *Towards a Theory of Instruction*, Cambridge, Mass: Harvard University Press.

Dienes Z. P. (1960). *Building Up Mathematics*. Hutchinson.

Donald, M. (2001). *A Mind So Rare.* New York: Norton & Co.

Fauconnier, G. & Turner, M. (2002). *The way we think: Conceptual Blending and the Mind's Hidden Complexities.* New York: Basic Books.

Filloy, E., & Rojano, T. (1989). Solving equations, the transition from arithmetic to algebra, *For the Learning of Mathematics*, Canada, 9 (2), 19–25.

Gray, E. M. (1991). An Analysis of Diverging Approaches to Simple Arithmetic: Preference and its Consequences*, Educational Studies in Mathematics*, 22, 551–574.

Gray, E. M. & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *Journal for Research in Mathematics Education*, 26 2, 115-141.

Gray, R., & Thomas, M. (2001). Quadratic Equation Representations and Graphic Calculators: Procedural and Conceptual Interactions. *Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia, Sydney, 30 June-4 July, 2001* (2), 115–141.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 317–326.

Lakoff, G. & Núñez, R. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. New York: Basic Books.

Lima, R. N. de, (2007). *Equações Algébricas no Ensino Médio: Uma Jornada por Diferentes mundos da Matemática*. (Doctoral Dissertation Catholic University of São Paulo.)

Lima, R. N., & Healy, L. (2010). The Didactic Cut in Equation Solving or a Gap Between the Embodied and the Symbolic Mathematical Worlds? In M. M. Pinto, & T. F. Kawasaki (Ed.), *Proceedings of the 34th Meeting of the International Group for Psychology of Mathematics Education*, 3, pp. 353–360. Belo Horizonte, Brazil.

Lima, R. N. de, & Tall, D. (2006a). The concept of equations: What have students met before? *Proceedings of the 30th Conference of PME* (vol. 4, pp. 233–241). Prague, Czech Republic.

Lima, R. N. de, & Tall, D. (2006b). What does equation mean? A brainstorm of the concept. In D. Hughes-Hallett, I. Vakalis, H. Arikan (Eds.) *Proceedings of the International Conference on the Teaching of Mathematics at Undergraduate Level*, Paper 157, Istanbul, Turkey.

Lima, R. N. de, & Tall, D. O. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, *67* (1), 3–18.

McGowen, M. C. & Tall, D. O. (2010). Metaphor or Met-before? The effects of previous experience on the practice and theory of learning mathematics. *Journal of Mathematical Behavior* 29, 169–179.

Matz, M. (1980). Towards a Computational Theory of Algebraic Competence. *Journal of Mathematical Behavior* 3 (1), 93-166.

Piaget, J. (1970). Piaget's theory. In P. H. Mussen (Ed.), *Carmichael's Handbook of child psychology* (pp. 703-732). New York: Wiley.

Payne, S. J., & Squibb, H. R. (1990), Algebra Mal-Rules and Cognitive Accounts of Error. *Cognitive Science* 14, 445–448,

Radford, L. & Guérette, G. (2000). Second degree equations in the classroom: A Babylonian approach. In V. Katz (ed.). *Using history to teach mathematics. An international perspective* (pp. 69–75). Washington: The Mathematical Association of America.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22, 1–36.

Skemp, R. R. (1979). *Intelligence, Learning, and Action*. Wiley: London.

Tall, D. O. (2004). The three worlds of mathematics. *For the Learning of Mathematics*, 23 (3). 29–33.

Tall, D. O. (2011). Crystalline concepts in long-term mathematical invention and discovery. *For the Learning of Mathematics*. 31 (1) 3–8.

Tall, D. O. (2013). *How Humans Learn to Think Mathematically*. New York: Cambridge University Press.

Thorpe, J. A. (1989). Algebra: What should we teach and how should we teach it? In S. Wagner, C. Kieran. (eds), *Research Issues in the Learning and Teaching of Algebra*. EUA: NCTM. v. 4, 11–24.

Van Hiele, P. M. (1986). *Structure and Insight*. Orlando: Academic Press.

Vaiyavutjamai, P., & Clements, M. A. (2006). Effects of classroom instruction on students' understanding of quadratic equations. *Mathematics Education Research Journal*. 18 (1), 47–77,

Vlassis, J. (2002). The balance model: hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, 49, 341–359.