Integrating History, Technology and Education in Mathematics

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Plenary Presentation at *História e Tecnologia no Ensino da Matemática*
July 15, 2013, Universidade Federal de São Carlos, Brazil

At the inaugural meeting on *História e Tecnologia no Ensino da Matemática* in Rio, I presented a theoretical framework (Tall, 2003) using technology in a visual and symbolic approach to the calculus and the later formal development of mathematical analysis. In the following decade this has expanded to a broad theoretical framework for the cognitive and affective long-term development of mathematical thinking in general that applies also to history and technology in mathematics and mathematics education (Tall, 2013).

Figure 1 shows one of several reviews of learning theory (Wees, 2009), from early behavourist theories to various combinations of cognitive, constructivist and social viewpoints.

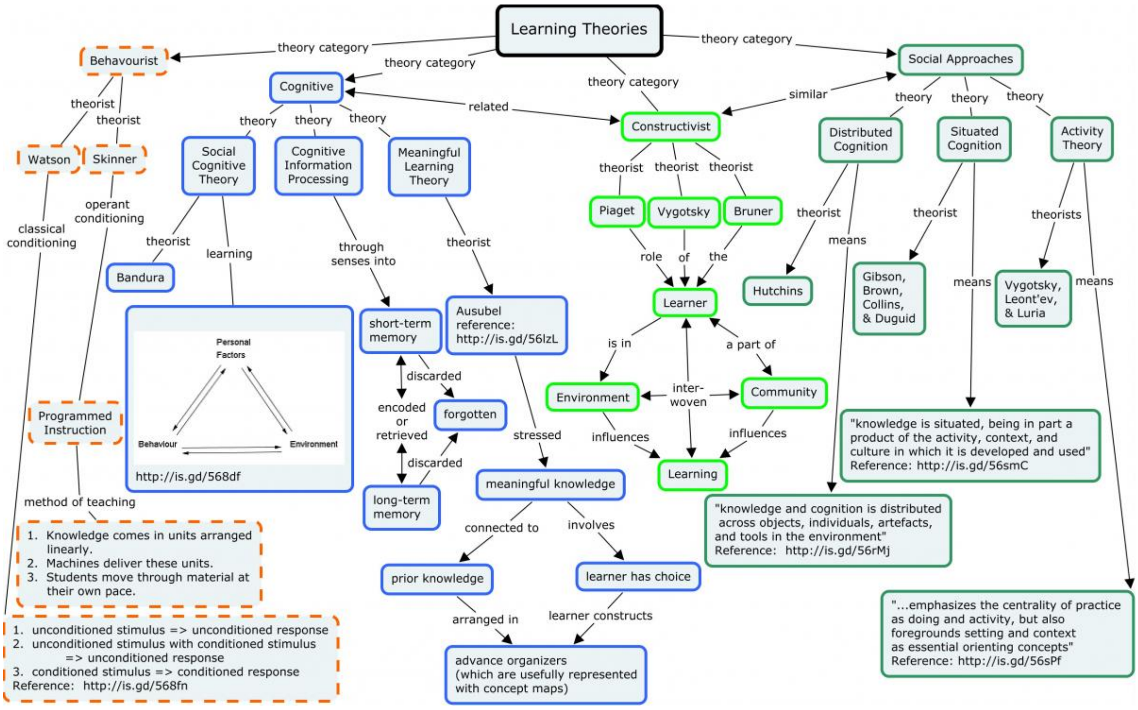

Figure 1: Learning Theories

Many of these theories have applications in mathematics, yet mathematics has its own characteristics related to the nature of its knowledge structure and the particular development of mathematical thinking in the individual and in society. To cope with the enormous complexity of the varied aspects of the whole picture, it is essential to find underlying structure that can focus on the most vital aspects of the nature of mathematics itself and the various ways that individuals build up mathematical ideas over time. The proposed framework focuses on how mathematics is constructed based on fundametal human attributes of perception (input), action (output) and the use of language and symbolism to build sophisticated knowledge structures in mathematics. Mathematics has flexible ideas that have necessary consequences that I term *crystalline concepts*. As new contexts are encountered, previous experience (termed a *met-before*) may be supportive and encourage generalization, or problematic and impede progress. The theoretical framework is self-contained but also offers the possibility of blending its theoretical constructs with those in other communities of practice to evolve new insights.

The development of mathematical thinking

Mathematics begins from our perceptions of and actions on the natural world around us, first through *practical mathematics* as we build on our perceptions of shape and space and our actions in counting and measuring that lead to the operations of arithmetic and the generalized arithmetic of algebra. We use language to describe objects, and perform operations, such as constructions in geometry, and counting, measuring and more sophisticated operations in arithmetic, algebra, calculus and other areas of mathematics. We recognise relationships in geometry, such as the fact that a triangle with two equal sides must have two equal angles, and regularities in arithmetic, such as addition being independent of order. Our reasoning develops in sophistication, leading to *theoretical mathematics*, based on theoretical definitions, with Euclidean proof in geometry and proof based on the 'rules of arithmetic' in arithmetic and algebra. This occurs in two parallel forms of development that I term *conceptual embodiment*, focusing on activities with objects, imagination and thought experiment, and *operational symbolism* arising from operations such as counting, measuring and more general operations of arithmetic, encapsulated as mental objects (such as numbers, algebraic expressions).

In school, making sense of mathematics develops through perception, operation and reason, leading to more formal levels of school mathematics based on definition and proof in the conceptual embodiment of Euclidean geometry and the operational symbolism of arithmetic and algebra. (Figure 2.)

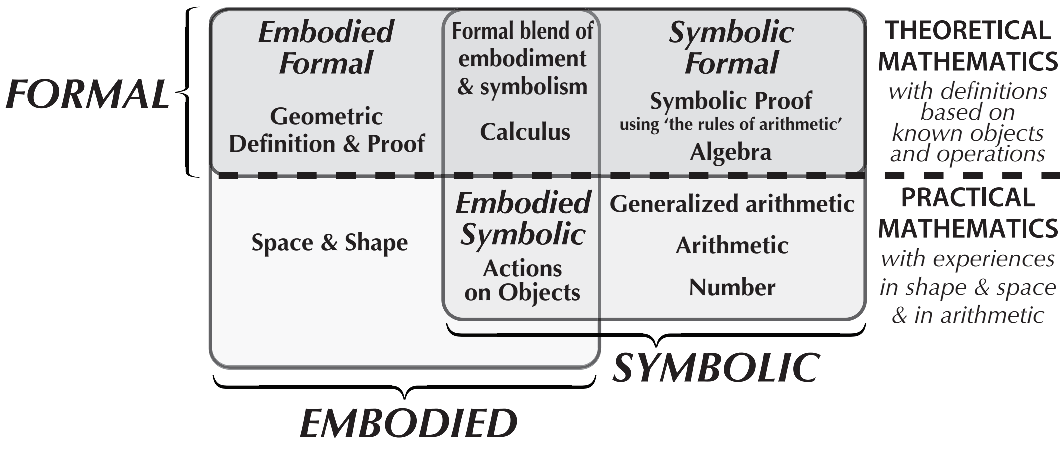

Figure 2: Natural Mathematics based on perception, operation and imagination

The embodied and symbolic mathematics in school extends to more sophisticated forms of applications of mathematics that formulate and predict the properties of natural phenomena. This relates to the historical development of 'natural philosophy' applying thought experiment and operational symbolism to real world problems that may be extended in the imagination to include perfect conceptions in Euclidean geometry and the potential infinity of repeated operations that cannot be realised in the finite life of human beings. The need to make new sense of infinite processes and more general concepts led to the transition in *formal mathematics* to a higher-level *axiomatic formal* world of proof based only on quantified set-theoretic definition and formal proof, independent of any specific natural foundation. Theoretical mathematics can now be seen either as the summit of natural mathematics (in Figure 2) or the transition to the axiomatic formal world of set-theoretic definition and formal proof (Figure 3).

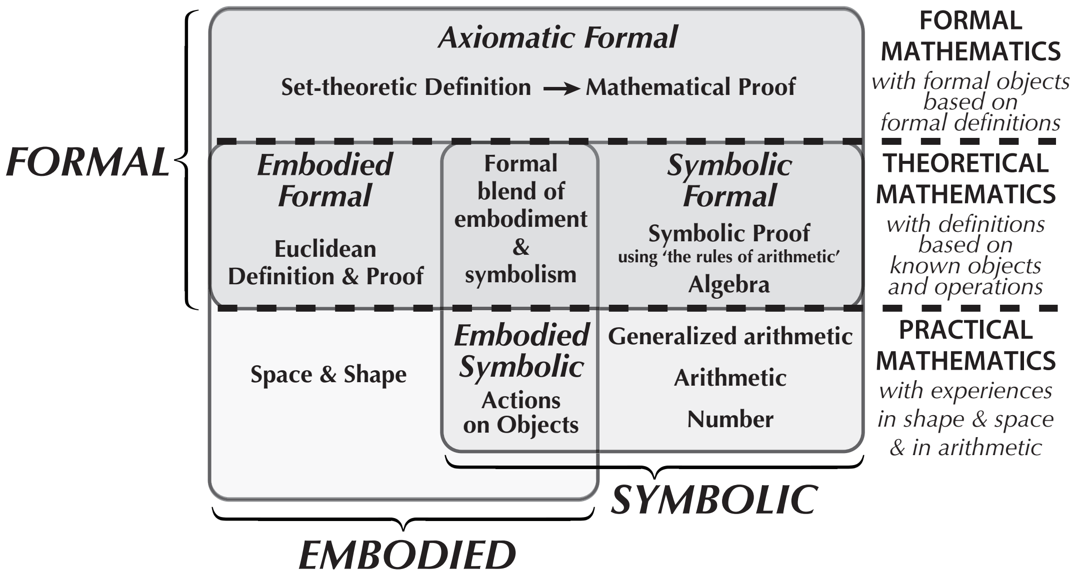

Figure 3: The Three Worlds of Mathematics (Tall, 2013)

I term these distinct developments of mathematics, three *worlds* of mathematics, each developing and blending together over the long term, beginning in embodiment, where a conceptual focus on properties of objects leads to verbal forms of reasoning such as Euclidean proof and an operational focus on the properties of operations leads to the formulation of 'rules of arithmetic' as a basis for symbolic proof. Embodiment and symbolism blend together in various ways including relationships between numbers and points on the number line and functions and graphs in the Cartesian plane. Embodiment offers perceptual experience and thought experiments to imagine perfection that is not attainable in the physical world, such as platonic figures in geometry and the potential infinity of counting in arithmetic. This level of theoretical mathematics is appropriate for most applications of mathematics in other areas of endeavour.

Structure theorems

The development of mathematical thinking at each stage incorporates earlier stages, with embodiment underpinning symbolism and both of them underpinning formalism. However, formalism is not the end of the development in sophistication. Formal structures such as groups, ordered fields, vector spaces, and so on, often lead to *structure theorems* that prove that formal structures have related embodied and symbolic representations. Some of these are unique, for instance the system of real numbers $\mathbb{R}$ is a complete ordered field that can be represented visually as a number line and symbolically as decimal expansions. Other axiomatic structures have more general forms of embodiment and symbolism. For example, a group can be proved to operate as transformations of a set, for example multiplying any element x in the group by a fixed element g, to give a transformation on the set of elements taking x to xg. This gives a permutation of the underlying set. so that a finite group can be considered as (a subgroup of) a group of permutations, represented visually and operationally.

Structure theorems return to embodiment and symbolism operating at a more sophisticated level, revealing the full development of mathematical thinking as an increasingly sophisticated succession of blends of embodiment, symbolism and formalism (Figure 4).

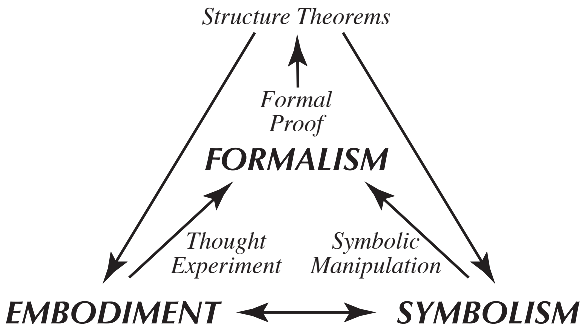

Figure 4: Blending embodiment, symbolism and formalism

Crystalline concepts

The increasing sophistication constructs mathematical concepts with built-in relationships within and between concepts in all three worlds. In the embodied world of Euclidean geometry, figures at the highest level are *platonic concepts* that have a built in structure of relationships revealed first by recognition and construction and then by definition and Euclidean proof. In the operational world of symbolism, the symbols operate dually as process and concept, which Gray & Tall (1994) defined to be *procepts*. These have symbols that operate flexibly as both process (such as $3+4$ as addition) and concept (such as $3+4$ as sum). Formally defined axiomatic structures also have necessary properties and relationships that follow from the definitions by formal proof. Each of these is an example of a more sophisticated idea that I term a *crystalline concept* (Tall, 2011). This is a concept that has a necessary structure and relationships with other structures as a consequence of its context.

Grasping the nature of crystalline concepts is essential in school mathematics. Differences that are noticed at one level—such as counting on eight after two is harder than counting two after eight—later become irrelevant as $2+8$ and $8+2$ are conceived as two different procedures to compute the same crystalline concept. This crystalline concept can be represented in many equivalent ways, such as $8+2$ or $5+3+2$ or $5+5$, or 2×5, and so on. Likewise 'equivalent fractions' $3/6$ and $4/8$ are seen as different ways of writing the same rational number $1/2$ and 'equivalent expressions' such as $2x+8$ and $2(x+4)$ have different procedures of calculation, yet $f(x)=2x+8$ and $g(x)=2(x+4)$ represent the same function.

Curriculum development currently lacks a terminology to describe equivalent concepts that are seen at a higher level as being the same. The name 'crystalline concept' describes the flexible entity that simplifies mathematical thinking. For instance, equivalent fractions represent the same crystalline concept of rational number.

The shift in meaning when processes are encapsulated as equivalent concepts occurs throughout the curriculum and allows the powerful thinker to operate in a simpler and more flexible way.

At university level, successively larger number systems, such as the natural numbers, integers, rational numbers, real numbers, complex numbers can be constructed where the next larger system is given by some construction using equivalence classes. The resulting construction gives a larger system that contains an isomorphic copy of the previous system. By using the notion of crystalline concept, we can conceive them as successive systems contained one within another: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

This offers simpler human ways of thinking about mathematics by seeing equivalent concepts flexibly as different representations of a single underlying crystalline concept. While some individuals develop more complicated ways of operating mathematically, flexible thinkers conceive of more complex ideas in simple ways.

Met-befores

The ultimate flexibility of thinking about sophisticated concepts requires the learner to build on successive experiences that affect the way the learner interprets new ideas. This has a significant effect on individual learning. A cognitive structure that we have *now* that depends on previous experience is called a *met-before*. In most curricula, successive experiences are designed so that needed concepts have been introduced in earlier learning. For example, one learns addition before multiplication, so that multiplication can be introduced as repeated multiplication. Previous experiences that support new learning are termed *supportive* met-befores. However, experiences that work at one stage, such as 'take away leaves less' or 'multiplication makes bigger' in whole number arithmetic, may become *problematic* in later learning, where taking away a negative numbe gives more and multiplying fractions can give a smaller result.

Supportive met-befores encourage generalization and lead to pleasure in making sense in new contexts. Problematic met-befores cause conflict that can either act as a spur to attack the problem with greater determination, leading to increased pleasure, or as an impediment to learning, leading to disaffection. Failing to make sense can encourage procedural learning, to gain at least the alternative pleasure of passing an exam rather than negative feelings of failure.

As the learner goes through successive experiences, supportive and problematic met-befores play their part in giving positive or negative feelings towards mathematics. This can have serious long-term effects, cumulatively causing a birfucation between those who succeed by seeking to make sense of new ideas, those who take the alternate route of learning by rote to seek pleasure by passing tests, or those who become disaffected.

Met-befores affect the conceptions and attitudes of mathematicians in history and cause differences between the approaches to mathematics in different communities of practice. For instance, the three broad categories of intuitionism, logicism and formalism at the turn of the twentieth century reveal corresponding preferences:

- intuisionism: natural mathematics based on human perception and construction;
- logicism: formal logic unrelated to natural intuition;
- formalism: formal set-theoretic mathematics acknowledged by Hilbert as inspired by natural intuitive experiences.

Grasping two levels of sophistication

The introduction of both crystalline concepts and met-befores into a single theoretical framework involves two distinct levels of development. Crystalline concepts require higher-level appreciation of mathematical structure while met-befores require a deeper understanding of the cognitive development of the individual (Figure 5).

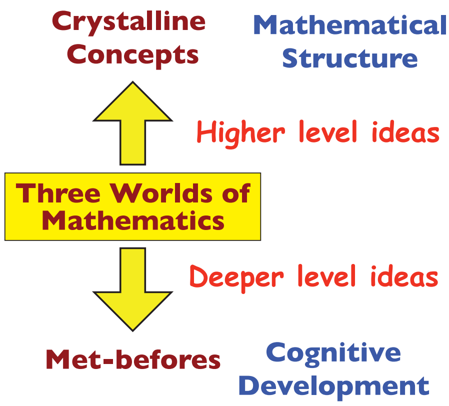

Figure 5: Higher and deeper levels of ideas in the framework

Individuals in various communities of practice will find such a theory makes different demands on their familiar expertise. The notion of 'met-before' applies not only to learners, *it applies to all of us*, including experts in different communities of practice. Those with experience in pure mathematics may have a grasp of structure theorems and higher-level relationships between formally defined concepts, but they may not yet have a grasp of the subtle met-befores affecting learners at various stages. Applied mathematicians may have expertise in embodying problems in symbolic ways to solve them and predict consequences but may have less experience of axiomatic formal mathematics or of the learning of young children. Teachers at various levels may have expertise appropriate for their teaching, but not necessarily of much earlier or later developments in the curriculum.

In the light of a novel theory, some experts may see the ideas as opening up new avenues of development, while others, who remain attached more firmly to their own area of expertise, may find the novel ideas problematic.

The theoretical framework presented in the book '*How Humans Learn to Think Mathematically'* (Tall, 2013), has been described as follows:

> Issues of mathematical thinking and learning can seem like the proverbial elephant vis-a-vis blind men – a huge topic about which varied perspectives offer partial views, but little sense of the whole.
>
> In contrast, this volume offers readers a sense of the "big picture" – a synthetic and holistic view of the development of mathematical understanding.
>
> Alan Schoenfeld, University of Berkeley, USA

On the other hand, a reviewer of the idea of crystalline concept claimed:

> [The notion of crystalline concept is] driven by an idea of universalizing truth …written within the well-known rhetoric of formalizing epistemologies that recent works in anthropology, sociocultural approaches, and ethnomathematics have proven as partial and biased. From a Constructivist

A review of the book by someone who was committed to the notion of metaphor stated:

> I wish I hadn't accepted the invitation to review this manuscript. … The term … met-before [is] not cleanly defined. Review (from a Philosopher?)

A historian opposed to the use of cognitive psychology to understand historical development rejected the cognitive view because it did not fit historical methodology.

A cognitive scientist, whose methodology involved carefully prepared experiments to seek statistical significance, criticised the use of case studies in the book to tell stories that are designed to evoke meaningful classroom experiences for teachers.

On the other hand, an appreciation of the book states:

> Here he offers the scholar's dream: a unified theory of mathematical development that accounts for positive and negative experiences at all levels, from infants to adults. Readers will find themselves drawn into his three worlds of mathematical experience. There they will see how compression of multiple experiences takes place which can lead to contact with the crystalline concepts that populate the world of mathematics. John Mason, Open University, UK.

The theory presented here has the hallmarks of a change in paradigm that challenges fondly held theories in a range of different areas. However, the proof of a theory lies in its eventual value in making sense of diverse ideas, by blending together essential element to produce a framework that can *explain* and *predict* phenomena. The evidence that will be presented using the theory of met-before will show that difficulties occurring at one stage of development may occur because an experience that was supportive in earlier learning becomes problematic in a new context. This may occur cummulatively over several stages. Research that focuses only on the problems at a specific level may be looking in the wrong place to solve the problem. Differing communities of practice may also look at aspects that do not tell the full story. It is like the joke about the man who lost something in a dark street and looked for his keys under a lamp post, where there was light rather than in the dark where they were lost.

Figure 6: Searching for a solution in the wrong place

No single framework can encompass the full picture. The framework offered here focuses on how the individual uses human attributes to build sophisticated crystalline concepts and how the long-term development is affected by previous experience that may be supportive or problematic. It has implications in education, in historical development, and in technological innovation and may be blended with other frameworks to produce emergent insights that are not evident in the original theories (as formulated by Fauconnier & Turner, 2002).

The evolution of theory and practice depends on individuals and communities of practice not only seeking stability that works for them in their current context, but also to keep an open mind about possibilities that may seem problematic at first but may later lead to new and more powerful ways of thinking. However, the simultaneous need to work with higher-level ideas of crystalline concepts and deeper cognitive issues arising through met-befores presents a challenge to those who are not familiar with both.

The need to cope with successive levels of sophistication was formulated by Van Hiele (1986) referring to the conceptual development of geometry. Renaming the four levels encountered in school geometry as *recognition*, *description*, *definition* and *deduction* gives a development that applies not only to geometry but occurs also in *all* long-term development of mathematical thinking (Tall, 2013). For example, it occurs in the development from practical mathematics to theoretical mathematics.

practical mathematics (simultaneous properties)		theoretical mathematics (deduction of properties)	
recognition	description	definition	deduction

Figure 7: Van Hiele levels in practical and theoretical mathematics

Over his lifetime, Van Hiele varied the description and number of levels ending up with three (corresponding to recognition, descripion and theoretical in figure 7). I understand that Freudenthal in his later years declared that at any one time there were only *two* levels of concern: the one that you are in at the moment and the next level that you are working towards. This perceptive remark suggests that attempting simultaneously to move up a level to crystalline concepts and dig deeper to a more fundamental level of met-befores places a strain on individuals who are new to both developments. It is therefore important for the reader to seek to gain a sense of the two levels even though details in one or the other may initially be less familiar.

The remainder of this paper will consider the framework in the contexts of history, technology and mathematics education in general to outline the ideas that are considered in greater detail in the references given.

History

The relationship between the development of concepts in history and in the cognitive development of the individual is complicated by the fact that history tells a story of development of exceptional adults at the forefront of mathematics in successive generations, while education concerns the growth of individuals representing the full spectrum of the population as they develop from childhood to maturity. Attempts have been made to relate the historical evolution of concepts to that of the individual. For example, ‘Haeckel’s law’ states that the development of the individual recapitulates the evolution of the species, or 'ontogeny recapitulates phylogeny'. Haeckel (1868) supported his theory with pictures of embryos made on woodcuts, but in his first edition he used the same cut for three different species, which some saw as falsifying his data. These were redrawn in later editions, but the damage to his reputation was already done.

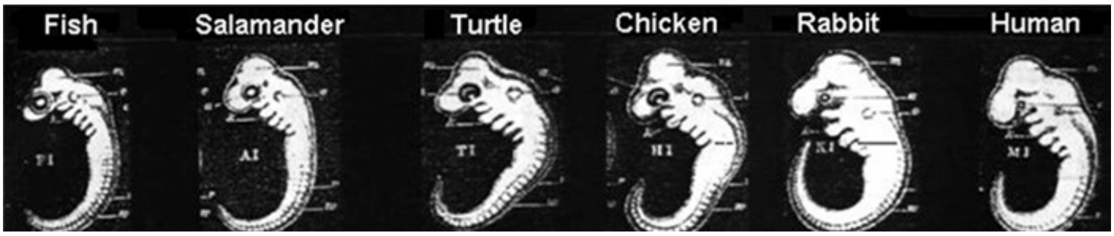

Figure 7: Haeckel's drawings of embryos

More modern photos reveal clearer differences between early embryos (Richardson et al, 1998). However, these include the yolk sac in the fish, salamander and human that complicate the issue.

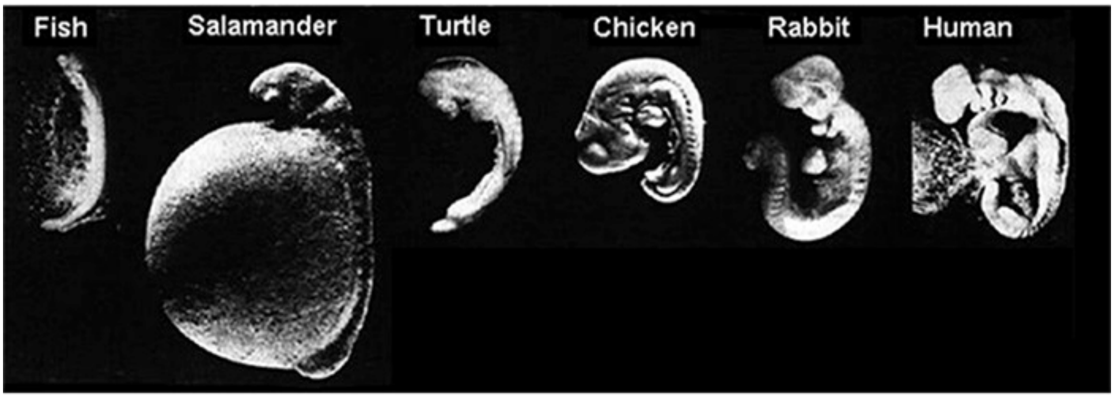

Figure 8: Photographs of actual embryos by Michael Richardson (1998)

Haeckel's theory continues to feature in modern textbooks, while the claimed deceptions in the initial publication continue to be cited by creationists in an attempt to discredit evolutionary theory and others who regard the original theory as a deception.

With such disputes over evolution and individual development, it seems foolhardy to develop a single framework that includes the development of mathematical thinking in history. However, the evolutionary argument is essentially irrelevant as the major developments of mathematical ideas in the last ten thousand years have occurred in a short period of evolutionary time in which genetic evolution is small compared with the way that corporate knowledge has developed over successive generations. There is no reason to consider Plato or Archimedes to be significantly different in evolutionary terms from modern mathematicians. The difference lies in the corporate knowledge available in the community and how it grows from the experiences of those involved.

Chapter 9 of *How Humans Learn to Think Mathematically* (Tall, 2013) performs an analysis of historical development growing from a natural blending of embodiment and operational symbolism. The two main forms of mathematics in geometry and arithmetic followed different paths for centuries, with Greek geometry having a ruler without a unit and Greek number theory developing a sophisticated theory of prime factorization and the infinity of primes. The schism continued over the ensuing centuries as measurement involved quantities such as lengths whose product is an area while the product of numbers is a number.

When Descartes (1637) blended together geometry and number with the simple expedient of arbitrarily choosing a unit length, he was able to use proportions to multiply lengths as numbers and to represent symbolic relations as graphs in the Cartesian plane.

In the early nineteenth century, Cauchy (1821) built his approach to the calculus on Greek geometry where points are marked on lines combined with Cartesian coordinates to assign points numerical values that could be either fixed or variable. This allowed him to consider null-sequences as arbitrarily small variable quantities that could be imagined as infinitesially small. He defined the value of $f(x+\alpha)$ where α is an infinitesimal given by a null-sequence (a_n) to be the sequence $(f(x+a_n))$. He could then calculate functions in an extended system that incorporated both fixed quantities and infinitesimals. The problematic aspect that arose was how to represent these infinitesimal points on a number line.

The nineteenth century development of the properties of numbers evolved into a formal definition of the real numbers as a complete ordered field. Now the real numbers are elements of a set that make up the line rather than points that lie *on* the line. A complete ordered field cannot include a positive infinitesimal quantity x that is smaller than any positive real number because $\frac{1}{2}x$ is smaller still. Infinitesimals were rejected from epsilon-delta analysis.

However, the devil is in the detail. Formal mathematics allows the study of any consistent axiomatic structure formulated as quantified set-theoretic axioms. It is simple to prove a structure theorem that *any* ordered field K that extends the real numbers *must* contain infinitesimals. (See Tall, 2013, chapter 13.) More precisely, any finite element x in K (meaning $a < x < b$ for $a, b \in \mathbb{R}$) is either a real number or uniquely of the form $a+\varepsilon$ where a is called the *standard part* and ε is a (positive or negative) infinitesimal. A map μ that takes x to the standard part of $(x-a)/\varepsilon$ maps the elements of the form $x+k\varepsilon$ where k is finite (called the *field of view*) onto the whole real line. If we denote the image of an element x in the field of view also by the same name x (which is precisely what we do when we mark a location on a map), then we see infinitesimal detail on the extended number line K magnified to become visible. (Figure 9.)

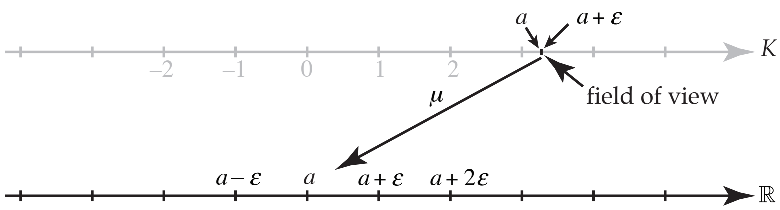

Figure 9: Magnifying infinitesimal detail to see a real picture

The case of Cauchy’s thinking interpreted in cognitive terms is detailed in Katz and Tall (2013). This paper was three years in review as one reviewer recognised its insights and accepted the paper but a historian rejected it twice, claiming that it did not make sense.

A broad spectrum of differing opinions continues to persist in the various communities of practice today. Some see the calculus in natural terms that builds on the perceptual notion of arbitrarily small quantities, some see mathematical analysis based on set theory, others prefer non-standard analysis that requires sophisticated logical machinery to underpin a view that is in the spirit of earlier pioneers working with infinitesimals.

Technology and Personal Development

The new technologies incorporate supportive ways of using dynamic visualisation and operational symbolism now being supported by enactive interfaces that offer new forms of embodied control. These extend the original vision of Alan Kay (1972) whose original design for a computer at Xerox Parc was based on Bruner's three modes of enactive, iconic and symbolic communication, leading to the use of a mouse as an enactive interface to select and control icons and a keyboard to input text and symbols. Now the blending of embodiment, symbolism and language moves to an entirely new level with direct manipulation of data on a tablet using our fingers. At the same time, the enormous power of computation, dynamic interaction and access to the Internet has changed, and continues to change, the way that the world operates. However, while technology changes rapidly, the neural structure of the human brain and the fundamental basis of human perception and action remains stable. As a consequence, the three-world framework of cognitive development continues to have long-term relevance as technologies evolve.

My earlier development of *Graphic Calculus* (Tall, 1986), based on the local straightness of differentiable functions (meaning that the graph of a differentiable function looks like a straight line under high magnification) offers a foundation for the natural development of the calculus from dynamic embodiment to operational symbolism. The later development of mathematical analysis studied by Marcia Pinto (1998) distinguished between students who 'give meaning' to the formal definition of limit by building on their existing concept imagery and those who 'extract meaning' by working formally with the quantified concept definition. This can now be related to the change in paradigm in the historical development from 'natural' mathematics built on human perception and operation to 'formal' mathematics built on set-theoretic definition and deduction with success and failure to grasp the ideas relating to supportive and problematic met-befores. The thesis of Victor Giraldo (2004), focusing on the conflicting meanings that arise in analysis can also be seen through the lens of supportive and problematic met-befores.

We now have available a broad view of calculus and analysis that offers a natural introduction to the calculus using dynamic embodiment and operational symbolism leading either to a pragmatic approach in applications using arbitrarily small quantities, or a formal approach in standard analysis or a logical approach in non-standard analysis.

Using the Genealogy Project on the web, it is possible to trace one's mathematical ancestry through doctoral supervisors and earlier mentors back in history. My own genealogy in mathematics (and also the genealogy of the students who studied for their doctorates with me) goes back through my supervisor, Sir Michael Atiyah, winner of the Fields Medal and the Abel prize, through 14 generations to Sir Isaac Newton and his inspiration, Sir Isaac Barrow, who in turn also inspired Leibniz:

1. Michael Atiyah, Cambridge, PhD 1955.
2. William Hodge (1903–1975), Edinburgh, BA 1923.
3. Edmund Whittaker (1873–1956), UCambridge, MA 1895.
4. Andrew Forsyth (1858–1942), PhD 1881.
5. Arthur Cayley (1821–1895), Oxford, PhD 1864.

6. William Hopkins (1793–1866), Cambridge, MA 1830, Mathematician and Geologist.
7. Adam Sedgwick (1785–1873), Cambridge, MA 1811, Geologist.
8. Thomas Jones (1756–1807), Cambridge, MA 1782, Mathematics Tutor.
9. Thomas Postlethwaite (1731–1798), Cambridge, BA 1753, Mathematician, Master of Trinity.
10. Stephen Whisson, Cambridge, MA 1742 BD 1761.
11. Walter Taylor, Cambridge, MA 1723, Mathematician and Regius Professor of Greek.
12. Robert Smith (1689-1768), Cambridge, MA 1715, Mathematician and Music Theorist.
13. Roger Cotes (1682-1716), Cambridge, MA 1706.
14. **Isaac Newton** (1642-1727), Cambridge, MA 1668.
15. **Isaac Barrow** (1630-1677), Cambridge, MA 1652 **(who also inspired Leibniz)**.
16. Vincenzo Viviani (1622-1703), Pisa 1642.
17. Galileo Galilei (1564-1642) Pisa, 1585.
18. Ostilio Ricci (1540-1603), Brescia. Galileo attended his lectures on Euclid and became his student.
19. Nicolo Fontana Tartaglia (1499-1500), Brescia, who mentored Ricci.

After 350 years of dispute over the meaning of limits and derivative, the vision of local straightness offers a resolution in terms of the difference between the limitations of human perception and action and the power of human imagination and formal theory building (Tall, 2013, chapter 11). What we see in our practical perception of the world and theorize in terms of thought experiments visualizing arbitrarily small quantities, we can formalize in terms of an extension of a complete ordered field and prove a structure theorem that allows us to *see* infinitesimals on an extended number line. This blends together radically different viewpoints within a single crystalline structure. Differing communities of practice accept or reject various aspects of the whole picture depending on their beliefs based on differing met-befores. These include natural approaches to the calculus using infinitesimals as arbitrarily small quantities, and formal approaches to analysis using standard or non-standard analysis.

As new forms of technology continue to be invented, they offer new ways of conceptualizing mathematics. Some lead to innovative ways of making sense of mathematics, while others use old ways of working that may maintain the problematic aspects of learning already present.

For example, there is a plethora of apps on phones and tablets that encourage students to practice tests with the main objective being to develop the techniques to pass specific examinations. Of course this is valuable for an individual who needs a certificate of competency to be admitted to new paths of development, but does such procedural learning encourage more powerful meanings and relationships? For example, an Algebra app on the iPad enables the user to move terms in a linear equation around; if two terms such as $2x$ and $3x$ are moved together, they combine to give $5x$, and if a term is passed to the other side of the equation, it changes sign. But what does this *mean*? As we will see shortly, conceptualizing algebra in this way is problematic.

Mathematics Education

In the last decade, the framework of three worlds and the effects of supportive and problematic met-befores have been applied throughout the mathematics curriculum. In early arithmetic, the notion of flexible use of symbolism as process and concept has revealed new insights, identifying the 'proceptual divide' in which some children become entrenched in practising procedures with an impoverished view of arithmetic that lacks the flexibility that others develop to make arithmetic so much more simple.

It is not a simple matter of procedural competence or conceptual understanding. *Both* are essential for long-term flexibiity of mathematical thinking. The framework of embodiment and symbolism highlights the blending of these two aspects. Arithmetic begins with embodied operations on physical objects. In this activity, the focus of attention can be on the objects, on the operations, or on a combination of the two. Focusing on the objects as operations are performed allows the learner to *see* subtle relationships. For example, counting a collection of 8 and 2 objects, a child who is developing the notion of 'conservation of number' can see that the total is the same. Other relationships may also be seen, such as the flexible idea that $8+2$ is the same as $7+3$ by shifting one object, or that 2 lots of 3 is the same as 3 lots of 2. Throughout the long development of mathematical ideas, a grasp of the crystalline structure of procepts offers greater flexibility and power. However, when focusing on the operation of counting itself, then counting 8 after 2 is different from counting on 2 after 8 and it may not initially be apparent that they give the same result.

A distinction is made in Tall (2013, chapter 7), between *embodied compression*, which focuses on the conceptual embodiment of operations on objects, and *operational compression*, which focuses on practising the procedures symbolically. Embodied compression allows the child to see flexible relationships while operational compression may not. However, embodied compression sees the properties in a particular context that may become problematic. For example, in a real-life situation, three ducks with two legs is different from two ducks with three legs. More generally, 2 lots of 3 ($3+3$) is different from 3 lots of 2 ($2+2+2$). Embodied compression therefore has two sides to it. On the one hand it enables the learner to see flexible relationships, but on the other hand it may link to specific contexts that have aspects that become problematic at a later stage. This tension contributes to the longer-term proceptual divide as embodied learning requires transformations of meaning as the contexts change and many children find this problematic.

The compression of knowledge from an operation as a procedure to a flexible procept makes mathematics simpler at every stage. Yet the distinction between numbers as mental objects and operations on numbers is maintained in most modern curricula. Carefuly analyzing the role of symbols in arithmetic and algebra, we find that, in a range of contexts, symbols *must* play a dual role as object or as operation.

This arises in handling fractions where the sum of two fractions can be performed by converting them to equivalent forms with a common denominator and then added *as objects*, but the product of two fractions requires one to be *an operation* applied to another, so that $\frac{3}{4}\times\frac{2}{5}$ is the operation to give $\frac{3}{4}$ 'of' $\frac{2}{5}$, where now 'of' is said to mean 'multiply'. The same phenomenon occurs again in adding and multiplying signed numbers where $3+(^{-}2)$, $3-(^{+}2)$ are different procedures but the same crystalline concept, simply seen as $3-2$, and multiplication becomes even more complicated with $2\times(^{-}3)$ seen as adding two lots of $^{-}3$ (or 2 debts of 3) and $(^{-}3)\times 2$ seen as taking away 3 lots of 2, or removing 3 credits of 2. These increasingly complicated meanings are liable to cause new levels of difficulty at successive stages.

Over time, as children encounter new contexts, problematic met-befores that work at one level but not at the next may impede progress and encourage the learning of procedures as routines to solve standard problems. Over the longer-term, successive changes in context may proliferate these difficulties. While some learners handle the ideas flexibly and gain pleasure in seeing ideas generalize in new contexts, others turn to procedural learning of ideas that are less well connected and are more likely to break down.

The research of Rosana de Lima Nogueira (Lima, 2007; Lima & Tall, 2008; Tall, Lima & Healy, 2013) reveals the solution of linear and quadratic equations are affected by long-term problematic met-befores that occur in the use of embodiment, symbolism and more general reasoning.

A symbolic approach to an equation of the form 'expression = number' can be undone as a process (for instance, one can solve $3x-2=10$ by undoing the subtraction of 2 to get $3x=12$, revealing the solution $x=4$). An equation of the form 'expression = expression' cannot be solved by 'undoing', leading to the 'didactic cut' (Filloy & Rojano, 1989). The representation of an equation as a balance can solve an equation such as '$2x+4=3x+2$' (which is difficult in the symbolic didactic cut) but is less evident with '$3x-2=10$' (which is easy with the didactic cut). The more general idea of 'doing the same thing to both sides' proves to have a level of generality that is not grasped by many students who tend to use 'procedural embodiment', which means moving the symbols around and incorporating extra actions such as 'change sides, change signs'. The problem gets worse with quadratic equations where virtually all the students in three separate classes were unable to respond to the question

> To solve the equation $(x-3)(x-2)=0$ for real numbers, John answered in a single line that:
>
> '$x=3$ or $x=2$.'
>
> Is his answer correct? Analyse and comment on John's answer.

Most students in the minority who offered a response attempted to solve the equation using the formula that they had been taught and the majority of these could not do the necessary manipulation to change $(x-3)(x-2)=0$ to the form $ax^2+bx+c=0$ (Tall, Lima & Healy, 2013). At the time of writing, this paper is still in review after more than two years, partly because it uses a theoretical framework to interpret existing data rather than a standard methodology to design an experiment to test a hypothesis.

The significance of this analysis is that it brings three separate sources of difficulty into a single overall framework that explains and predicts phenomena observed in the literature. It also suggests that the problem of teaching algebra in the current curriculum has its origins in experiences in learning over many years, with procedural teaching and learning of arithmetic and algebra making it progressively more difficult to teach students a subject at a later stage when they have already been damaged by previous experiences.

This has a serious consequence for the whole of mathematics research. Studies often focus on a specific teaching experiment, seeking to find new ways of teaching that improves the learning in a given context. The theoretical framework given here suggests that the problem may lie not just in the context under consideration, but may relate back to earlier experiences that may have their origins in several contexts met years before.

It is no longer sufficient just to focus on the learning at a single stage, detailing the 'misconceptions' that learners appear to have. *It is now essential to balance the need for a grasp of the flexible crystalline concepts that make mathematical thinking simpler while identifying the supportive and problematic aspects that operate in learning*. This involves a new approach to mathematics education that takes account of the complementary role of supportive met-befores that encourage generalization and problematic met-befores that impede long-term learning.

There is already ample evidence of successive build-up of problematic aspects of learning mathematics in school that can cause long-term disaffection. In addition to studies in arithmetic, algebra, calculus, mathematical analysis, my current PhD student, Kin Eng Chin has identified the changing meanings of trigonometry as it builds from the ratio and proportion of triangle trigonometry (with unsigned lengths and angles less than 90° in a right angle triangle), to circle trigonometry (with signed lengths and visual properties of periodic symmetrical behaviour) and on to analytic trigonometry (with power series and complex numbers that bring together disparate ideas in trigonometry, exponential and logarithmic functions). (Chin & Tall, 2011.)

Mercedes McGowen has traced the longer-term changing meanings of the 'minus sign' through the curriculum, from the operation of 'take away' or 'subtraction' to the concept of 'negative number' that can be represented as an object on the number line, to the concept of 'additive inverse' that is a unitary operation, mapping x to $-x$. In the latter case, x can be positive, negative or zero, or even be in a system that is not ordered, such as the complex numbers or the integers remainder n. (McGowen & Tall, 2013.)

However, this research focused not just on problematic met-befores; it complemented the study of problematic aspects with the use of the function machine as a supportive strategy to reflect on the various meanings of the minus sign and its use in algebra. The function machine may be considered as an input-output box that can be imagined as a unary process (with one input x to output $-x$) or a binary process (with two inputs, x and y to output $x-y$) and also as a mental object: a box, where two boxes can be placed end-to-end to combine two functions. The teaching involved reflections on the learning by students and by teachers, revealing a broad benefit for both, although for some students, the deep embedding of problematic aspects from earlier years proved too difficult to dislodge.

This leads us on to a seek a more productive *connectionist* approach to learning in which the teacher acts as mentor in well-designed contexts to encourage learners to share the construction of new ideas in ways that make sense to them. For example, this occurs in Lesson Study where I have had the privilege of working with Masami Isoda and his APEC study group. (See, for example, Tall, 2008.) Lesson Study is particularly good at encouraging children to build their own ideas in a well-organised curriculum. It already contains aspects of embodiment and symbolic reasoning in problem solving, encouraging children to make flexible links and to form their own conclusions about alternative approaches to ask if they are understandable, efficient and accurate. However, I believe that it can benefit from blending with the ideas of embodiment, symbolism and development of reasoning, taking into account the emotional affects of supportive and problematic met-befores.

As I reflected on the bigger picture that became clearer as the framework became more focused over the last ten years, I initially saw competing theories proposed by differing communities of practice. Then I began to realise how each theory has value in its own context and that, in addition, theories could be blended together to give new insights that were not available in each individual theory alone. It is not a matter of *competing* different theories to see which one is best. It is essential to reflect deeply on the essential aspects in differing contexts and to seek to blend fundamental ideas together to evolve theory and practice that function well in various contexts in the future.

The theory presented here is already a blend of quite different theoretical frameworks: the mathematical theory comes from my doctoral supervisor in Mathematics, Michael Atiyah, who taught me the value of seeking the perfection of geometric ideas and their relationship with symbolic thinking and formal proof; the psychological theory of instrumental and relational learning and the related emotional effects from my doctoral supervisor in mathematics education, Richard Skemp.

Richard, often used to say 'there is nothing so practical as a good theory.' The test of a theory is whether it works in practice. The theoretical framework offered here has possibilities not in competing with other theories, but in blending with other theories to gain a fuller picture of the enterprise of mathematical thinking. Its purpose is offer a picture of the cognitive development of mathematical thinking in the individual and in history that supports learners of every kind to overcome problematic aspects that stand in the way of the fuller appreciation of the crystalline concepts of mathematics. Because it focuses on the essential qualities of human thinking that take advantage of any given context, it has applications in the use of new technology and in the evolution of mathematical thinking into the future.

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