FLEXIBLE THINKING & MET-BEFORES: IMPACT ON LEARNING MATHEMATICS

Mercedes A. McGowen
William Rainey Harper College
Palatine, USA
mercmcgowen@sbcglobal.net

David O. Tall
University of Warwick
Coventry, U.K.
david.tall@warwick.ac.uk

Responses to questions on different uses of the minus sign—as an operation on numbers, the sign of a number, and the additive inverse of an algebraic symbol—reveal the fragility of knowledge and lack of flexible thinking in algebra students in two year colleges and universities that arise from experiences that the students have met before. The analysis of transcripts of interviews uncovers problematic aspects that arise as a consequence of mathematical meanings that worked satisfactorily in earlier learning but become problematic in new contexts, causing confusion and disaffection.

Key Words: Flexible thinking, met-befores, mathematical symbols, minus sign, problematic aspects of signed numbers and algebra, assessment in mathematics.

Introduction

What makes reconstruction of one's existing knowledge, such as the minus symbol, so difficult for so many students? The difficulty facing instructors in undergraduate pre-college courses involves clarifying the reasons for the students' previous lack of success and identifying what precisely is lacking in each individual student's development. Ausubel, Novak and Hanesian (1968) wrote: "the most important single factor influencing learning is what the learner knows. Ascertain this and teach accordingly." New experiences that build coherently on prior experiences are much better remembered and what does not fit into prior experience is either not learned or learned temporarily and easily forgotten. When existing knowledge is not appropriate in a new situation, the learner needs to adapt their approach to cope with the new knowledge. We identify this need for adaptation as a major factor in causing a range of difficulties for students learning mathematics, in particular in the interpretation of mathematical notation, which, contrary to received opinion, often involves subtle changes of mathematical meaning.

Undergraduate students in introductory, intermediate and college algebra courses have not learned to distinguish the subtle differences symbols play in the context of various mathematical expressions. Many of them could be characterized as victims of "*the proceptual divide*" (Gray & Tall, 1994) in

Mercedes McGowen & David Tall (2013). Flexible Thinking and Met-befores: Impact on learning mathematics, With Particular Reference to the Minus sign. *Journal of Mathematical Behavior* 32, 527–537.

which they resort to the "more primitive method of routinizing sequences of activities–rote learning of procedural knowledge" (Gray & Tall, 1994). In the long run, procedural knowledge proves to be inflexible and more difficult to use in solving problems.

This is an issue that is paid little attention by a majority of mathematics instructors, despite the fact that the ability to think flexibly, develop conceptual links between and among related concepts, curtail reasoning, generalize, and modify improper stereotyped learning strategies are all components of the structure of mathematical abilities, essential for success in learning mathematics (Krutetskii, 1969).

We begin by reporting the results of studies that investigated the extent to which undergraduates enrolled in remedial courses (introductory algebra and intermediate algebra) and college algebra demonstrate the ability to think flexibly. Following this is a description of the various changes in meaning of the minus symbol and of the notion of met-before. A brief overview of the duality and ambiguity of mathematical notation is then given. The problematic met-before, the minus symbol, is shown to be an underlying cause contributing to student difficulties dealing with ambiguous notation throughout the pre-college algebra curriculum, limiting development of more flexible thinking. Analysis of particular difficulties experienced by students as they attempt to modify their initial arithmetic interpretations of the minus symbol (subtraction and negative number) to more appropriate algebraic interpretations of ambiguous mathematical notation are reported and examples of other met-befores that affect learning negatively are identified.

Developing conceptual links between and among related concepts

A questionnaire dealing with linear equations, intercepts and slope was given to university students enrolled in sections of an introductory algebra course at the beginning of the semester. Responses examined singly and in combination revealed noticeable differences in the percentage of correct responses on combinations of questions dealing with related concepts and offer evidence of students' failure to utilize knowledge and skills learned in one context in a different situation. Asked to explain the difference in meaning of the expressions $2n$ and n^2, 43 of 120 undergraduates (36%) gave a correct response—but when asked to explain the difference in meaning of the expressions n^2 and 2^n, only 8 students (7%) correctly responded (McGowen, 2007). Some students enrolled in math-intensive courses at a nearby two-year college, given the same questionnaire, also interpreted the notation incorrectly. A pre-calculus student claimed that "There is no difference; you get the same answer" and a Calculus III student responded: "n^2 is n times n; 2^n is the same as $2n$."

In a subsequent study at a large two-year college, introductory, intermediate and college algebra students' responses to questions dealing with other related concepts were also analyzed singly and in combination. When compared, responses of related questions revealed a lack of robust, connected knowledge of elementary algebraic concepts and skills and the absence of meaningful understanding of basic mathematical terminology. The pattern of noticeable differences in the percent of correct response to related questions is an indicator of the failure to think flexibly. A majority of the two-year college students could substitute values for m and b to get the equation of a line in slope-intercept form (question A), but were unable to use that knowledge to select the correct equation of the line given its graph (question B):

(A) The equation of the line with slope –3 and y-intercept (0,5) is:

a. $y = -5x + 3$ b. $y = 5x - 3$ c. $y = -3x + 5$ d. $y = 3x - 5$

(B) Which of the following equations has the given graph? Circle and justify your choice.

a. $6x + 4y = 12$ b. $2x - 3y = 12$ c. $6x - 4y = 12$ d. $3x + 2y = 12$

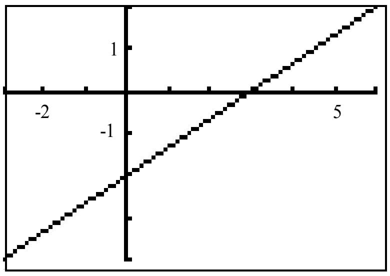

FIGURE 1: Linking graphs and symbolism

Seventy-one percent (65/92) of introductory algebra, 83% (454/554) of intermediate algebra, and 89% (102/114) of college algebra students selected the correct equation, given the slope and y-intercept in question (A), but less than 1% (14/92) of introductory algebra, 22% (12/54) of intermediate algebra and 43% (48/114) of college algebra students selected the correct equation for the graph of the line in question (B). Given a linear equation in slope-intercept form, only 40% (37/92) introductory algebra students were able to identify the vertical-intercept; 41% (38/92) correctly identified the x-intercept of an equation in standard form.

Students revealed a lack of meaningful understanding of basic mathematical terminology such as "solving" versus "evaluating in responses to the following items on a questionnaire:

(A) Finding the input when the output is known is the process of __________.
a. *simplifying* b. *evaluating* c. *factoring* d. *solving*

(B) Finding the output when the input is known is the process of __________.
a. *simplifying* b. *evaluating* c. *factoring* d. *solving*

Seventy percent (50/71) of introductory algebra students identified the process of solving in (A) but only 40% (29/71) identified the process of evaluating in (B). College Algebra students' results were similar: 73% (83/114) identified the process of solving but only 41% (47/114) identified the process of evaluating correctly.

Responses of graduate students majoring in mathematics reveal that they too sometimes have an inadequate understanding of mathematical terminology. A faculty member at a two-year college who had given the questions on solving versus evaluating to her students was invited to pose as a student by a textbook publisher's representative and submit a question to an online tutor. This was a new free service being offered by a textbook publishing firm to students whose school adopted the publisher's textbook. Graduate students working on their PhDs in mathematics were employed as online tutors. The graduate student tutor working that afternoon was asked to explain the difference between solving an equation and evaluating a function or expression. He replied:

> "If a book asks you to evaluate $x^2 - 2x + 1$, what they are asking for is a simplified version of this polynomial which would be $(x - 1)^2$. Solving an equation or expression is actually plugging in a particular value to come up with a solution.
>
> For example:
>
> $f(x) = x^2 - 2x + 1$. Solve for $f(4)$.
>
> $f(4) = 4^2 - 2(4) + 1 = 16 - 8 + 1 = 9$.
>
> Is this helping you feel a little bit better about the difference between the two?"
>
> (McGowen, 2006, p. 25).

Changes in meaning of mathematical concepts throughout the curriculum

In this section we will illustrate a general principle that mathematical concepts change meaning as the curriculum progresses into more general contexts and this changing meaning causes difficulties that lead to disaffection with mathematics. Essentially, a way of working that is perfectly satisfactory in one context no longer works in a more sophisticated context. We begin with a focus on the particular example of the use of the minus sign.

Initially the minus sign is encountered in whole number arithmetic where it is used to evoke the operation of subtraction, or 'take away'. Here the minus sign in 5–3 means 'start with five and take away three'. The result is 2 and it is visibly less than the initial quantity 5. This leads to the conception that 'take away' leaves less. It is even 'a common notion' explicitly listed in the foundations of Euclidean geometry in the form 'the whole is greater than the part.' In whole number arithmetic, numbers are magnitudes that do not have a direction or sign.

The second meaning of the minus sign occurs when negative numbers are encountered. The symbol –3 refers to the negative number –3. Now numbers are of three distinct kinds, positive numbers where +3 is essentially the same as the familiar number 3, negative numbers such as –3 and the single number 0 which is neither positive nor negative. On the (horizontal) number line, 0 is in the middle, positive numbers are on its right and negative numbers are on the left. On the vertical axis, positive numbers are above zero and negative numbers below.

A third meaning occurs in algebra where $-x$ is the additive inverse of x. It is the number which, when added to x gives zero. The meaning is different from that in dealing with a number such as –3, for the latter carries with it the meaning that they symbol is *a negative number*. However, when the variable x is given a particular numerical value, the value of $-x$ need no longer represent a negative number. When evaluated as a number, the result can be negative (if the numerical value of x is positive), negative (if the value of x is negative) or even zero (if the value of x is zero).

There is a fourth meaning which is encountered in systems such as the complex numbers or the integers modulo a whole number, which satisfy the rules for a 'commutative ring'. These include the standard arithmetic of addition and multiplication including the existence of an additive inverse written as $-x$ for every element x in which $x + (-x) = 0$ but may not satisfy the rules for order. (The order properties require that every element must be classified into three exclusive possibilities: 'positive', 'negative' or the zero element where an element x is either 'positive' or 'negative' or zero and the sum and product of two 'positives' are both 'positive'.) In the case of the integers modulo 3, if we write n_3 as the remainder of n modulo 3, then we have $1_3 + 1_3 = 2_3$ and $2_3 + 1_3 = 0_3$ so 2_3 is -1_3 and if the system were given an order, then this term is simultaneously 'positive' and 'negative'. A similar argument can be made to show that the complex numbers cannot be ordered because i is not zero and if it is 'positive', so is the product of i times i and if it is 'negative', then $-i$ must be 'positive' which again shows that –1 is 'positive because it is the product $(-i)(-i) = -1$. Both possibilities will lead to the necessity that both –1 and its square +1 are 'positive' which violates the order axioms.

This fourth meaning involving the algebra of commutative rings, which includes the integers, rationals, reals, complex numbers, integers modulo n, and other general systems is a higher level concept again which is not encountered in school or in elementary college algebra courses, but it does extend the notion that concepts may fundamentally depart from meanings that are perfectly satisfactory in one context but become problematic in a more sophisticated context.

These four changes in meaning also have affects in other areas of the curriculum, for instance, the meaning of the negative sign in the power notation 3^{-2}, or in the notions of inverses not only in addition but in multiplication, where 7 divided by 3 has a remainder when dealing with sharing of whole numbers, but can be a fraction $\frac{7}{3}$ or $2\frac{1}{3}$. There are also other quite different uses of the minus sign in the function notation f^{-1} which impede understanding.

The notion of 'met-before'

McGowen and Tall (2010) argue that the general idea of metaphor, when used to perform an intellectual analysis of how concepts are conceived, does not necessarily give a complete view of how students learn. The notion of *met-before* (Tall, 2004, Nogueira de Lima & Tall, 2008, McGowen & Tall, 2010) was introduced to focus on how new learning is affected by experiences that the learner has met before. A *met-before* is 'a mental structure that we have *now* as a result of experiences we have met before' and applies to *all* current knowledge that arises through previous experience, both positive and negative. What students bring to their learning, both in terms of previous experience that is supportive and previous experience that may be problematic describes what the student thinks now as a consequence of experiences met-before. Inappropriate use of personal met-befores can lead to subtle difficulties in learning for the student.

Met-befores are a *personal* construct of the individual that apply in different ways in different individuals in different contexts. Some fortunate individuals at certain stages of their development have experiences that they interpret in more general ways than others. For instance, one child may realize that addition is independent of order without being told, while another child, counting on fingers, finds a sum like 8+2 much easier than 2+8 because counting on 2 after 8 is much easier than counting on 8 after 2. This can lead to very different developmental paths for individuals but it should not be assumed that the spectrum of success and failure is set in stone. So much of individual development in mathematics depends not only on the innate ability or the hard work of the learner, but also *at every stage* on how the learner uses their current knowledge to make sense of new ideas and how they are mentored by a perceptive support that takes account both of the mathematical meanings and the needs of the learner.

A wider picture of the issues can be found in the book *How Humans Learn to Think Mathematically* (Tall, 2012). In this paper we concentrate on the specific example of the long-term changes in meaning of the minus sign.

Duality and ambiguity of the minus symbol

Ideas that have been evolving in the literature in recent years are steadily becoming more focused to explain why some aspects of mathematics give pleasure to students as they make sense of them and pain as they become disaffected by new ideas that do not make sense (Skemp, 1979). Gray and Tall (1994) suggested that the mathematician's desire for precision and rejection of ambiguity, has led to the failure of realizing the underlying duality and ambiguity of symbolism which gives it such flexibility, particularly in the teaching of mathematics. Symbols such as 3–2 have a dual use as process (take 2 away from 3) and also as concept (the difference between the two numbers which is 1). Gray and Tall hypothesized that the ability to think flexibly in mathematics depends on this dual use of symbolism for both procedure and concept, a duality found throughout mathematics. They defined the blending of process and concept represented by the same notation to be a *procept* as "symbolism that inherently represents the amalgam of process/concept ambiguity" to explain the divergence and qualitatively different kind of mathematical thought evidenced by more able thinkers compared to the less able (Gray and Tall, 1991, 1994).

The symbol –3 is an example of a procept which can be interpreted in several ways, depending upon the context. As indicated in the previous section, the minus sign could be an operation representing subtraction, or part of the symbolism for a negative number, –3. Demarois, McGowen & Whitkanack (1996, p. 169) used the term 'opposite' of a number to speak of its additive inverse as a unary operation on a function in their intermediate algebra textbook. They introduced the third meaning of the minus sign given earlier in this paper as a function inputting a number x and outputting $-x$. If these operations are analyzed using the notion of function then the first three meanings for the minus sign given in the previous section can be interpreted in terms of:

(a) The process of subtraction which is a binary process requiring two inputs where the expression 7–3 has inputs 7 and 3 (which are initially unsigned numbers),

(b) a mathematical object, negative three,

or

(c) a unary function inputting 3 and outputting the value –3.

Students first encounter the minus symbol when they are introduced to the arithmetic binary operation of subtraction. When a number such as –3 is introduced, students encounter the second interpretation of the minus symbol. This leads to difficulties in interpreting the notations for arithmetic operations. For example, sixty-two percent of 160 university students enrolled in an introductory algebra course evaluated $(-3)^2$ as 9 but only 26% correctly evaluated -3^2 as –9. In subsequent studies at two-year colleges, 516 intermediate algebra students were asked the same two questions. Eighty-one percent of the students (418/516) correctly evaluated $(-3)^2$ but only forty-nine percent (251/516) correctly evaluated -3^2. Evaluation of -5^2 included: –25, 25, –10, 10, (–5)(–5) = 25, (–5)(5) = –25, –3, and $1/5^2$. Evaluation of $(-5)^2$ included (–5)(–5) = 25, 25, –25, $5 \times 5 = 25$, –10, 5^{-2}, –3, and $1/10$.

Interpreting mathematical notation involving the ambiguity of the minus symbol when squaring a negative number versus taking the opposite of a number squared requires flexibility to deal with both process-object ambiguity and notational ambiguity involving order of operations as well as the ability to switch one's train of thought. Comparing -3^2 with $(-3)^2$ requires students to deal with both types of ambiguity flexibly. The large discrepancies in correct responses to related questions, 62% versus 26% and 81% versus 49%, reveal the difficulty of interpreting ambiguous notation experienced by students with fragmented knowledge and lacking flexibly to distinguish the subtle differences symbols play in the context of various mathematical expressions.

The initial arithmetic interpretations of the minus symbol (subtraction and negative number) is a problematic met-before for many students when they encounter algebraic interpretations of ambiguous mathematical notation. When numbers are replaced by variables and interpretation of the minus symbol as "additive inverse" is introduced, the student's arithmetic interpretation of the minus symbol when confronting the ambiguity of $-x$ needs to be restructured. This requires flexibility of thought and a realization that context must be considered. Broadly speaking, undergraduate students entering pre-college courses, shown the minus symbol on the blackboard or overhead and asked what comes to mind when they see the symbol, first list "subtraction," followed by "negative number." In fourteen years of research, not one of these students has included "the additive inverse", "the multiplicative inverse " or the more advanced notion of "the inverse of a function" as interpretations of the minus symbol.

Skemp (1987) identified position, as well as size, as components of a symbol system which contribute to students' difficulties. The expression $-2^{1/2}$ requires the mutual assimilation of separate schemas, each of which has a structure of its own. If the relationship between ambiguous symbols and the conceptual structure is such that they are in equilibrium, or in which the conceptual structure is dominant, symbols help us use the power of mathematics. Some students therefore find these ideas natural and easy to use. If, however, the procedural use of symbols dominate the conceptual system, students will become "progressively more insecure in their ability to cope with the increasing number, complexity, and abstractness of the mathematical relations they are expected to learn" (Skemp, 1987, p. 186).

Skemp (1987, p. 188) claimed that "symbols are magnificent servants, but bad masters, because by themselves they don't understand what they are doing." He cautioned that new material needs to be presented in such a way that it can always be assimilated conceptually and defined symbolic understanding to be "the ability to connect mathematical symbolism and notation with relevant mathematical ideas". The definition of a symbol system as "a set of symbols corresponding to a set of concepts, together with relations between the symbols corresponding to relations between the concepts" (Skemp, 1987, p. 177) is similar to that put forth by Backhouse (1978) and by Byers and Herscovics (1977).

Mathematics instructors generally fail to recognize the difficulties students experience in interpreting the minus symbol. Its meaning is so obvious and trivial to them that it does not appear to be a difficulty even worthy of examination. The introduction and growing use of technological tools which seek to implement the mathematician's intuitive understanding of the minus symbol with the computer scientists' traditional programming practices challenge us to rethink our own understandings as well as our instructional practices. Mathematicians, using the traditional power notation, interpret the algebraic expression $y = -x^2$ as $y = -(x^2)$, when given a negative number input. The graphical representation of $y = -x^2$ may be described as "the opposite of the graph of $y = x^2$.

A software developer has the sometimes difficult task of transforming ambiguous mathematical notation into unambiguous programming code when designing routines which are supposed to reflect accepted mathematical practice. The general computer programming convention for the notation -3^2 is that the number includes its sign, thus -3^2 is thought of as meaning $(-3)^2$, i.e., negative three squared, and $y = -x^2$ is interpreted as $y = (-x)^2$. In Tall's original *Graphic Calculus* software, an input string of characters was processed internally so that if a minus sign occurred at the beginning of the expression, such as $y = -3x^2 + 2$ or even as the first element inside a bracket, such as $y = 4(-3x^2 + 2)$ then the string was internally processed to place a zero before the minus sign, so that the expression was interpreted as $y = 0 - 3 * x ^ 2 + 2$ or

$y = 4 + (0 - 3 * x \wedge 2 + 2)$ so that it was interpreted internally in the standard mathematical sense without the user even being aware of the problem. Texas Instruments programmed its graphing calculators to implement the mathematician's intuitive, traditional power notation interpretation in their software. These calculators include separate keys for the binary operation of subtraction [–] and the unary operation of additive inverse [(–)]. Entry of [–][3][^][2] yields the answer –9, but entry of [(–)][3][^][2] results in the positive answer, 9. Inclusion of both the subtraction and additive inverse keys, with their different functionality, places the burden of interpretation on the user, as well as focusing attention on the need to understand the role of language, context and grouping symbols.

This is an example of a situation in which the use of technology requires mathematics instructors to clarify their own understandings and re-examine their assumptions, as they incorporate the use of these technological tools into their courses. Using graphing calculators necessitates explicit acknowledgment and discussion of the ambiguity of the notation and of an awareness of the language used (minus, difference, opposite of, additive inverse), and of activities that might better effect reconstructions of students' arithmetic schemas and the met-befores of associating the minus symbol with the operation of subtraction and negative number.

How do students interpret $-x$? Is their interpretation dependent upon the words used with the symbol(s)? What comes to mind when one hears "minus", "difference", "negative *x",* "the additive inverse of *x*" or "the opposite of *x*"? Does $-x$ denote a process (taking the additive inverse) or an object (negative number)? What are they prepared to notice? Do they see two symbols, "–" and "*x*", or one symbol, "*–x*"*?* How do their prior experiences and existing knowledge of the minus symbol impact their understanding and interpretation of it? Studies of college students' difficulties identified the minus symbol as a problematic met-before that, if not explicitly addressed, continues to cause difficulties for many students as they advance to subsequent courses (McGowen and Tall, 2010). In the book *How Humans Learn to Think Mathematically* (Tall, 2012), this is shown as a major reason for disaffection throughout the whole curriculum as more and more students become disillusioned because they either find new ideas too complicated to cope with, or they find them problematic because they interpret new ideas in old ways that are no longer appropriate. The result is that almost everyone ends up teaching and learning how to *do* mathematics in ways that impede long-term learning.

Students are generally taught that "we don't like to start an algebraic expression with a minus sign," thus when we write $y = mx + c,$ for $m = -1$, we tend to write $y = c - x$, and avoid confronting the ambiguity directly. However, when evaluating a quadratic function such as $y(x) = -x^2 + 1$ given $x = -3$, do students interpret $–x^2$ as subtraction $–(x)^2$ or as squaring the negative $(–x)^2$? Do they see the evaluation of $–x^2 + 1$ as –9+1 = -8 or as 9+1 = 10 ?

What do students know and how do they know it

The initial study designed to provide information about students' ability to interpret function notation and the minus symbol in various contexts, evaluate functions, and translate among representations included a pre-test given to twenty-six two-year college students enrolled in an intermediate algebra course during the first week of the semester (McGowen, 1998). Only six of twenty-six students correctly evaluated both $–3^2$ as –9 and $(–3)^2$ as 9, using the standard conventions. Interview transcripts revealed that most students were unaware of the difference between finding the additive inverse of a number squared, i.e., $-x^2$, interpreted as "finding the opposite of 'x squared'," and squaring a negative number, $(–x)^2$.

Students build up their mental images of a concept in a way that may not always be coherent and consistent and they do not experience cognitive conflict when the context is changed. When asked to square the binomial $(t–2)^2$ students frequently write $t^2– 4$, They generally fail to recognize that the same process of squaring a binomial is invoked when they are given a quadratic function such as $f(x) = x^2 - 3x + 5$, and asked to evaluate $f(t-2)$. They fail to execute the procedure correctly in either task, sometimes writing $t^2 + 4 - 3t + 5$ in the second instance, while writing $t^2 - 4$ in the first instance. Two different, incorrect answers to the same task embedded in different contexts is indicative of a compartmentalization of knowledge and path-dependent logic due to prior experiences. When interviewed, students expressed surprise that they were being asked to square a binomial as part of the process of evaluating a function. They were unaware that they had given two different answers, both incorrect, for squaring a binomial, until, during the interview and examining their work, they examine what they had previously written. The fact that students experience no cognitive conflict when executing procedures suggests that they routinize the procedures, developing mechanical skills, not cognitively-based methods of operation.

The pre-test documented how met-befores from prior experience in arithmetic involving grouping symbols and the minus symbol becomes problematic when evaluating a quadratic function. The failure to recognize that a negative value is being squared is so established and stable that its selection and retrieval is automatic for many students. Asked to evaluate $f(x) = x^2 - 3x + 5$, given $x = –3$, fifteen students wrote $f(-3) = -3^2 - 3(-3) + 5$. Six of those students interpreted $–3^2$ as $(–3)^2 = 9$ and evaluated the function as

$f(-3)=23$. Though none of these students used parentheses to indicate they were squaring a negative number, they all used parentheses when substituting -3 for x in the linear term. Another six students of the fifteen showed the same initial work but evaluated -3^2 as –9, writing –9 + 9 + 5 = 5; two students wrote (–3)(3) – (3)(–3) + 5 = 5 One student interpreted $f(-3)$ as a multiplication and proceeded to divide both sides by –3. Eleven of the twenty-six students used consistent, correct notation and completed the evaluation correctly.

The majority of errors were initially assumed to be the result of 'a lack of understanding about the algebraic order of operations' and 'a failure on the part of students to use grouping symbols consistently'. The iconic representation of a function was introduced as an organizing lens, and using the graphing calculator, students were encouraged to compare the binary process of subtraction with the unary process of finding the opposite (additive inverse). They investigated the role order of operations and grouping symbols play in the processes of squaring a negative number $(-3)^2$ and in finding the additive inverse (opposite) of a number squared, -3^2. Figure 2 illustrates the use of the function machine for these investigations.

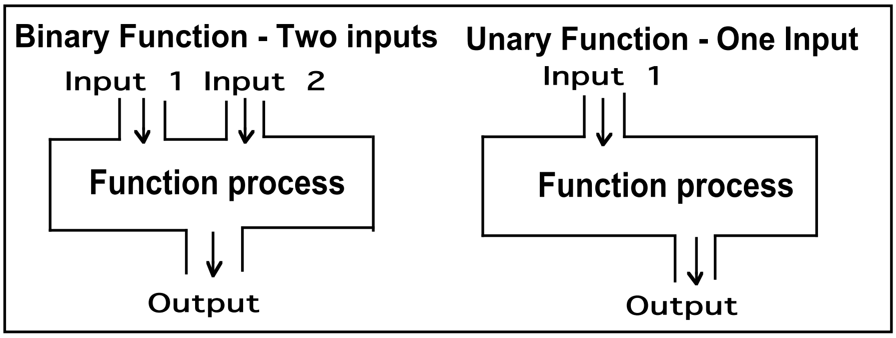

FIGURE 2: Function Machine Representations: Binary & Unary Processes

Since TI calculators are themselves function machines, they automatically supply the missing input when the binary operation of subtraction is selected and only one input is entered. They display what the student enters (input) as well as the result of the computation (output). The iconic representation of a function machine and the graphing calculator provided students with tools for visualization and analysis.

Students were given three tasks: (1) subtract three, (2) find the additive inverse of three; and (3) to enter -3^2. The calculator displays are shown in Figure 3.

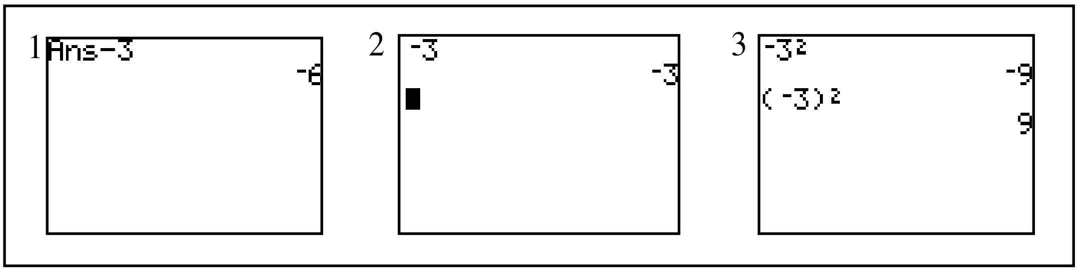

FIGURE 3: TI-83 View Screen of Binary and Unary Operations

As subtraction is a binary operation, Screen 1 automatically supplies the missing first input, displayed as "ans," when the subtraction key is pressed first. "Ans" uses the last result in memory as first input (in this case, –3). The subtraction performed was -3 – 3, with the result, -6. On screen 2, -3 is displayed (upper left) when the "opposite" or additive inverse key is pressed, followed by 3. The result of the unary operation of taking the additive inverse of 3 is -3, displayed on the right. Screen 3 displays two calculations and their results: (a) calculating the opposite of three squared, which is -9 and (b) squaring a negative 3, which is 9.

Class discussion focused explicitly on the input-output process conception of function. The need for consistency in the use of notation and the role of context were also topics of discussion. Students analyzed other arithmetic operations using the iconic input-output function machine representation, characterizing them as either binary or unary functions. Re-conceptualizing arithmetic operations as unary or binary functions provided students with a framework within which to clarify their understanding of the difference between the operations of subtraction and finding the additive inverse of a number. The arithmetic investigations were followed by investigations evaluating symbolic representations such as $f(x)=x^2$ for $f(-3)$ and $f(x)=-x^2$.

As homework, students were asked to submit a written reflection about their investigations. In the reflection they were to complete three sentences: (1) "I used to think … (2) Now I realize …, and 3) I've changed my mind about …". Four students who evaluated -3^2 incorrectly during the classroom investigation, wrote on their reflections that they had made a careless sign error and now understood the problem. One student wrote that she knew her answer was correct (it was actually incorrect) because the other members of her group agreed with her. These students consistently evaluated either the numerical expression or a quadratic function with a negative-valued input incorrectly throughout the remaining twelve weeks of the semester. Written reflections, together with other investigations and interviews, resulted in modifications of other students' existing concept images (McGowen, 1998, pp. 104-107).

David Ausubel, along with Richard Skemp, Robert B. Davis and others, maintained that meaningful learning results when the student consciously and explicitly ties new knowledge to relevant concepts within his/her schema. The most important element of meaningful learning is how new information is integrated into an existing knowledge base. Using criteria proposed by David Clark et al. (1996), learning in which "students are actively involved in integrating, or linking, new concepts and skills into an already existing conceptual framework, not simply accumulating isolated facts and procedures," is characterized by evidence that the student:

- claims to have learned something new;
- can articulate what it is they think they have learned, with some degree of clarity and accuracy;
- can demonstrate formation of links with an existing framework that the student already possesses.

Meaningful learning occurred for several students. Interview transcripts provide insights into their met-befores, as well as evidence of the extent to which they modified their existing knowledge. Students who were successful focused on qualitatively different features of the processes than did those students who modified their prior experiences inappropriately. Typical of the students who were successful is the reflection of Student B:

> I realized that the problem was looking for the opposite of 3^2...but I didn't understand the rationale. When I see the sign (–) it is a change for me to know that it means "the opposite of." I always though it meant a negative number or $-(-x)$, a positive x. The reflection assignment enhanced my understanding of the opposite of a square by looking at it as two functions, and then order of operations would have exponents first, then the opposite of the value. I didn't know what the order of operations was in relation to exponents and oppositing.... I do know this now. Exponentiation takes precedence over oppositing in the absence of grouping symbols.

This articulate response provides evidence that meaningful learning has occurred. The student has assimilated newly-acquired knowledge, re-conceptualizing the two processes of squaring a negative number and taking the opposite of a number squared as functional processes, accompanied by a change from insecurity to confidence—a changed mental state that gives the student a degree of control over the situation not previously had (Skemp, 1987).

Other responses reveal the complexity of interpreting ambiguous notation and the difficulties inherent in trying to re-construct one's existing understanding of the minus symbol as a result of cognitive conflict. Tall and Vinner (1981, p. 152) have noted: "Only when conflicting aspects are evoked simultaneously need there by any actual sense of conflict or confusion"). The confusion two students experienced are documented in written reflections:

I learned that without parentheses you cannot make $-3^2 = 9$. The change of thinking I've had since this assignment is drastic! I began to realize how crucial parentheses are. The parentheses show that there is only one operation being done. Without parentheses, two operations are being taken. Ex: $-3^2 = -9$ means take the opposite and square; $(-3)^2$ [means] just square –3. I find this a bit hard getting used to! [Student M]

Any two negative numbers that are multiplied by each other must result in a positive answer. After discussing the assignment I felt that even though I may not have been able to find the correct answer, I still learned that I have to go about a few different ways to try to find an answer and by discussing with someone else I am able to check my answers...sometimes my old ways of thinking like to butt in and I have a hard time saying no and to keep on trying the problem. [Student A]

How students assimilate new knowledge depends upon their prior experiences and previously-constructed cognitive images, how the new problem is represented, how they retrieve and represent relevant existing knowledge, and what they focus attention on based on the visual cues they pick up from scanning the written symbols. Met-befores, true in a given context, can sometimes lead to cognitive conflict in another context. When asked to evaluate -3^2, some students focused on the fact that the answer must be negative and ignored what it means to square a number. They attempted to resolve the cognitive conflict they had experienced by focusing on getting the correct answer:

Now I know that when you square a –3 it stays negative. -3^2 is always negative. [Student C]

I didn't understand that when you multiply -3^2 that it is $(-3)(3)$ which will give you the answer –9. I always though it was $(-3)(-3)$ regardless of parentheses. Now I realize that was wrong. [Student K]

These responses are examples of the spatial problem of size and position described by Skemp (1987). He attributed students' difficulty to the task of having to deal with two schemas: the symbol system and the structure of mathematical concepts. The students' responses suggest that it is the symbol system which dominates their conceptual structure and mathematics is nothing more than the manipulation of symbols.

Students' efforts to interpret ambiguous notation document the bifurcation that occurs as a consequence of the qualitatively different ways of thinking and constructing knowledge as individual students assemble bits and pieces of knowledge into their existing cognitive collages based on prior experiences. How one's met-befores are modified are not only based on one's prior experiences, but on the initial focus of attention, how what is perceived is classified, and what relevant knowledge is retrieved from memory and used in the present situation. Sfard (1991, p. 17) reminds us that "Algebraic symbols do not speak for themselves. What one actually sees in them depends on the requirements of the problem to which they are applied. Not less important, it depends on what one is *able* to perceive and *prepared* to notice." Skemp (1987, p. pp. 10-11) claims: "We classify every time we recognize an object as one which we have seen before […] once it is classified in a particular way, we are less open to other classifications." Edelman (1992, p. 87) argues that the ability to carry out categorization is embodied in the nervous system and that perceptual categorization is "the selective discrimination of an object or event from other objects or events for adaptive purposes […] that does not occur by classical categorization, but rather by disjunctive sampling of properties."

Analysis of students' comments in the subsequent study supports these claims. Student B focuses on the two processes, comparing and contrasting them, combining the visual cues of parentheses, exponents, and minus symbols into a coherent, appropriate reconstruction of her knowledge and growing awareness of the role of context. Student C disjunctively focuses on the exponent and the squaring process, which, once she is aware that the minus symbol denotes a negative answer in this context, causes conflict. Two students focused attention on the presence/absence of parentheses—M successfully and A, unsuccessfully. K, like C, disjunctively samples multiple cues (squaring indicated by the exponent, the minus symbol indicating a negative number, and the minus symbol indicating the answer should be negative), combining them inappropriately.

Interview transcripts also provided clues about the initial focus of attention. Student D appears to have focused initially on squaring a number and the role of the order of operations. Student L focused on the arrangement of symbols and interpretation of the task, relating prior experiences with new knowledge about the role of parentheses.

> I never thought about the order of operations when I was supposed to square three first then put in the opposite. [Student D]

> I was confused because before whenever a variable was to be substituted for a particular number it was expressed like this: $x = -1$, not $f(-1)$. I used to think that $-3^2 = 9$. Now I realize that the answer is –9. I used to think f times (–1). Now I realize what the problem asks for. I used to think the substitution was correct. Now I realize that the parentheses are missing and my notation is incorrect. [Student L]

Concept maps created in the fourth week of the semester reveal other student conceptions that do not coincide with mathematical practice, including the association of $y(x) = -x^2$ with a graphical representation of a parabola opening upwards and $y(x) = x^2$ with the graphical image of a parabola opening downward as indicated by arrows in the lower left corner of the concept map in Figure 4. Perhaps being 'positive' is associated with having a maximum and being ‘negative’ links to a minimum. We do not know. But something caused the student to respond in this way.

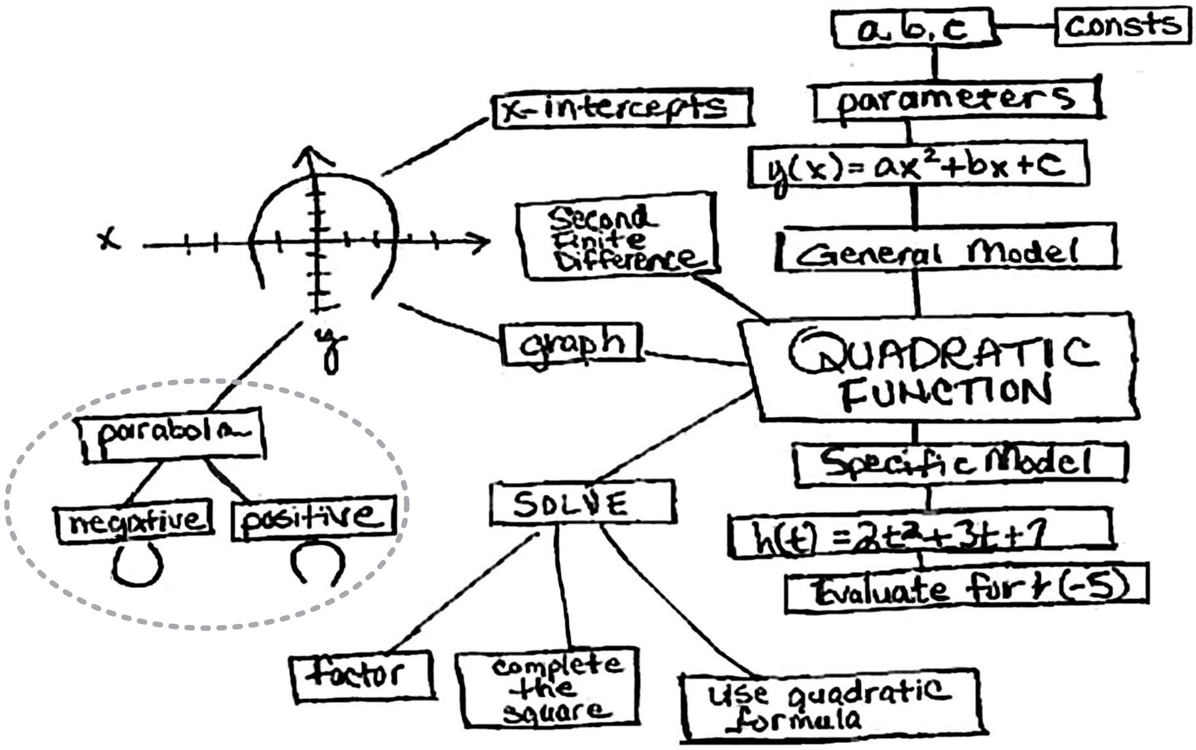

FIGURE 4. Week 4 Concept Map with Inappropriate Connections

Class members encouraged the student to test her beliefs by examining the graphs and input/output table values of the two functions. These investigations demonstrated that her initial drawings were incorrect but the student needed several additional investigations before she modified her existing knowledge. The student’s concept image then remained stable throughout the semester as indicated on her final concept map completed during Week 15. The reconstructed concept image is indicated in the portion of the final concept map outlined in Figure 5.

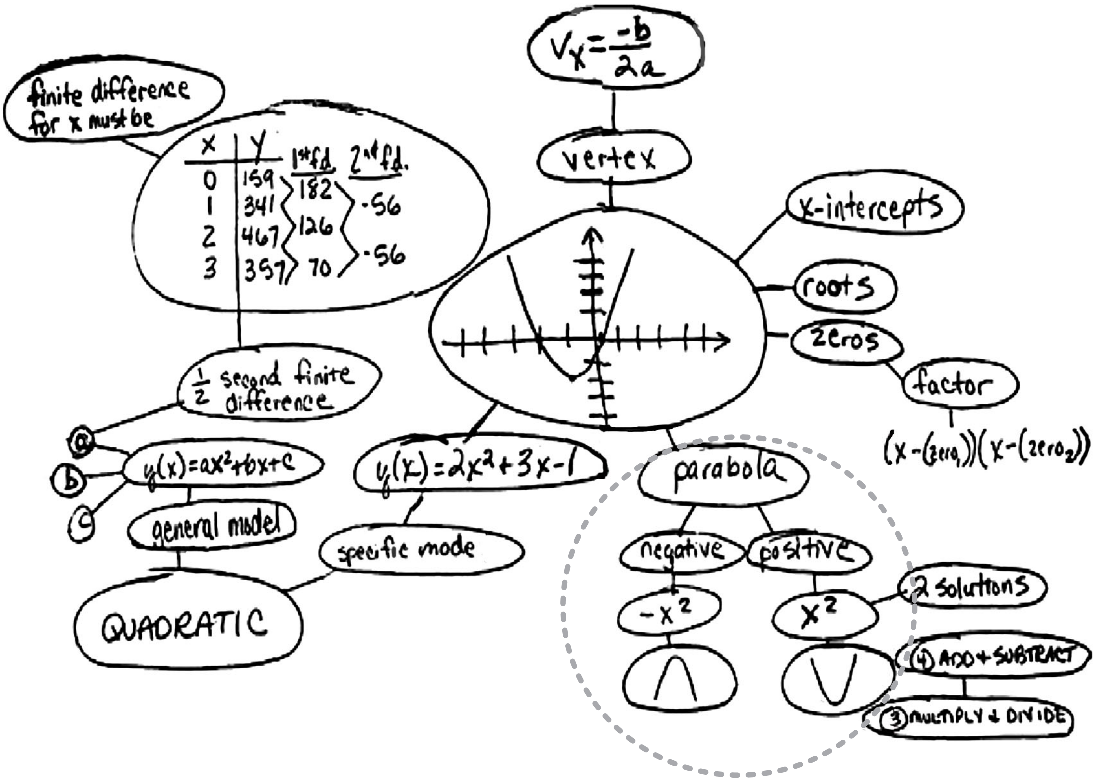

FIGURE 5. Week 15 Concept Map showing Reconstructed Concept Image

Despite reconstructing the graphic representations of x^2 and $-x^2$ appropriately, this student continued to interpret $-x$ as a negative number and $-(-x)$ as a positive value, assuming that the numerical value of x was always positive. Her previous experiences of the minus symbol used to indicate a negative number was deeply embedded into her existing conceptual schema and was not impacted as a result of the graphing investigations. During the final interview, this student when asked the meaning of $-f(x)$, responded "negative output, the answer is negative." and when asked "What first comes to mind when you see $f(-x)$", answered "negative input, the input is negative." Asked how she knew this, the student replied, "I just assumed it would be negative because the minus sign is in front of x." The interviewer wrote down –5 and $-x$, then asked "Does it makes a difference if the minus symbol is in front of a number or in front of a variable?" The student responded: "Being in front of a variable, it would be a negative answer. And negative five is just that, negative five."

Earlier in the interview the student had indicated that c in the expression $(x-c)$ could be either positive, negative or zero. Asked how the minus symbol in the expression $(x - c)$ differs from $f(-x)$ or $-f(x)$, the student replied: Because they [pointed at $-f(x)$ and $f(-x)$] are by themselves." The physical arrangement of the minus symbol preceding a variable appears to be perceived initially as a cognitive unit by many students interviewed. This symbol pattern apparently activates a path of selection and retrieval based on an arithmetic conceptualization of a negative number—an object, not a process, which cues the retrieval of a schema that includes the met-before of the minus symbol preceding a number or variable as always indicating a negative number. This concept image is so refined and stable, its selection and retrieval is automatic (McGowen, 1998, pp. 108-114).

The cognitive demands on students as they attempt to make sense of ambiguous mathematical notation, both arithmetic and functional, are far more formidable in their complexity than has generally been recognized. Comparative investigations, explicit class discussion, the use of function machine representations and the graphing calculator and reflective writing assignments generated cognitive dissonance that challenged most students to re-conceptualize previous understandings after reflecting on what they had done and thought. As the result of teaching interventions designed to address the lack of understanding about the minus symbol, many difficulties experienced by students were revealed. Students voiced their confusion, and described their struggles to attempting to determine which interpretation of the minus symbol was appropriate in a given context. One student expressed the difficulty he and many in the class were struggling with, as a result of the interventions: "How do I know what the negative sign means in a given problem? Which way do I think about it?"

The most successful students demonstrated significant growth in their mathematical abilities over the semester. However, their improvement in ability to deal flexibly with conceptual questions was not as great as their improvement in their ability to deal flexibly with ambiguous notation in procedural questions. Students at the other extreme, the least successful, were somewhat more able to deal flexibly with procedural questions involving ambiguous functional notation than they were with traditionally formatted questions. The least successful demonstrated almost no growth during the semester and what little growth did occur was very inconsistent, both within individual students as well as between members of the group.

This seemingly evident remark has highly significant consequences. Much of current research data reports how students perform in typical classes. If the teaching and learning fails to take account of the complex changes in mathematical meaning and the needs of individual students, then it suggests that the less successful students operated in increasingly complicated ways. Thus the analysis of errors becomes more complicated and the variety of research reported proliferates in different directions. A more coherent view may be sought not by simply compiling statistics how different students perform but by seeking to encourage students to make more sense of the changing nature of mathematics, which, in turn, requires teachers, curriculum designers and mathematics educators to seek to make more sense of the long-term development of mathematical ideas and of the cognitive development of learners.

The Minus Symbol and other Met-befores

Students' prior arithmetic operational experiences include other met-befores that can cause difficulty when variables are introduced. Some are evident met-befores such as 'the difference is the larger number minus the smaller', 'multiplication makes bigger', 'addition gives a bigger result while subtraction makes smaller' and problems coping with the minus sign. However, there are also more complicated errors relating to mis-remembering rules learnt by rote. Student responses to the following question (Bright and Joyner, 2003) provide additional evidence of ideas that were perfectly satisfactory in their original arithmetic context but are now recalled as met-befores that interfere with construction of new knowledge. Students were given several pairs of variables representing positive or negative values according to their respective positions on the number line pictured below and asked the following survey question to determine which had the greater value.

The numbers 0, 1, x, y, and $-z$ are marked on the number line below.

FIGURE 6: quantities on a number line

For each pair of numbers circle the number that has the greater value. If the two numbers are equal, circle both numbers.

A. y	$-y$	D. $x-y$	$y-x$
B. z	$-(-z)$	E. $\lvert y \rvert$	$-y$
C. $2y$	y	F. $x+y$	$x-y$

A majority of 128 two-year college students enrolled in an introductory algebra course recently surveyed claimed that $2y$ was larger than y (Part C). The two most common explanations for this choice were "$2y$ is greater because it is two times more than y" and "$2y$ because it has a number in front of the letter." One student added "and because it has more variables" (again perhaps because the more (whole) numbers you multiply together, the more you get.) Nearly one-third of the students responded that the two expressions $x-y$ and $y-x$ (Part D) were equal. Reasons given included: "same problem, just switched around," "because they are both subtracting a variable," and "because you don't know if the variable was an opposite or not." Less than one-third of the students indicated that $x-y$ is greater than $x+y$ (Part F) and were able to provide a valid reason. The most common response was: "$x+y$ because the numbers were added," which may relate to the met-before with whole numbers that the sum is bigger than the difference. One out of every five students responded that "$x + y = x - y$." Reasons given for this choice included: "because adding a positive and a negative is the same as subtracting a positive and a negative number" and "because in both equations you are really adding the numbers." Note, in this case, that y is visibly negative, so subtracting a negative means changing its sign and adding the positive (McGowen & Tall, 2010, p. 174).

This is consistent with the hypothesis that changes in context lead to difficulties in making sense of the mathematics and consequently, learning by rote may result in more fragile knowledge that is likely to break down. There may be a further unintended consequence. By failing to help students make sense of the new ways of thinking appropriate for a new context, we may give ourselves a proliferation of even more new difficulties that make later teaching and learning even more complicated. When teachers understand what students know and how they think, and then use that knowledge to make more effective instructional decisions, significant increases in student learning occur. Black & Wiliam (1998) examined approximately 250 studies and found that gains in student learning resulted from a variety of methods all of which had a common feature: formative assessment. This is a form of assessment that uses the data acquired to adapt instruction to better meet student need. They concluded that (1) improving formative assessment resulted in noticeable increases in student learning; (2) there is room for improvement; and (3) there are ways to improve the effectiveness of formative assessment:

What is needed is a culture of success, backed by a belief that all can achieve. Whilst it can help all pupils, it gives particularly good results with low achievers where it concentrates on specific problems with their work and gives them both a clear understanding of what is wrong and achievable targets for putting it right. Pupils can accept and work with such messages, provided that they are not clouded by overtones about ability, competition and comparison with others. In summary, the message can be stated as follows:

Feedback to any pupil should be about the particular qualities of his or her work, with advice on what he or she can do to improve, and should avoid comparison with other pupils (Black & Wiliam, 1998, p. 143).

In addition, self-assessment by students is an essential component of formative assessment. Before a student can take action to improve their learning, feedback about their efforts has three elements: the desired goal, the evidence about their present position and some understanding of a way to close the gap between the two and all three must to some degree be understood before one can take action to improve their learning. Black & Wiliam concur with the research into the way that people learn:

"New understandings are not simply swallowed and stored in isolation—they have to be assimilated in relation to pre-existing ideas. The new and the old may be inconsistent or even in conflict, and the disparities have to be resolved by thoughtful actions taken by the learner. Realizing that there are new goals for the learning is an essential part of this process (Black & Wiliam, 1998, p. 143).

In an attempt to better understand what students know and how they think, three instructors at a two-year college agreed to participate in a formative assessment project and investigate the extent to which introductory, intermediate, and college algebra students could demonstrate understanding of and ability to apply both concepts and skills in different contexts. Prior to instruction on a new unit, students completed questionnaires dealing with related concepts and skills on topics addressed in that unit. Analysis of students' responses provided additional evidence of students' inability to utilize knowledge and skills learned in one context in a different situation. Differences in the percent of correct responses to related questions on a given topic/concept revealed little or no conceptual understanding and a superficial degree of students' fragmented knowledge.

Though the percent of correct student responses to a given question varied from section to section and course to course, the lack of robust, connected understanding was common among all students participating in the study and revealed the shallowness of their understanding of elementary algebra concepts and skills. Students had a much better understanding of the relationship between slopes of parallel lines than they did of perpendicular lines. The added complexity of dealing with the minus symbol in front of the multiplicative inverse of m_1 when dealing with perpendicular lines is another instance of the problematic met-before of the minus symbol as an underlying cause contributing to student errors. Eight-two of 114 college algebra students (72%) identified $m_1 = m_2$ as the relationship between slopes of parallel lines, but only 41% (47/114) correctly identified $m_2 = -1/ m_1$ as the relationship of slopes of perpendicular lines. Fewer students (38%) were able to write the equation of a line perpendicular to a specified line through a given point and only 31% students answered both questions correctly. Forty percent (29/71) of introductory algebra students correctly identified the relationship between slopes of parallel lines correctly and 41% (29/71) identified the relationship between slopes of perpendicular lines.

The problematic met-before of the minus symbol continues to be an underlying cause of errors when students encounter quadratics. Asked to evaluate the function $y = x^2 - 5x + 3$ for $x = -3$, 28 of 127 intermediate algebra students (22%) correctly evaluated it. However, when $(t-2)$ is substituted for x, only 7% evaluated the function correctly. Their inability to correctly interpret the minus symbol given the different meanings of the minus symbol continues. Some students believe that they have "used up" the negative sign. One student, after writing $y = -3^2 -3(-3) + 5$, explained: "I have to do parentheses first."

Beneath his initial work of $-3^2 -3(-3) + 5$, he wrote 9 + 5. Pointing at the first term, -3^2, he said, "Now I have to do this but I can't remember if it's negative nine or just nine. I never can remember which to use." He wrote down –9 and stopped. "There's no sign in front of this (pointing at 9 + 5), so I need to multiply," and wrote: –9(14) = 136.

Translating between symbolic and graphic representations was another area in which many students demonstrated inflexibility. Given $y = x^2 + 2x - 35$, 50 of 54 (93%) intermediate algebra students were able to determine its factors but asked to identify the factors of a quadratic given the graph and view window, only 52% (28/54) could do so, and even fewer, 26% (14/54) could solve the equation $3x^2 + 8x = -5$.

One of the teachers who participated in the formative assessment project described what she had learned about her college algebra students when analyzing their responses to related questions:

I see more clearly how my students view concepts and how well they really understand concepts. They could answer questions from one direction, but not from another. Many students have very fragmented knowledge. The project has shown me that although many of my students can "do" the mathematics, their fragile grasp of the language of mathematics doesn't allow them to know WHAT to do WHEN. They say things like, " I know how to do that. I just didn't know I was supposed to do it here.

Analysis of related questions on the topics of linear equations/slope, linear systems, and linear inequalities provided more evidence of her students' fragmented learning and failure to utilize knowledge and skills learned in one context in a different situation. She observed:

Few students had a robust knowledge of any given topic. In every section there were great differences in what students knew coming into the course. Most students lacked clear understanding of distinctions between slope, coefficients and intercepts. For some students, the intercept is a number, not a point (ordered pair). For others, the *x*- and *y*- intercepts were the *x*- and *y*-coefficients. Still other students thought the slope was an ordered pair, i.e.,(numerator, denominator), not a ratio. And finally there were those who thought the slope was an intercept.

A large percentage of the College algebra students had a very fragile knowledge of linear functions. They could substitute values for *m* and *b* to get the equations of a line in slope-intercept form but given the graph of a line only 1 in five could determine whether *m* and *b* were positive or negative. They could state that slope is "rise over run" and "b is the *y*-intercept, but were unable to use that knowledge to select the correct graph of the line. They thought the *x*- and *y*-intercepts were coefficients and could not choose the correct equation of a line in standard form when the graph is given. I did not expect to face this problem with College Algebra students. The formative assessment project helped me recognize the problem.

She identified the problematic minus symbol met-before as an underlying cause of student errors in the class.

Students did not distinguish context when interpreting the different meanings of the "–" symbol as subtraction, or the unary operation of "taking the opposite" (additive inverse). They used the subtraction operation of a linear factor as the sign of the zero.

Using the visual image of a function machine with two inputs to represent a binary operation and a function machine with one input to represent a unary operation helped students notice context and interpret the "–" symbol correctly. I started saying "*subtract the zero*" and writing (x – ZERO), which resulted in a significant improvement in students' ability to write the factored form of a polynomial correctly.

Conclusion

Changing what students value and how they view learning mathematics are frequently much harder challenges than teaching them mathematical procedures and application of formulas. If we are to change the severely procedural orientation to mathematics focused on 'correct answers' students have learned to value above all, curriculum materials must offer an alternative approach to learning algebra for students who have taken one or more pre-college courses (introductory, intermediate, college algebra) previously—in high school or at college—and have failed to place into a college-level mathematics course. Setting expectations for growth in flexible thinking at the beginning of a course plays a major role in determining how students grow mathematically.

Instructors need to reflect on their own met-befores and consider not only supportive met-befores viewed as pre-requisites for learning new mathematics but also problematic met-befores that impede learning and cause mathematical dysfunction. It may be helpful to look at problematic met-befores in a positive light, giving students confidence in their prior knowledge. Acknowledging how a met-before operated satisfactorily in an earlier context and finding positive new ways of addressing the changed situation, provides opportunities to encourage students to develop new ways of working in a new context. Mathematics educators and teachers at all levels need to acknowledge the difficulties that learners experience in their mathematics classes as a result of inflexible ways of thinking, difficulties interpreting ambiguous notation and the need to cope with changing contexts, with the resulting fragmentation of strategies. They need to recognize the met-befores that underlie the fragmentation of strategies that occurs as a result of initial perceptions, inappropriate categorization, and retrieval of inappropriate schemas leading to the divergence of performance, that arises from the differences in the students' development of flexible thinking. This should all be taken into account when planning instructional tasks that encourage students not only to build their confidence through success but also to address the underlying causes of their difficulties.

The nature of each and every student's long-term development in new situations—as the complication of new ideas and the problematic nature of earlier experience cause difficulties—is one that needs to be addressed not just in handling college students' remedial problems, but throughout the whole curriculum that has led to these difficulties at successive stages, causing disaffection for so many rather than the pleasure of seeking to make sense of mathematics at every stage of development.

References

Ausubel, D. P., Novak, J. D. & Hanesian, H. (1968). *Educational Psychology: A Cognitive View* (2nd Edition, 1978). Holt, Rinehart & Winston. New York, NY.

Backhouse, J. (1978). Understanding School Mathematics—A Comment. *Mathematical Teaching*. 82.

Black, P. & Wiliam, D. (1998). Inside the Black Box: Raising Standards Through Classroom Assessment. *Phi Delta Kappan*. Vol. 80, 2, 139–148. Published online: http://www.pdkintl.org/kappan/kbla9810.htm.

Bright, G. W, & Joyner, J. M. (2003). Dynamic Classroom Assessment: Linking Mathematical Understanding to Instruction. Vernon Hills, IL: ETA/Cuisenaire.

Byers, V. & Herscovics, N. (1977). Understanding School Mathematics. *Mathematical Teaching*. 81, 24–27.

Clarke, D., Helme, S. and Kessel, C. (1996). *Studying Mathematics Learning in Classroom Settings: Moments in the Process of Coming to Know*. A paper presented at the National Council of Teachers of Mathematics Research Presession. San Diego, CA.

Davis, G. E. & McGowen, M. A. (2007). Formative feedback and the mindful teaching of mathematics. In *The Australian Senior Mathematics Journal*. Vol. 21, No. 1., 2007. pp. 19-29.

Davis, R. B. (1989). Three Ways of Improving Cognitive Studies in Algebra. In Wagner, S. & Kieran, C. (Eds.), *Research Issues in the Learning and Teaching of Algebra* (pp. 115–119). National Council of Teachers of Mathematics and Lawrence Erlbaum Associates.

Davis, R. B. (1986). Conceptual and Procedural Knowledge in Mathematics: A Summary Analysis. In Hiebert, J. (Ed.) *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp.265–300). Hillsdale, NJ: Erlbaum.

DeMarois, P., McGowen, M., & Whitkanack, D. (1996). *Applying Algebraic Thinking*

to Data. Preliminary Edition. Harper Collins Publishers: Glenview, IL.
Edelman, G. (1992). *Bright Air, Brilliant Fire: On the Matter of the Mind*. New York, NY: Basic Books, A Division of Harper Collins Publishers.
Gray, E.M. & Tall, D. O. (1994). Duality, Ambiguity, and Flexibility: A "Proceptual" View of Simple Arithmetic. In *Journal for Research in Mathematics Education*, 25, 2, 116–140.
Gray, E. M. & Tall, D. O. (1991). *Success and Failure in Mathematics: Procept and Procedure: A Primary Perspective*. Mathematics Education Research Centre. University of Warwick.
Krutetskii, V.A. (1969). An Analysis of the Individual Structure of Mathematical Abilities in Schoolchildren. In Kilpatrick, J. & Wirszup, I. (Eds.). *Soviet Studies in the Psychology of Learning and Teaching Mathematics*, (pp. 59—104). Chicago, IL: University of Chicago Press.
McGowen, M. A. (1998). *Cognitive Units, Concept Images, and Cognitive Collages: An Examination of the Process of Knowledge Construction*. Ph.D. Dissertation, University of Warwick, Coventry, UK.
McGowen, Mercedes. (2006). Who are the students who take pre-calculus? In Nancy Baxter-Hastings, (Editor). MAA Notes 69: A Fresh Start for Collegiate Mathematics. Mathematical Association of America: Washington, D.C., pp. 15-27.
McGowen, Mercedes & Tall, David O. (2010). Metaphor or Met-before? The effects of previous experience on the practice and theory of learning mathematics. *Journal of Mathematical Behavior* 29, 169–179.
Nogueira de Lima, Rosana and Tall, David (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*. 67 (1) 3-18.
Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics 22*(1), 1–36.
Skemp, R. R. (1987). *The Psychology of Learning Mathematics* (Expanded American Edition). Hillsdale, NJ: Lawrence Erlbaum & Associates.
Tall, D. O. (2004). The three worlds of mathematics. *For the Learning of Mathematics*, 23 (3). 29–33.
Tall, D. O. (1994). Understanding the Processes of Advanced Mathematical Thinking. An invited ICMI lecture at the International Congress of Mathematicians, Zurich, Switzerland.
Tall, D. O. (2012). *How Humans Learn to Think Mathematically*. In preparation for publication by Cambridge University Press (USA).
Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*. 12, 151–169.