MAKING SENSE OF MATHEMATICS THROUGH PERCEPTION, OPERATION & REASON: THE CASE OF TRIGONOMETRIC FUNCTIONS

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This paper evolves a framework for making sense of mathematics through perception, operation and reason and uses it in a specific case of trigonometry. It is based on a fundamental theory of how humans learn to think mathematically from early childhood to the frontiers of mathematical research. It is intended to be of theoretical value in mathematics education and of practical value for teachers and learners. The example of trigonometry develops from visual and symbolic relationships in specific right-angled triangles to circle trigonometry involving signed quantities and dynamic functional relationships. The data relates to student teachers preparing to teach secondary mathematics.

THEORETICAL BACKGROUND

In this paper we continue the evolution of theories of mathematical thinking in terms of *perception*, *operation* and *reason*. Similar frameworks have developed over the years, including that of Fischbein (1987), the first president of PME, whose theory considered three different approaches that he termed *intuitive*, *algorithmic* and *formal* and the second president Skemp (1979) involving *perception*, *action* and *reflection*. Each of these have elements in common with Bruner (1966) who referred to *enactive*, *iconic* and *symbolic* modes of communication where the symbolism included natural language and the special mathematical languages of arithmetic and logic. We continue the theoretical evolution by focusing on the sensori-motor foundations that we share with other species and the special quality of language and symbolism that is peculiar to *Homo Sapiens*. In doing so we are mindful of the evolutionary biology approach (e.g. Leron & Hazzan, 2008) which distinguishes the fundamental sensori-motor thinking that occurs in an immediate response and the deeper longer-term thinking that occurs in more sophisticated mathematics.

Our research builds on the theoretical framework presented in PME 30 by Tall (2004) that we now interpret as formulating the long-term development of mathematical thinking as different individuals grow from child to adult, based on the fundamental foundations of human *perception*, *operation* and *reason*. The original framework focused on three distinct long-term developments of mathematical thinking. Two develop in school mathematics. The first is based on *conceptual embodiment* building from human perceptions and physical actions through increasingly sophisticated practical activity and thought experiment to imagine perfect platonic concepts within

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the mind. The second involves physical actions, such as counting, being symbolised as manipulable mental concepts in the *operational symbolism* of arithmetic and algebra. The relationships may be formulated verbally, such as the geometric definitions of Euclidean figures and more general ideas of congruence and parallel lines that lead on to Euclidean proof and the arithmetic properties of operations such as commutative. associative and distributive properties of arithmetic that later offer a basis for algebraic proof. Much later, in pure mathematics at university, these lead to more advanced forms of mathematical reasoning in *axiomatic formalism* based on formal set-theoretic definitions and mathematical proof.

In subsequent years, two further elements have been incorporated. The first reflects the nature of the thinkable concepts that develop in all forms of mathematics including geometry, arithmetic and algebra, and in formal axiomatic structures. A *crystalline concept* is given a working definition as 'a concept that has an internal structure of constrained relationships that cause it to have necessary properties as a consequence of its context' (Tall, 2011). The second involves the effect of previous experience in new situations where a *met-before* is given a working definition as 'a trace that it leaves in the mind that affects our current thinking' (Lima & Tall 2008, McGowen & Tall, 2010). These experiences may be *supportive* in a new context where they continue to make sense and may be used to generalize existing knowledge to more general situations, or they may be *problematic* and impede progress.

Problematic met-befores reflect established ideas of *epistemological obstacles* (Bachelard 1938, Brousseau, 1983), however, we now see them as part of a wider vision in which impediments caused by problematic met-befores complement supportive met-befores that encourage generalization. As in the goal-oriented theory of Skemp (1987), supportive and problematic aspects emotionally affect progress. Achieving a goal gives pleasure and continuing success encourages the learner to confront problems with a determination to succeed, leading to a positive cycle of reinforcement. Failure to achieve a goal may lead to an alternative goal, such as learning 'how to do' the mathematics to achieve alternative success such as passing tests. Repeated failure can lead to a downward spiral as failure leads to anxiety and less engagement, then less engagement leads to increasing anxiety, and so on.

We suggest that, to make sense of mathematical thinking, the teacher should be aware of the changing needs of the student in new situations, to build on previous success and to realise that what worked before will need a new approach to make sense of the new situation. To do this we consider how the learner makes sense through perception based on fundamental conceptual embodiment and thought experiment, then through the coherent relationships in operational symbolism, and later in terms of reasoning based on definition and deduction. In school mathematics, reasoning develops in various forms: through practical definition and principles such as congruence to deduce theorems in geometry, through general principles such as 'doing the same thing to both sides' in solving equations and refining ways of thinking by formulating observed regularities as principles such as the 'rules of arithmetic' as a basis for algebraic proof. In university mathematics these are refined further into set-theoretic definitions proving theorems in an axiomatic framework.

THE CASE OF TRIGONOMETRY

The development of trigonometry builds from embodied practical and experimental experiences in drawing and construction and symbolic operations such as calculating the sine of an angle as 'opposite over hypotenuse' as a numerical quantity and then considering flexible relationships between lengths and trigonometric ratios. It may then shift to the broader context of circle trigonometry where the angle can be greater than 90°, sine and cosine may be interpreted dually as a ratio of lengths or as the horizontal and vertical components of a point on a unit circle, and lengths may now be signed numbers that vary as the point moves dynamically round the circle to give rise to the concept of trigonometric functions. By measuring the angle in radians, a relationship arises between the angle and the length of the arc it subtends, enabling further conceptual links to be made between the change in the trigonometric functions and their rate of change in the calculus.

Michele Challenger's thesis (2009) studied the teaching of trigonometry in school. She found that students often described the ideas as being complicated and spoke of a difference between two distinct forms of trigonometry, as in the following comments:

> I hate trigonometry. There is just so much to remember: all the diagrams and formulas. I never know which one to use.
>
> Are we talking about triangle trigonometry or circle trigonometry here?
>
> I used to understand it when it was just triangles but now I don't know where to start.
>
> What is sine exactly? I thought I knew but now it is so confusing.

This suggests two distinct contexts for trigonometry in school:

(i) *triangle trigonometry* involving problems related to the relationship between the sides and angles of specific right-angled triangles using trigonometric ratios.

(ii) *circle trigonometry* involving variable angles of any size at the centre of a circle, with trigonometric ratios involving signed numbers and the properties of trigonometric functions.

In our study of students in teacher training prior to entering teaching, we found that they also responded at a third level, relating to their later undergraduate study:

(iii) *analytic trigonometry* involving trigonometric functions expressed as power series and the use of complex numbers to relate exponential and trigonometric functions.

This more sophisticated viewpoint could easily affect the way in which the students as future teachers viewed the teaching and learning of their pupils unless they developed an awareness of the way in which the pupil's knowledge structures affected their learning.

EXPERIMENTAL DATA AND ITS ANALYSIS

Our data was collected in a questionnaire given to post-graduate students preparing to teach mathematics at secondary school level. Its purpose was to gain an initial picture of how the respondents make sense of trigonometry. Here we only have space to consider selected responses to three of eight questions by three students.

Respondent A is a male PGCE student with a first class mathematics degree. To the first question 'Describe $\sin x$ in your own words', he responded:

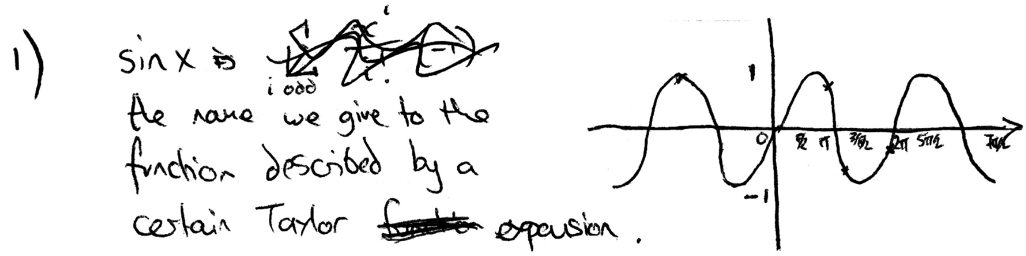

Figure 1: Respondent A, Question 1

He said (the name of) a Taylor expansion and drew a graph. His immediate reaction was therefore a combination of analytic and circle trigonometry (levels (ii) and (iii)).

Respondent B is a female PGCE student with a 2(i) bachelor degree in physics with previous employment as a medical physicist. Her response for Item 1 is as follows:

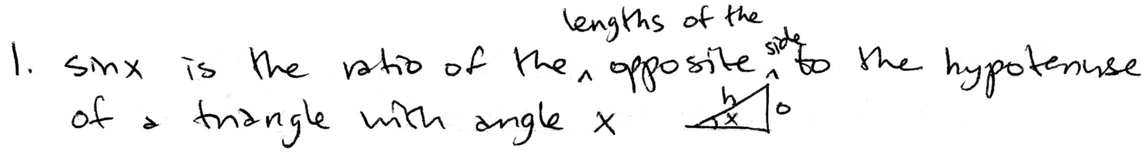

Figure 2: Respondent B, Question 1

This response is level (i), in particular, it describes sin x as the ratio of the lengths focusing on the first stage of compression from operation to symbolic concept.

Respondent C is a female PGCE student with a 2(ii) degree in mathematics. She offered a verbal relational description with little detail.

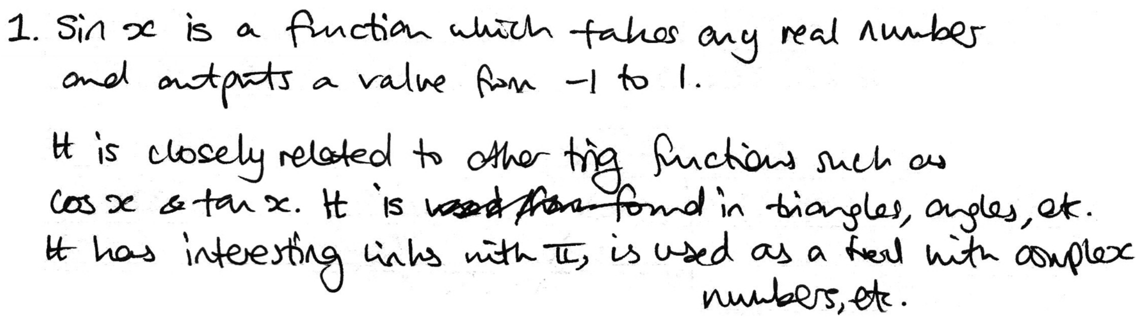

Figure 3: Respondent C, Question 1

This response operates at levels (ii) and (iii), speaking of $\sin x$ as a function defined for a real number, and refers to wider links involving complex numbers.

These brief remarks do not reveal the full extent of the students' knowledge. However, other responses clarify a broader picture.

Item 3 of the questionnaire asked "What is the value of sin 270°? Explain why it is this value." Respondent A gave the following response:

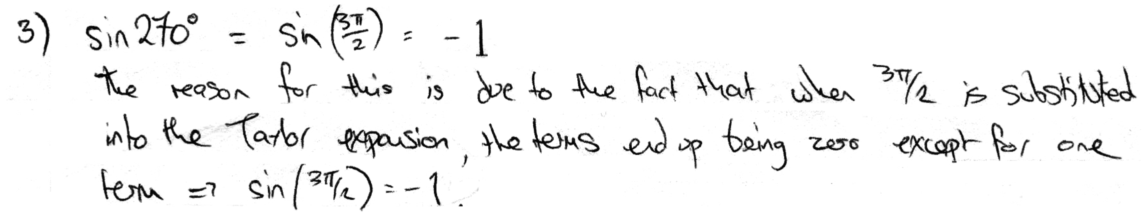

Figure 4: Respondent A, Question 3

First, the student clearly sees that sin 270° is –1. Yet his reasoning suggests that 'when 3π/2 is substituted into the Taylor expansion, the terms end up being zero except for one term.' This is patently untrue if the actual number is substituted as the later terms would be clearly non-zero. So he must, in some way, have in mind a series such as that for $\cos x$ where $x = 0$ and $\cos 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = 1 - \frac{0^2}{2!} + \frac{0^4}{4!} - ....$

Skemp (1979) distinguishes between conceptual links (C-links) and associative links (A-links). This link is clearly associative, for it does not work computationally. This relates clearly to the dual processing theory reported in Leron & Hazzan (2008) where the immediate response operates at an intuitive, non-analytic level.

Respondent B replied as follows:

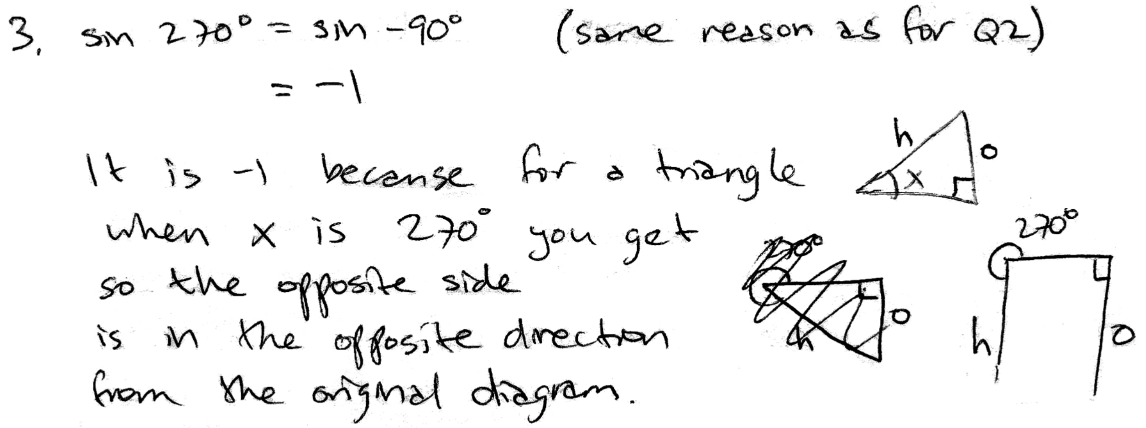

Figure 5: Respondent B, Question 3

Initially the student sees that $\sin 270° = \sin -90° = -1$ using the graph as in the previous question (not shown here). However, when attempting to draw the angle turning anti-clockwise through 270°, the first attempt has the opposite side *o* clearly drawn downward but is scribbled out because the angle is not the right size. When it is redrawn with the correct angle 270°, the radius is now vertical, with the (unsigned) hypotenuse *h* drawn over the (signed) opposite side *o*, so there is no longer a visible triangle as occurs at level (i). This becomes problematic at level (ii) and is solved ingeniously by the student drawing separate lines for *o* and *h* in a manner that is clear but is not the expected picture that occurs as the radius moves around the circle.

Respondent C drew the sine graph and read the value from it as follows:

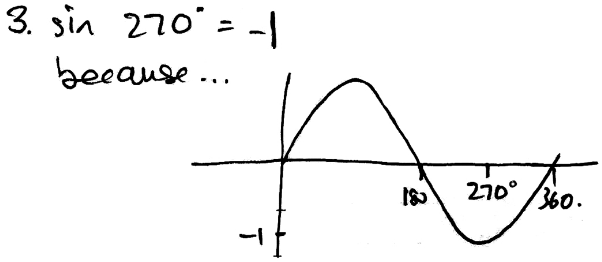

Figure 6: Respondent C, Question 3

Essentially this is an intuitive embodied response operating graphically at level (ii).

Item 5 of the questionnaire asks “Explain why $\sin\theta$ can never equal 2.”

Respondent A replied:

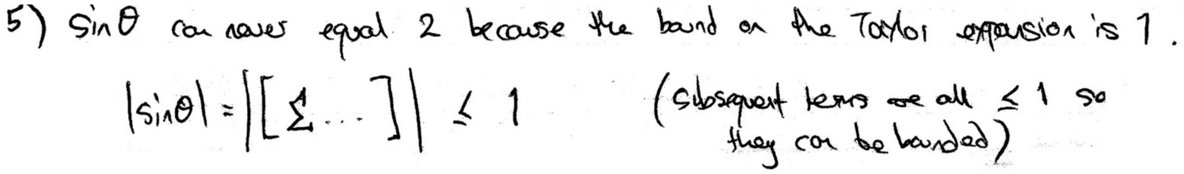

Figure 7: Respondent A, Question 5

Again he responded with immediate associative links between the function as a Taylor series and its behaviour as a function at level (iii) and level (ii) with the conceptual links not yet reflected upon. As we considered the situation, we realised that we too could not give an immediate analytic response involving the Taylor series without switching to the visual representation of the graph.

Respondent B made sense of the question by working at level (i) using the observation that the hypotenuse is the longest side of a right-angled triangle.

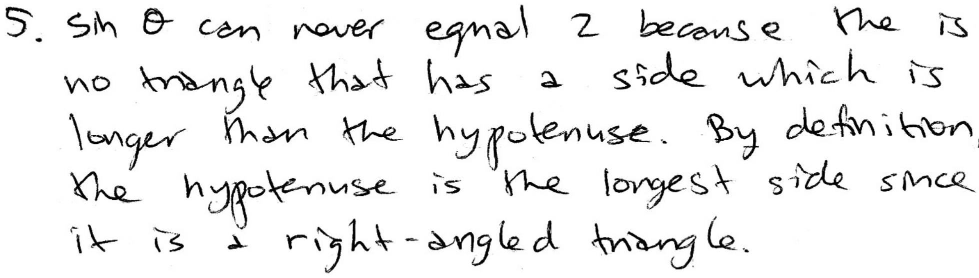

Figure 8: Respondent B, Question 5

Respondent C simply wrote an inequality without further comment.

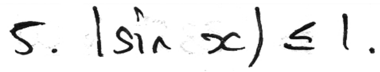

Figure 9: Respondent C, Question 5

Comparing this with the student's previous responses in figures 3 and 6, reveals a level (ii) response from a student who has also responded at levels (ii) and (iii).

These immediate responses tell a story. Student (A) with a first class degree refers to analytic ideas such as power series at level (iii) and circle trigonometry at level (ii) with analytically faulty associative links between them. Student (B) with an upper 2nd degree and practical experience responds by combining level (i) triangle trigonometry and level (ii) circle geometry with a good grasp of both yet with a minor problem in visualising what happens when the angle is 270°. Student (C) with a lower 2nd mathematics degree operates visually at level (ii) with indications of possible links at level (iii).

This reveals the distance that all three student teachers need to travel to become sensitive to the issues that will arise in their classrooms. The much deeper question is whether we as mathematics educators, and others involved in mathematics teaching as teachers or curriculum designers, are explicitly aware of this phenomenon.

REFLECTIONS AND FUTURE DEVELOPMENTS

All around the world there is concern about raising standards, about competing in a global market place faced by problems of long-term availability of natural resources, economic crises and global warming. A broad understanding of the underlying mathematics is essential in quantifying these problems both by politicians and citizens. Yet governments seek to measure success through testing and teachers are tested by the success of their pupils on these tests. The problem is formulated succinctly in the acronym WYTIWIG: What You Test Is What You Get (Burkhardt, 1987). The problem may be that what we are not focused on the major goal of helping students to *make sense* of mathematics, instead so much teaching focuses on the lesser goal of procedural success on tests. The theory of supportive and problematic met-befores reveals this only as a partial success in the spectrum from mathematical sense-making to alienation in terms of mathematics anxiety.

We hypothesise that what is important is *making sense* of mathematics, and that this essentially requires the teacher, as mentor, to be aware of the needs of the student, encouraging learners to cope with changes of meaning by building confidence in earlier successes and making sense of problematic aspects in new contexts. Such developments are partially achieved through encouraging learners to voice and reflect on their own knowledge, for instance in Japanese Lesson Study (Isoda et al, 2007; Ong et al, 2010). A major factor, as we see it, is whether we can develop confidence in ourselves as theorists and teachers, to think deeply about making sense in mathematics, so that we may encourage learners to have confidence in their own sense-making and use that confidence to complement the power of supportive experience to have the determination to make sense of problematic new ideas.

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