In Honour of my Friend
Ted Eisenberg

Making Sense of Reasoning and Proof

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Mathematician
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Making Sense
Railing against Behaviourism
Aesthetics of Mathematics
Aesthetic Blindness
"Getting Things *Right*"

This presentation:

making sense of mathematical reasoning & proof,
as it develops throughout our lives through

Perception, Operation & Reason
on to **Mathematical Proof.**

The Sensori-Motor Language of Mathematics

Bruner: enactive, iconic, symbolic → arithmetic → logic

Conceptual Embodiment
(enactive, iconic) perception & action, thought experiments

Operational Symbolism
actions as symbolised objects that can be mentally manipulated

Axiomatic Formalism
set-theoretic definition and formal proof

Detailed analysis and relationships with many other theories in:
ICMI Handbook on Proof and Proving
How Humans Learn to Think Mathematically (CUP forthcoming)

The Development of Reasoning and Proof

Reasoning and Proof

evolve in both ***Embodiment*** *and* ***Symbolism***

developing through Euclidean definition and proof

& in symbolic proof based on observed properties of arithmetic, recast as ‘rules’ as a basis for deduction

and are restructured in **Axiomatic Formalism** in terms of set-theoretic definition and formal proof.

These need to be analysed in much greater detail.

Van Hiele theory in Geometry

Recognition **Description**	} **Practical Space & Shape**
Definition **Deduction**	} **Theoretical Euclidean Proof**
Rigor	**Formal Mathematical Proof**

Van Hiele theory in Geometry

Recognition Description	} Visual Descriptive
Definition Deduction	} Theoretical
Rigor	(Theoretical)

Van Hiele theory in Geometry

Recognition	Visual
Description	Descriptive
Definition	Definition & Construction (QEF)
Deduction	Euclidean Proof (QED)
Rigor	Hilbertian Proof

Van Hiele theory in General

Recognition Description	} Practical Mathematics
Definition Deduction	} Theoretical Mathematics
Rigor	Formal Mathematics

Process - Object Theories

Mathematical Operations are symbolised and conceived as mathematical objects that can themselves be operated upon.

APOS Theory
Action - Process - Object - Schema

Operational leading to Structural
Process is condensed, routinized, then reified as an object

SOLO Taxonomy
Uni-structural - Multi-structural - Relational - Extended Abstract

Procept Theory
Procedure - Multi-procedure - Process - Procept

Process - Object Theories

APOS Theory
Action - Process - Object - Schema

Operational leading to Structural
Process is condensed, routinized, then reified as an object

These theories begin with actions/operations
that become objects that have structures.

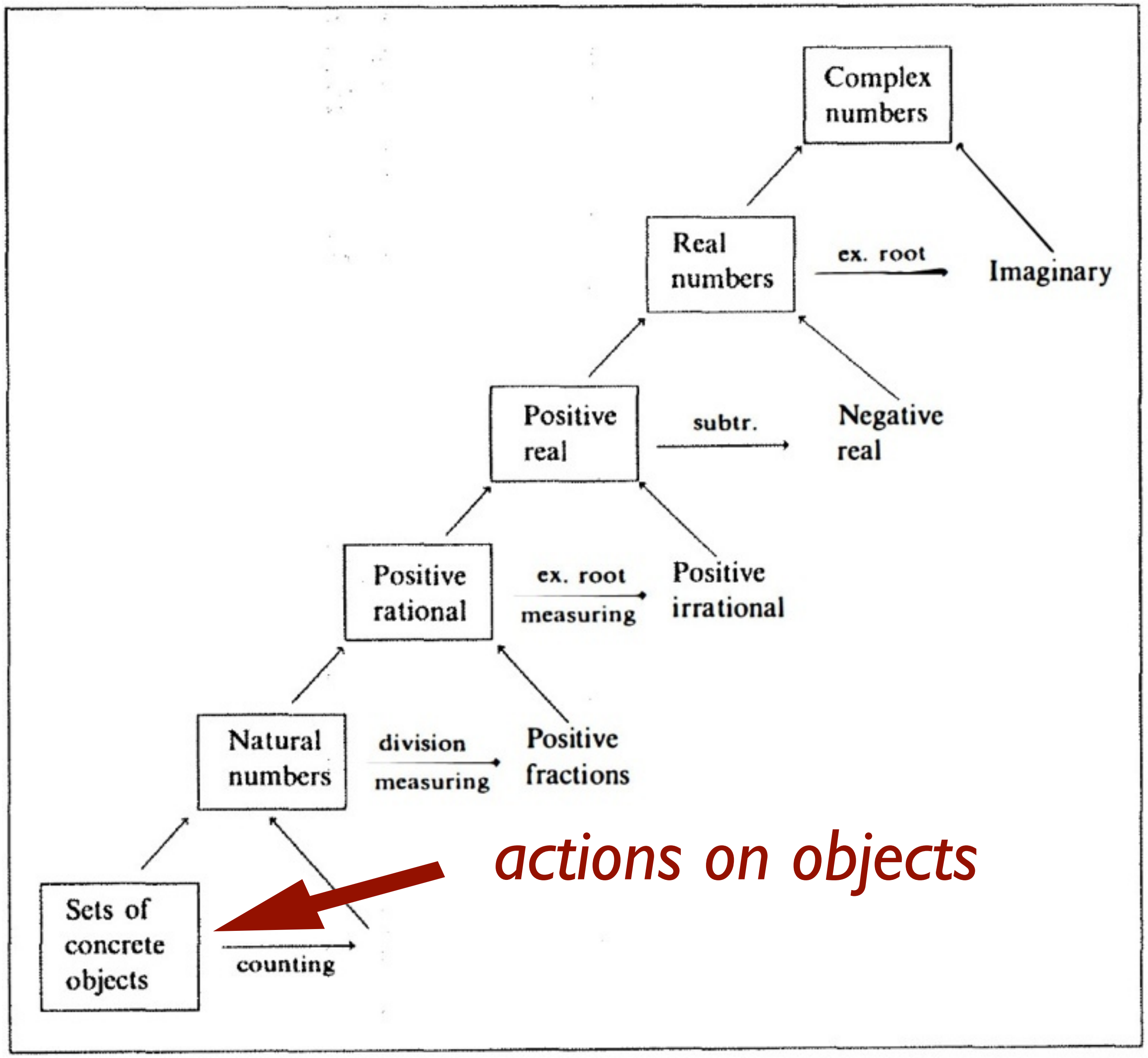
APOS Theory starts with Actions.

Operational conceptions usually come before Structural.

preliminary problem: is the proposed model of concept formation in force also when individual learning is concerned? Or, in other words, is it true that when a person gets acquainted with a new mathematical notion, the operational conception is usually the first to develop? The odds are that the answer to this question should be *yes*. Let me put it even more clearly: it seems that the scheme which was constructed on the basis of historical examples can be used also to describe learning processes.

Process - Object Theories

Expert View
Top-Down



Learner's View
Bottom-Up

Two forms of compression

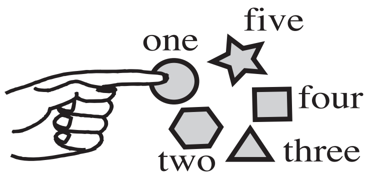
Learner's View
Bottom-Up

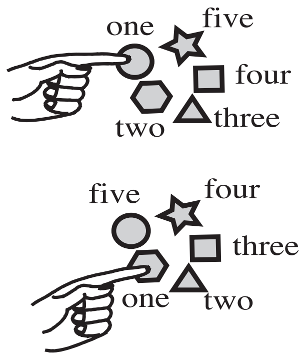
The operations performed are always performed on *existing* objects.
This gives two possibilities:

1. Focusing on the *objects* and the *effect* of the operations. — **Embodied compression**
2. Focusing on the *operations* and the resulting *symbolism*. — **Symbolic compression**

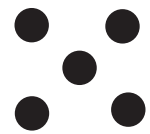
Two forms of compression

Embodied compression



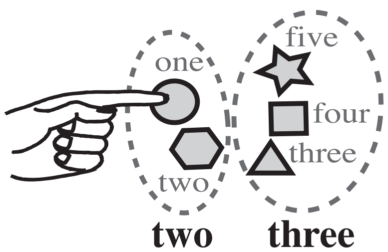


‘five’



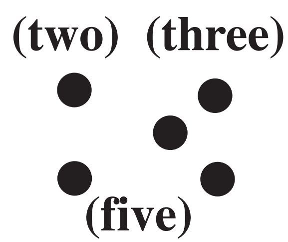
Procedure (as embodied action) | Equivalent Procedures | Effect | Embodiment of Effect

Compression from action to embodied object



count-all
count-both
count-on
count-on-from larger

‘two & three is five’



Procedure (as embodied action) | Equivalent Procedures | Effect | Embodiment of Effect

Compression from action to embodied object

Two forms of compression

Embodied compression

Focusing on the effects of the actions on the objects it is easy to see that 4+2 is the same as 2+4 or that 3 rows of 2 is the same as 2 rows of 3.

Embodied compression gives a sense of the relationships.

Symbolic compression

4+2 by count on from 4 to get ‘five, *six*’ is different from 2+4 as ‘three, four, five, *six*.’

Focus on procedures — *Sense of relationships*

Procedural — *Spectrum of performance* — *Conceptual*

Two forms of compression

Embodied compression

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Symbolic compression

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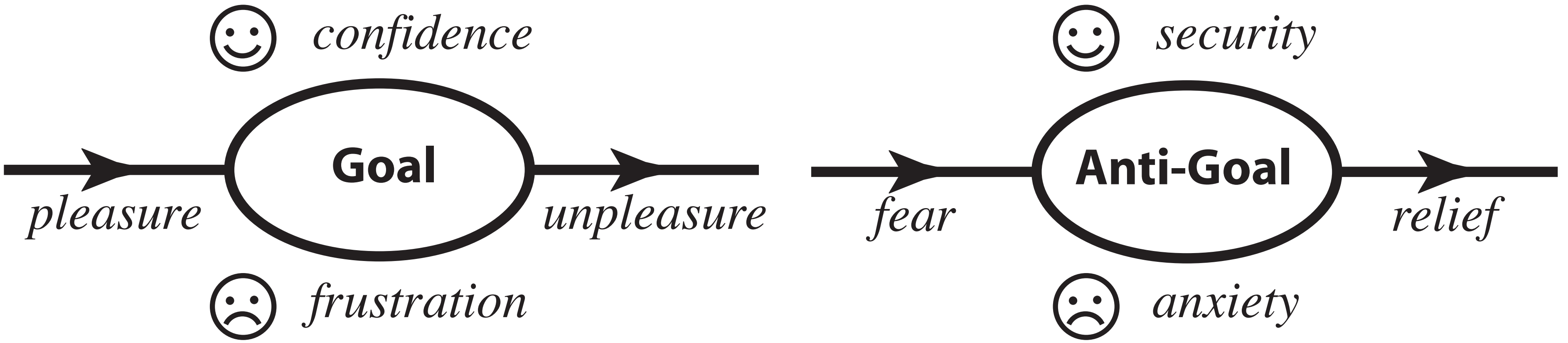
Focus on procedures & Sense of relationships

Long-term Pleasure and Pain



Richard Skemp 1979

The goal of making sense of mathematics gives pleasure and confidence and willingness to tackle new problems.

Lack of success for a confident person causes frustration and renewed effort to succeed.

Failure can lead either to the goal of procedural competence which can have its own success

or the anti-goal of avoiding failure and anxiety.

Supportive & Problematic Met-befores

A met-before is 'a cognitive structure we have *now* as a result of experiences met before.'

Analysing successive mathematical topics, e.g.

- whole number arithmetic,
- fractions with new properties (e.g. equivalence),
- signed numbers (including negative numbers),
- real numbers (including irrationals),
- infinite decimals that cannot be precisely calculated,
- complex numbers (with imaginary parts),

there are supportive met-befores that encourage generalization and problematic met-befores that impede progress which have consequences ...

Supportive & Problematic Met-befores

Supportive and problematic met-befores cause a continuing bifurcation between the increasingly smaller number of students who cope with the generalizations, often by having a sense of structure to guide their ideas

and those immersed in rote-learnt procedures that impede each future development and cause them to seek procedural competence as a default.

This affects teachers as well as learners.

Having a sense of relationships

The long-term growth of mathematical thinking is enhanced for those who have a 'sense of relationships' that guide their thinking.

Embodied compression can give a sense of relationships.

This can lead to

- a focus on meaningful *perception*,
- a blending of embodiment and symbolism,

or

- a sense of relationships between *operations*.

Having a sense of relationships

The long-term growth through embodiment, symbolism and reason can, *if successful*, lead to:

Embodiment:
flexible relationships in geometry, e.g. a triangle with 2 equal sides is also a triangle with 2 equal angles.

Symbolism:
flexible relationships between numbers as *procepts*, e.g. 2+4, 4+2, 3+3, 2x3, 3x2 are all ‘the same’.

Formalism:
flexible relationships in formal mathematics, e.g. for an ordered field there are several equivalent definitions of completeness that give the same structure.

Crystalline concepts

Working definition: A ***crystalline concept*** is a concept that has an internal structure of constrained relationships that cause it to have necessary properties as a consequence of its context.

platonic objects in geometry

procepts in operational symbolism

defined concepts in axiomatic formal mathematics

In the long-term development of mathematical thinking, properties are first *recognized*, then *described*, then they are *defined* in a way that can be used for the *deduction* of consequences that one property implies another.

Van Hiele levels and proof

Structural abstraction and proof
in embodiment, symbolism & formalism

Recognition **Description**	**of properties in a given context**
Definition **Deduction**	**of properties as a basis for deduction** **of theorems using proof (Euclidean, Algebraic, Formal)**

Van Hiele levels and proof

Structural abstraction and proof in geometry

Recognition Description	Practical Geometry: Space and Shape
Definition Deduction	Theoretical Geometry: Definition & Construction Euclidean Proof

Van Hiele levels and proof

Structural abstraction and proof in arithmetic

Recognition Description	}	Practical experience in arithmetic
Definition Deduction	}	properties of operations & numbers theorems in arithmetic

Van Hiele levels and proof

Structural abstraction and proof in algebra

Recognition **Description**	**generic properties of operations in arithmetic**
Definition	**'rules' of arithmetic**
Deduction	**algebraic proof**

Van Hiele levels and proof

Structural abstraction and proof in axiomatic formalism

Recognition	}	**of properties in a given context**
Description		
Definition	}	**of properties as a basis for deduction**
Deduction		**of theorems using formal proof**

Van Hiele levels and proof

Structural abstraction and proof in axiomatic formalism

often presented mainly as:

Definition	}	**of properties as a basis for deduction**
Deduction		**of theorems using formal proof**

Structure Theorems

Formal Mathematics deduces certain theorems that reveal *structure*

An equivalence relation can be embodied as a partition.

A finite dimensional vector space over F is isomorphic to F^n.

A complete ordered field is uniquely the real number line and decimal numbers.

A finite group is isomorphic to a group of permutations.

Structure Theorems

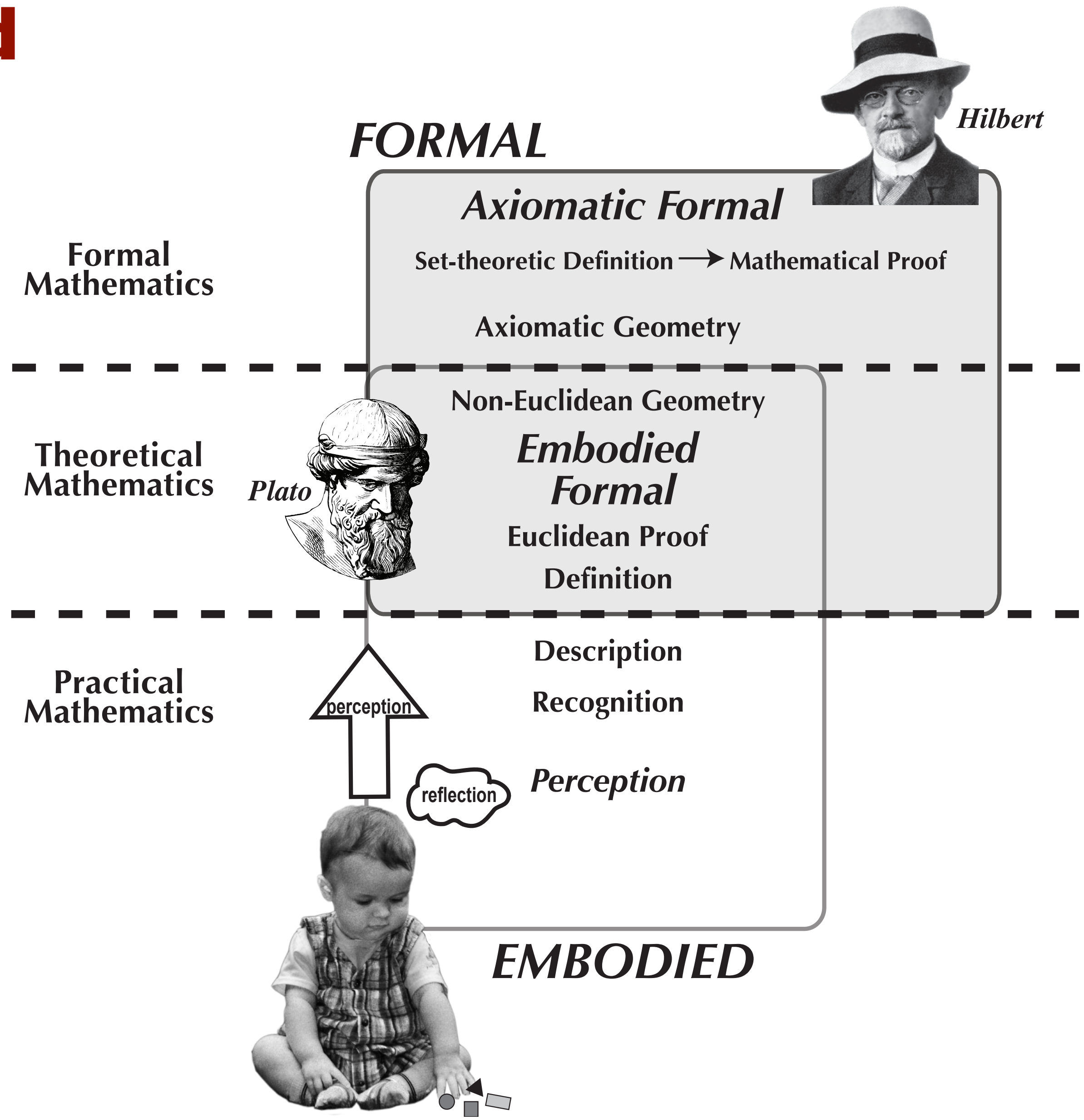
Formal Mathematics deduces certain theorems that reveal *structure*

Consequence: A structure theorem shows that an axiomatic structure has embodied/symbolic representations.

At the highest level, formalism leads back to embodiment and symbolism, now supported by formal proof.

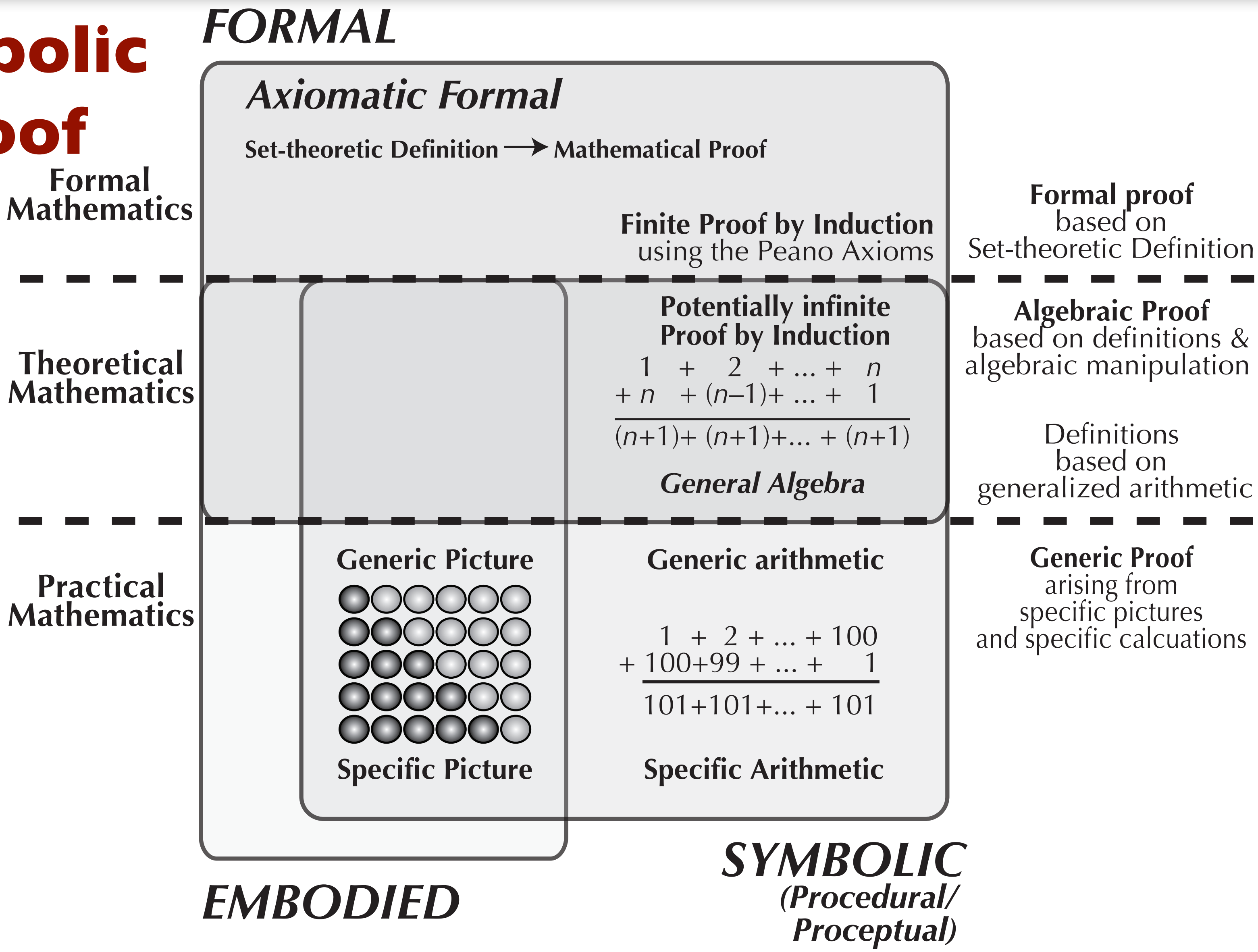
The full structure

Embodied Proof



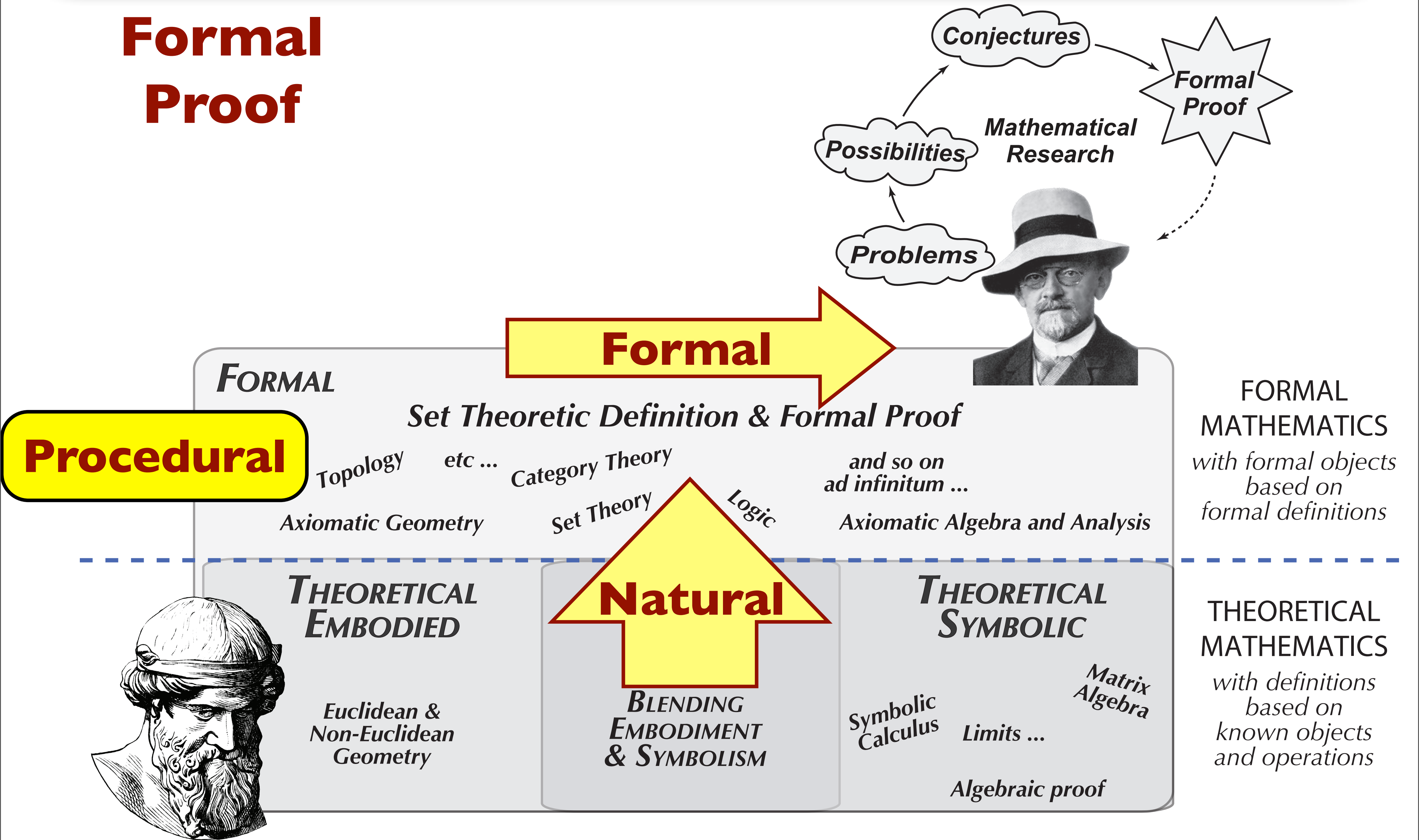
The full structure

Symbolic Proof



Proof that the sum of the first n whole numbers is $\frac{1}{2}n(n+1)$

The full structure



The full structure

FORMAL
MATHEMATICS

with formal objects
based on
formal definitions

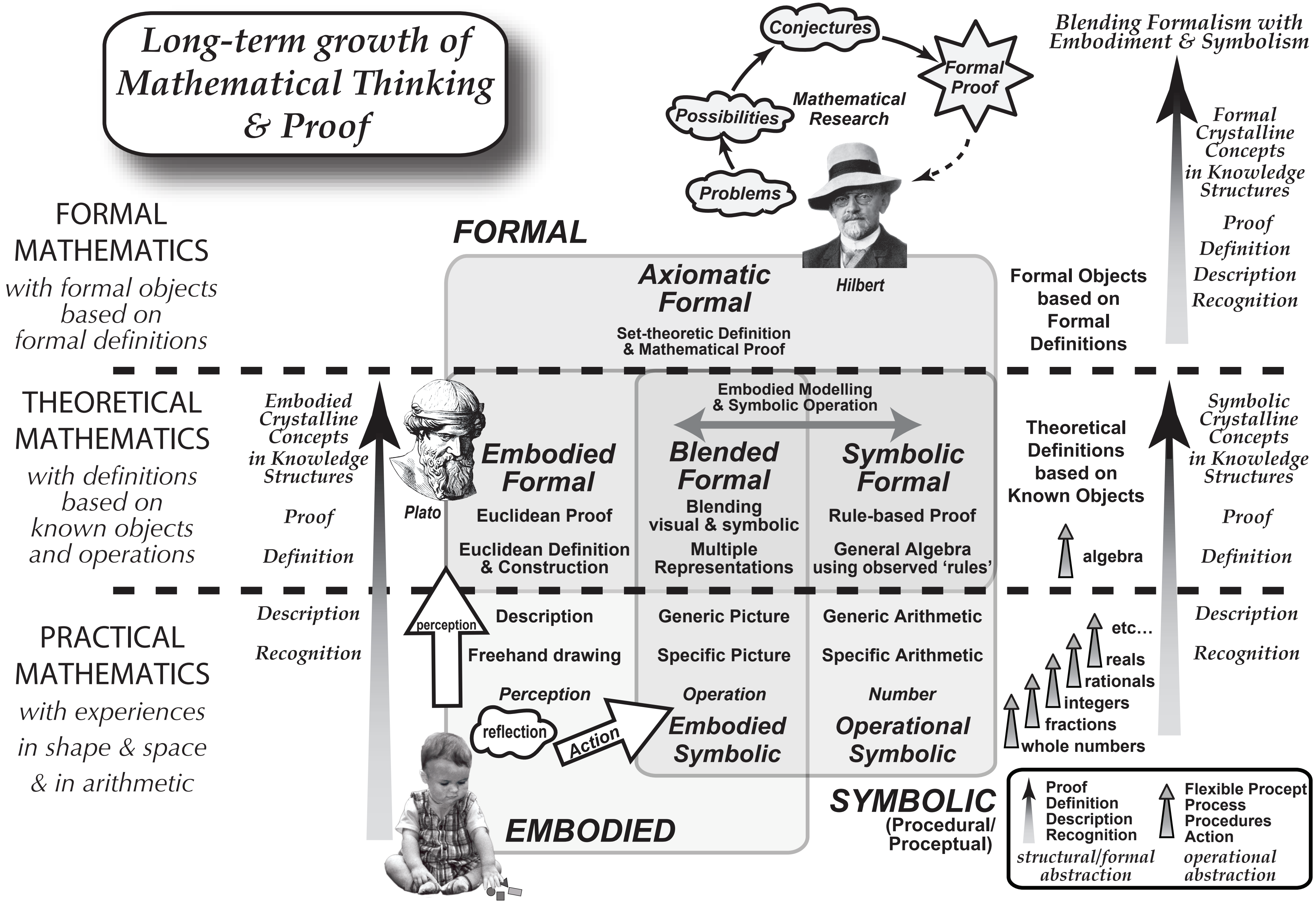
THEORETICAL
MATHEMATICS

with definitions
based on
known objects
and operations

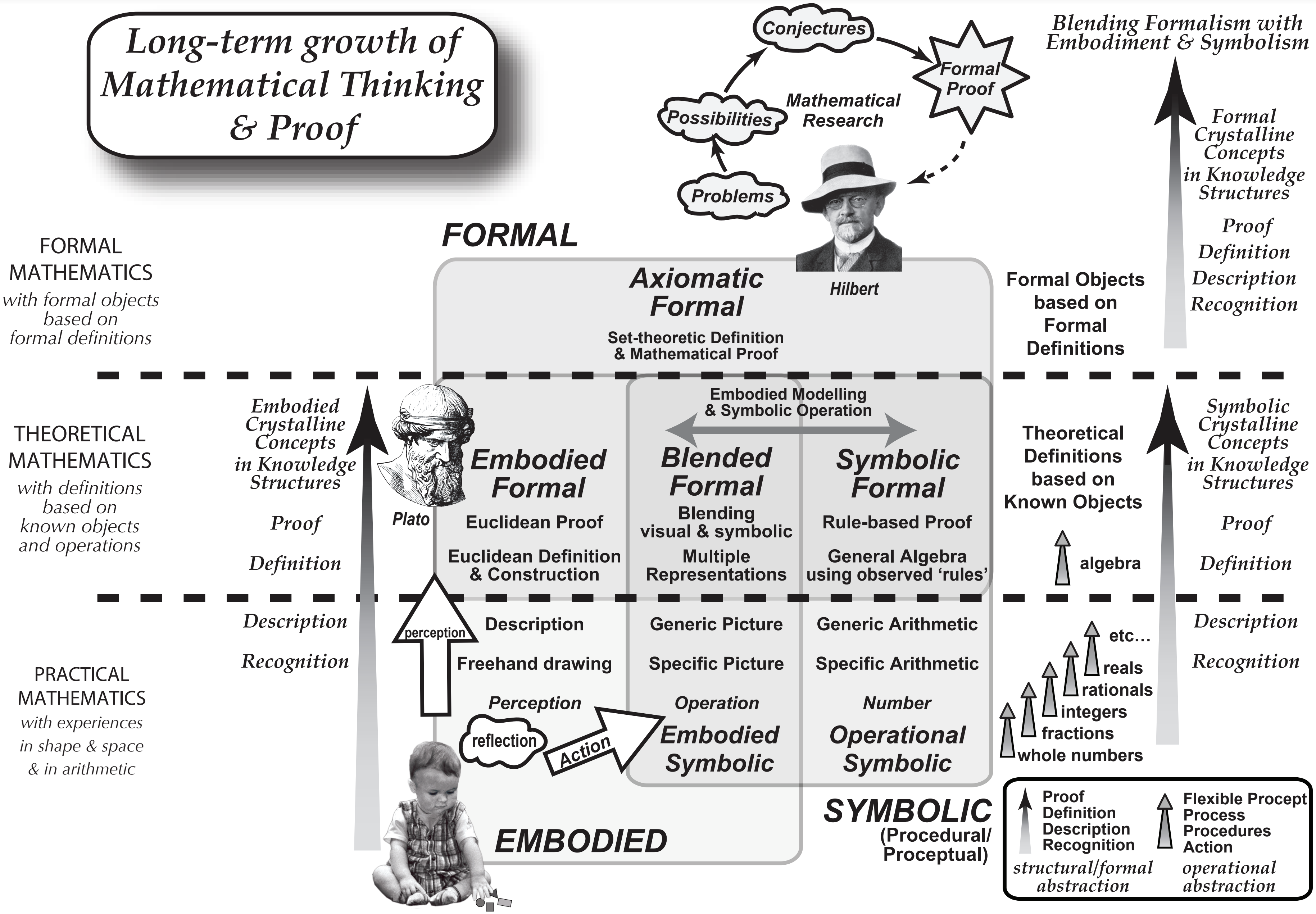
PRACTICAL
MATHEMATICS

with experiences
in shape & space
& in arithmetic

The full structure



The full structure



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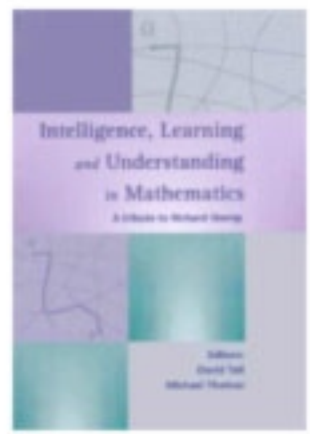
David Tall - Professor in Mathematical Thinking

Welcome to the **HOME** page of my website. A number of new papers (and drafts) have been added recently that refer to my latest developments on ***How Humans Learn to Think Mathematically***. Feel free to use information about my **research** as a resource, or **download** a paper. There is NEW **news** about recent changes on this site (made on **Tuesday 24th April, 2012**), and also **drafts** of earllier papers and **links** to other sites of interest.
See below for more information, including my students and my supervisors/mentors back via Newton and beyond.

- information on several **research themes** with links to relevant papers:
  - cognitive development | concept image | cognitive units | cognitive roots | generic organisers
  - procepts | algebra | limits, infinity & infinitesimals
  - visualization | calculus (& computers) | computers in school and college
  - problem solving | advanced mathematical thinking | proof
  - *three worlds of mathematics*
  - *lesson study*
  - *How Humans Learn to Think Mathematically* NEW
- **a glossary of terms**
- **published books**
- **research students** and joint publications
- **my supervisors and their supervisors/mentors** back to Isaac Newton, Galileo and Tartaglia



Downloads: **Research Papers** in PDF format, **Current writing in draft**, **Selected Lectures**, **Curriculum Vitae**.

News: A continual list of information on updates to focus on the most recent additions, including new papers, information on books including **Fermat's Last Theorem** (3rd Edition with Ian Stewart) and Intelligence, Learning and Understanding: **A Tribute to Richard Skemp** (ed. with Michael Thomas).

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Published Articles on Mathematics & Mathematics Education+recent drafts

See also other pages for **Earlier Drafts**, **Selected Lectures**, **Curriculum Vitae**.

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Go directly to papers published in 1975, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 2000, 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12

Some papers are also organised under themes or, where appropriate, with research students.

See also: **Earlier draft papers** | **Selected lectures** | **Curriculum Vitae**

Items marked x are drafts still under development and may later be UPDATED

2012x *How Humans Learn to Think Mathematically*, Chapter I. [from forthcoming book, CUP (USA)].

2012x David Tall (2012). Making Sense of Mathematical Reasoning and Proof. Plenary to be presented at *Mathematics & Mathematics Education: Searching for Common Ground: A Symposium in Honor of Ted Eisenberg, April 29-May 3, 2012, Ben-Gurion University of the Negev, Beer Sheva, Israel.* UPDATED **Overheads**.

2012x *How Humans Learn to Think Mathematically,* Chapter I. [from forthcoming book, CUP (USA)].

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2012x Mercedes McGowen & David Tall (2012). Flexible Thinking and Met-befores: Impact on learning mathematics, With Particular Reference to the Minus sign. (Draft).

2012x Kin Eng Chin & David Tall (2012). Making Sense of Mathematics through Perception, Operation & Reason: The case of Trigonometric Functions. (Draft).

2012x David Tall & Mikhail Katz (2012). A Cognitive analysis of Cauchy's conceptions of function, continuity, limit, and infinitesimal, with implications for teaching the calculus. (Draft) UPDATED

2012x Nellie Verhoef & David Tall (2012). The Complexity of Lesson Study in a European Situation. (Draft)

2012x David Tall, Rosana Nogueira de Lima & Lulu Healy (2012). Evolving a three-world framework for solving algebraic equations in the light of what a student has met before. (draft).

2012x David Tall (2012) A Sensible Approach to the Calculus. To appear in *Handbook on Calculus and its Teaching*, ed. François Pluvinage & Armando Cuevas.

2012x David Tall (2012). The Evolution of Technology and the Mathematics of Change and Variation. To appear in Jeremy Roschelle & Stephen Hegedus (eds), *Democratizing Access to Important Mathematics through Dynamic Representations: Contributions and Visions form the SimCalc Research Program*. Springer.

2012b Mikhail Katz & David Tall (2012). The tension between intuitive infinitesimals and formal analysis. In Bharath Sriraman, (Ed.), *Crossroads in the History of Mathematics and Mathematics Education, (The Montana Mathematics Enthusiast Monographs in Mathematics Education 12)* pp. 71–90.

2012a David Tall, Oleksiy Yevdokimov, Boris Koichu, Walter Whiteley, Margo Kondratieva, Ying-Hao Cheng (2011). The Cognitive Development of Proof, (ICMI 19: *Proof and Proving in Mathematics Education.*)