AN EXAMPLE OF THE FRAGILITY OF A PROCEDURAL APPROACH TO SOLVING EQUATIONS

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In this paper we consider the performance of students on quadratic equations after they had developed procedural methods of solving linear equations by shifting symbols using rules such as 'change sides, change signs' (Lima & Tall, 2008). Knowing their difficulties with algebraic manipulation, the teachers had focused first on simple quadratics with only two non-zero terms and then on the use of the formula which they considered as a universal method to solve all quadratic equations. The consequence was that longer-term only a few students could solve a quadratic equation using the formula, and a further procedural embodiment was adopted by a minority of students to solve an equation such as $x^2 = 9$ *by 'passing the power over the other side and changing it to a root' to get the single solution* $x = \sqrt[2]{9} = 3$*. Other than this all other methods involved procedural symbol manipulation, often leading to error. While it is the duty of mathematics educators to improve student learning, it is also a responsibility to understand why so many students end up performing 'rules without reason' that lead to failure.*

We discuss this phenomenon in terms of embodied cognition in which the procedural embodiment of symbol-shifting is preferred to any embodied model (such as a balance to represent an equation) and the manipulation of symbols fails to have the richer meaning proposed by theories of process-object encapsulation. This long-term failure is is placed within a wider theoretical framework incorporating human embodiment and symbolic manipulation.

INTRODUCTION

In this paper we consider data collected from students who had learned procedural ways of solving linear equations (Lima and Tall, 2008) and were now dealing with quadratic equations. Their responses to the solution of linear equations was described as a 'procedural embodiment' in which the symbols were shifted around accompanied by rules such as 'change sides, change signs.' Their teachers were concerned with the students' difficulties and focused on teaching the students to perform procedures that focused on being able to solve quadratic equations and to perform well on the tests. The syllabus specified that the students should be introduced to three methods of solution—by factorization, by completing the square and by using the quadratic formula. The teachers covered all three but moved on quickly to the use of the formula in the belief that this would enable them to solve any quadratic equation that would be given in a test. As we shall see, most of the students concerned continued to use procedural symbol-shifting and were unable to make sense of the solution of an equation such as $(x-2)(x-3)=0$.

The data collected in this study came from a collaboration in which the first author worked with a group of teachers and their students who were aged 14 to 15 and had first studied the solution of linear equations at least two years before and quadratic equations for at least a year. She encouraged the teachers to cooperate by developing tests to investigate what the students remembered about the solution of linear and quadratic equations.

In solving linear equations, although the teachers built their approach initially on ‘doing the same thing to both sides’, the students remembered not the general principles but the specific acts that they performed as they solved the equations. This involved shifting symbols around in their imagination and on paper, such as ‘move a term to the other side and change its sign’. Lima and Tall (2008) termed such operations ‘procedural embodiments’ because the procedures involved the embodied movement of symbols as mental entities being moving around, with additional rules to get the right answer.

Two specific rules dominated:

1) ‘change sides change signs’

in which, for instance, the equation $3x-1=3+x$ is operated upon by shifting the 1 to the right and the x to the left and changing signs to get:

$$3x-x=3+1$$

$$2x=4.$$

2) ‘change sides and put it underneath’

in which the 2 is moved over and put underneath to get

$$x=\frac{4}{2}=2.$$

In an attempt to use such rules, some students made mistakes, including changing $2x=4$ to

(a) $x=4-2$ (b) $x=\dfrac{4}{-2}$ (c) $x=\dfrac{2}{4}$

In (a) the 2 is passed over the other side and its sign is changed; (b) correctly ‘shifts the 2 over and puts it underneath’ but also ‘changes the sign’; (c) shifts the 2 over and puts the 4 underneath. This reveals the fragility of using procedural embodiments that may be misremembered and lead to a wider range of errors.

For instance, one student began the solution of the above equation by changing sides in an incorrect manner to get:

$$3x-1=3+x$$

$$3x-3x=+1.$$

Here the -1 on the left is shifted to the right to give $+1$, but the $3+x$ on the right becomes $-3x$ on the left and the left hand side reduces to $3x-3x$. The student then writes

$$0=\frac{1}{0}$$

which may involve 'moving the number 0 over the equals sign and putting it underneath'. This equation is now problematic, but the student 'moves towards a solution' by completing the line as

$$0=\frac{1}{0}=0$$

to declare the (erroneous) 'answer' to be zero.

Analysing all the responses, we found that no student explicitly verbalised the principle of 'doing the same operation to both sides' (although this may have been implicit in some solutions). Instead, all written solutions, whether successful or not, were written in a manner consonant with the use of procedural embodiments.

The majority of students involved had learned procedural methods which enabled a few to produce correct solutions but the majority either offered no solution or used the methods in fragile ways that led to error. In this paper we consider what happened when these students moved on to study quadratics, basing our analysis within a context of relevant research literature.

LITERATURE REVIEW

The literature considered here involves two distinct strands: the specific literature on linear and quadratic equations, and relevant theoretical frameworks related to embodiment and symbolism. The latter include process-object theories and embodied cognition.

Research studies on linear and quadratic equations

Lima & Tall (2008) reviewed the broad literature regarding the teaching and learning of linear equations. Some studies diagnose students' mistakes (e.g. Matz, 1980; Sleeman, 1984; Payne & Squibb, 1990; Freitas, 2002) in terms of mal-rules that involve erroneous forms of operations; others discuss the understanding students have about equations (e.g. Dreyfus & Hoch, 2004); and others show students from different countries making similar (if not the same) mistakes, related to their misinterpretation of solution techniques, and the lack of meaning attributed to the mathematical symbols (Linchevski & Sfard, 1991; Cortés & Kavafian, 1999). We also considered research studies that attempted to minimise students' difficulties by using concrete models (e.g. Vlassis, 2002; Filloy & Rojano, 1989). According to Vlassis (2002), these models have been shown to be effective in helping students to understand the equality between the two sides of an equation in simple cases, but they do not support more general situations in which negative or non-integer numbers are involved.

Research on students solving quadratic equations is much less extensive. Vaiyavutjamai and Clements (2006) reported that they found little research addressing the cognitive challenges students encounter with quadratic equations. Their research on *Effects of classroom instruction on students' understanding of quadratic equations* analysed written tests and interviews to compare the responses of students in Thailand and Australia. They found that students could produce correct solutions while still having serious misconceptions.

In addition, even when students could successfully write down a solution, Thorpe (1989) reported that the '±' sign in expressions such as $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ may not be meaningfully understood.

Gray and Thomas (2001) reported a teaching approach to quadratic equations using paper and pencil and graphic calculators. The students were asked to plot the graph of functions associated with the equation, and to find its solutions in various ways. They were able to perform a range of individual tasks yet lacked the flexibility to move easily from one representation to another.

Our students in this current study already have a history of lacking flexibility with symbol manipulation and understanding the procedures to solve equations. Instead of any conceptual insight such as a balance or an overall principle of 'doing the same thing to both sides', they move symbols around in a procedurally embodied manner in an attempt to simplify the equation to give a solution. As we analyse their work with quadratic equations, we draw on the literature of embodiment and of mental manipulation of symbols.

Theories of cognitive development

Process-object theories state that individuals learn by *encapsulating* (Dubinsky, 1991) or *reifying* (Sfard, 1991) a process into an object. APOS theory (Dubinsky, 1991), states that this transformation goes from ACTIONS to PROCESSES to OBJECTS, which are then organized in SCHEMAS; reification theory (Sfard, 1991) suggests that operational conceptions are condensed and reified into structural conceptions. Both theories observe that encapsulation (reification) might not occur, so that students continue only to use procedures manipulating symbols which Sfard calls 'pseudo-structural objects' that lack flexible meaning (Linchevski & Sfard, 1991).

The erroneous solutions we observed in solving linear equations certainly involved what Matz (1980) called mal-rules. However, underlying these rules are mental acts of shifting symbols around in a manner that involves picking a term up and putting it somewhere else, with an extra ingredient (such as 'change sign') to get the correct answer. This in turn involves a sensori-motor action of moving objects, now performed imaginatively in the mind and the result transferred onto paper.

The term 'embodied cognition' refers to cognitive theories that give priority to bodily experiences as sources of conceptual meaning (Lakoff & Johnson, 1980; Lakoff 1987; Lakoff & Núñez, 2000). Lakoff and Núñez (2000, p. xii) state that *"human ideas are, to a large extent, grounded in sensori-motor experience"*. This suggests that mathematical reasoning, which involves human ideas, is also grounded in sensori-motor experience.

Lakoff and his colleagues argue that the relation between mathematical reasoning and bodily experiences is made by conceptual metaphors, *"a cognitive mechanism for allowing us to reason about one kind of thing as if it were another"* (Lakoff & Núñez, 2000, p.6); they propose that it is by means of these metaphors that individuals learn *all* mathematical concepts.

In the solution of linear equations, the students in our study do not use the *conceptual* metaphor of a balance, they use a *functional* metaphor relating to the sensori-motor shifting of symbols as objects, including additional aspects (such as 'change sign') to give a correct solution. This strategy is capable of being integrated into a perfectly coherent method of solving equations that gives correct results. However, if the procedural solution process does not link to any appropriate conceptual meaning, it may be fragile and the student may begin to make errors attempting to remember the 'correct' rule, as happened above in the solving of $2x = 4$.

Process-object theories and theories of embodied cognition are each able to give insight into some aspects of the thinking processes involved. However, process-object theories focus on the shift from process to object with far less attention to bodily experiences in learning. At the same time, the embodiment of Lakoff and his colleagues does not refer explicitly to the compression of process into mental object through encapsulation.

To analyse our data in a way that offers a clearer insight into the meanings students give to equations and how they understand the rules they use to solve them, we realised the need for a theoretical framework that integrates embodiment with theories of process-object encapsulation.

THE THREE WORLDS OF MATHEMATICS

A theoretical framework integrating embodiment, process-object encapsulation and formal mathematical proof has its origins in the early nineties, with the development of the book on "Advanced Mathematical Thinking" (Tall, 1991). In its last chapter, Tall proposed the existence of (at least) three different kinds of mathematics, one (as in Euclidean geometry) through focusing on properties of objects and the relationship between those properties, one encapsulating processes into concepts as in arithmetic and algebra, and the third being Hilbert's formalist view of mathematics based on set-theoretic definitions and proof. This has subsequently developed into a practical framework to explain the cognitive development of mathematics of individuals from birth to adulthood, focusing on three distinct ways of thinking that mature over the years (e.g. Gray & Tall, 1994; Tall, 1995; Gray, Pitta, Pinto, Tall, 1999; Gray & Tall, 2001; Watson, Spirou & Tall, 2003; Tall 2004; Tall, 2006; Tall, 2008).

The first kind of mathematics is termed 'conceptual embodiment' and refers to the way in which an individual begins by interacting with physical objects and matures by thinking about them as thought experiments, focusing on their properties and building up relationships. One branch gives the conceptual embodied world that builds towards Euclidean geometry and beyond. Another branch focuses on actions, initially on physical objects, but later on mental objects, such as counting, sharing, adding, subtracting, multiplying. These processes are symbolised as mathematical operations and may be compressed into thinkable concepts (procepts) such as number, sum, product, fraction, algebraic expression, and so on. This gives a new kind of mathematics that develops long-term through what was initially termed the 'proceptual symbolic world' (e.g. in Tall, 2004). This incorporated the desire to build *flexible* thinking with symbols as processes or manipulable objects, but, as we see in this study, many students do not develop flexible ways of operation. They develop procedural ways that involve the successive steps of a learned procedure, often supported by an underlying procedural embodiment. Thus the world of symbolism in arithmetic and algebra is more properly defined in terms of both proceptual (flexible) thinking and procedural opoerations.

While some students continue to see the growth of arithmetic and algebra in a simple flexible way, compressing sequential operations into flexible manipulable concepts, others build increasingly complicated procedures that are likely to become increasingly unstable. It is the latter that is happening with most students in this study.

The two worlds of (conceptual) embodiment and (procedural/proceptual) symbolism interact throughout school mathematics. In what follows, these two developing worlds of mathematics will be termed 'embodied' and 'symbolic', on the understanding that these terms are used with the extended meanings given above. When we say 'embodied', we mean 'conceptual embodied' and when we say 'symbolic' we mean 'procedural or proceptual symbolic'. Our main concern in this study refers to the combination of procedural embodiment and procedural symbolism that appears in the working of the students concerned.

A fuller explanation of the framework of the three worlds of mathematics can be found in Tall (2008). In formulating this framework we attempt to remain as consistent with various other theoretical frameworks as possible, even though these frameworks may use the same terminology in different ways. For instance, in mathematics education the term 'formal' is often used in a Piagetian sense (for instance, in Fischbein's (1987) three-part theory of intuitive, algorithmic and formal approaches to mathematics). However, in mathematics, the term 'formal' is usually reserved for a more sophisticated form of thinking characterised by Hilbert in terms of axiomatic structures and mathematical proof.

In outline, the three-world model is represented in figure 1 as cognitive growth begins in the bottom left hand corner with the child interacting with the environment, building increasingly sophisticated descriptions and definitions upwards until reaching an embodied form of proof characterised by Euclidean geometry.

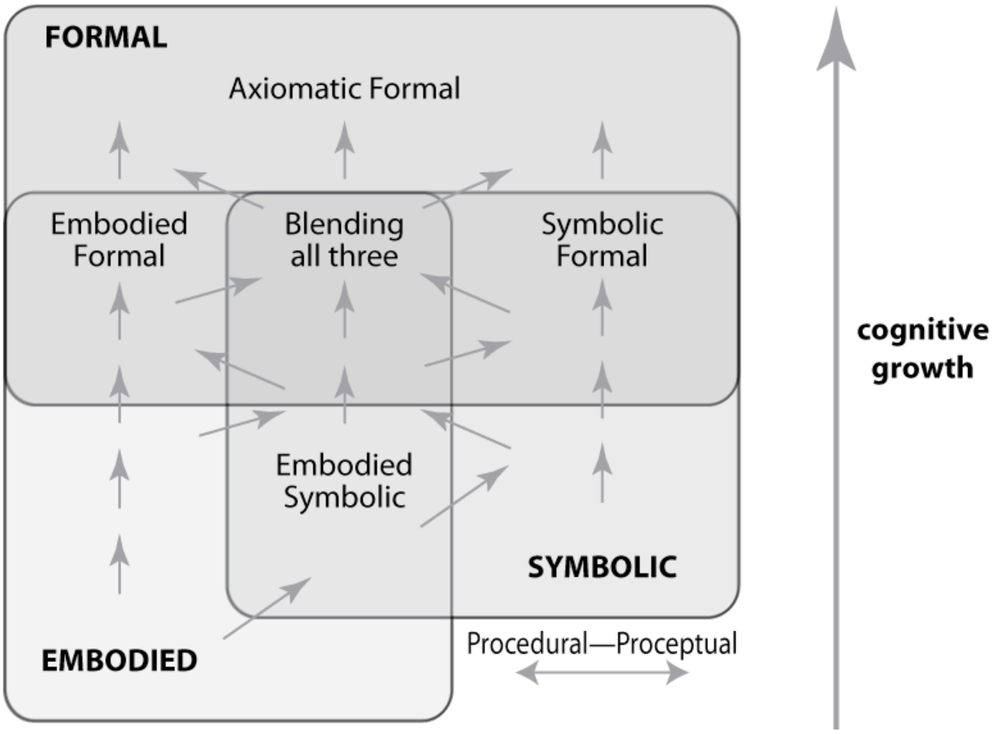

Figure 1: The three worlds of mathematics

Embodied actions such as counting operate in a combination of embodiment and symbolism which may shift over to the encapsulation of symbolic entities such as number. On the right-hand side, increasingly sophisticated encapsulation of process into mental object gives various successive forms of number: whole numbers, fractions, signed numbers, rationals, real numbers and so on and generalised arithmetic processes give rise to algebra and a symbolic form of proof based on the specified 'rules of arithmetic'.

The formal world is characterised by making deductions from definitions, in at least three different ways, as embodied formal in Euclidean geometry based on definitions of figures and principles such as congruence and parallel lines, as symbolic formal in algebra based on the rules of arithmetic and as axiomatic formal based on set-theoretic definitions and mathematical proof. Some forms of proof blend together two or more forms, for instance, spherical geometry is a blend of embodiment of space and the symbolism of spherical trigonometry.

This framework is in keeping with the viewpoint of the Advanced Mathematical Thinking Group of PME[2] (Tall, 1991), in which formal ideas introduced in school mathematics may function as a transition to the full axiomatic formalism of university pure mathematics. It also brings the use of the term 'formal' closer to that

[2] The International Group for the Psychology of Mathematics Education

of Fischbein (1987), which incorporates embodied, symbolic and axiomatic formal aspects.

In this study we see that the development of algebraic solution of equations could be based formally on the principle of 'doing the same thing to both sides'. However, few of the students involved use such a principle, instead they base their solutions of equations on embodied actions shifting symbols around, with additional aspects to produce the desired answer.

What students have met before

In considering the development from linear to quadratic equations, we need to examine how the earlier learning impacts on the later experiences. Tall (2008) introduced the term 'met-before' to describe 'a mental structure we have now as a result of experiences we have met before.' In order to cope with an unfamiliar task, different met-befores may be blended together (Tall, 2008; Lima & Tall, 2008). However, such a blending can lead to two quite different effects. There may be aspects of the blend that give increased power from putting new ideas together, giving great pleasure to the learner. On the other hand, there may be aspects that do not fit together neatly that cause anxiety and confusion. Thus blending can offer both a positive advantage and a negative impediment, yielding a broad spectrum of possible performance from those building powerful ideas with pleasure, through to those who struggle to cope with the conflict, to those who can make no sense of the situation at all.

Fauconnier and Turner (2002) focus mainly on the positive side of blending that leads to creativity and continual development of new mathematical ideas. However, the blending of old experience that does not fit has long been known as an epistemological obstacle (Bachelard, 1938). The notion of 'met-before' therefore includes both aspects that are supportive and also aspects that are problematic.

The transition from arithmetic to algebra involves both embodied and symbolic met-befores. Symbolic met-befores come from familiarity of the operations of arithmetic and a sense of the generalized operations of arithmetic that are symbolized in algebra. Problematic aspects arise from various sources, such as the met-before that an arithmetic expression is always a cue to calculate an answer, while an algebraic expression cannot give an answer unless the numerical values of the variables are known (the 'lack of closure' obstacle, Collis, 1978). Another met-before is the experience that the equals sign involves an expression on the left to be evaluated to give an answer on the right (as observed by Kieran, 1981). Embodied met-befores may arise from the use of physical or mental representations, such as the notion of balance to represent an equation. While the notion of a balance is often initially helpful, it may become problematic when dealing with equations with negative terms that no longer fit the specific embodiment. It was this effect that was noted by Vlassis when the balance idea was generally helpful in the first stages of simple equations but became problematic as the equations became more complex.

In this study we will see the influence of supportive and problematic met-befores as students shift from linear equations that already have problematic aspects to quadratic equations that introduce new features.

THE RESEARCH STUDY

The data presented in this paper is part of a doctoral study (Lima, 2007), developed at PUC/SP[3] (Brazil) and the University of Warwick (UK). The research arose from a combined study with the first author sharing ideas with a group of high-school teachers whose objective was to examine their current teaching practices to seek ways to improve their teaching. The researcher encouraged the teachers to carry out their own ideas and shared in the design of research instruments and the collection of data. The data came from 80 high school students in three groups, one of 32 14-year-olds, one of 28 15-year-olds, both from a public school in the city of Guarulhos/SP; and one group of 20 15-year-olds from a private school in São Paulo/SP, all of them had already been taught how to solve linear equations at least two years before this research took place, and quadratic equations at least one year before this research.

In the wider study, there were three data collections, each one administered by the class teacher in a lesson lasting 100 minutes. The first invited the students to construct a concept map of their knowledge of equations, the second was a questionnaire and the third was an equation-solving task. After an initial analysis of data, twenty students were selected for interviews, conducted by the researcher, in the presence of an observer, and tape recorded for further analysis. Of these twenty students, 14 were female and 6 were male. They were not chosen by gender, but by the kind of work they presented – including either typical mistakes or correct answers. In the interviews, we wished to investigate why students performed as they did. In particular, they were asked to explain what kind of symbol manipulation they had performed and why they believed it was a proper way to proceed. In this paper, we focus specifically on the work students performed when they had to solve quadratic equations. (Detailed analyses of other parts of the study can be found in Lima & Tall, 2006a; Lima & Tall, 2006b; Lima, 2007; Lima & Tall, 2008.)

Teaching quadratic equations

The teachers in this study reported that they were deeply concerned about the difficulties their students had encountered with linear equations but had to move on to keep up with the syllabus. They therefore decided that two specific objectives were necessary. First, that the general idea of solving quadratic equations should be addressed by considering simple examples. Then, in the knowledge that the students already had considerable difficulties in manipulating algebra, that they should focus on the method most likely to give success in the examination.

In general, the teachers in the study reported that students were taught three symbolic methods of solving quadratic equations:

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[3] The Pontifical Catholic University of São Paulo.

1) Factorizing the expression into two linear factors and using the principle that if the product is zero, then one of the factors must be zero.
2) Completing the square for the given quadratic.
3) Manipulating the equation to get a quadratic expression equal to zero and solving a general quadratic $ax^2 + bx + c = 0$ using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Teachers in Brazil are expected to teach all three symbolic forms of solving quadratic equations mentioned above. However, to simplify the complexity, knowing the difficulties already experienced by the pupils, the teachers decided that the general method of using the formula would work in *all* cases and so this would be emphasized. Method (1) was used to mainly to solve equations involving just two terms, such as $x^2 = 4$ or $x^2 - 3x = 0$. Method (2) was shown, but not emphasized.

The two-term equation $x^2 = 4$ can be solved directly by taking the square root

$$x = \pm\sqrt{4}$$

and therefore, $x = \pm 2$.

The second equation may be factorised as

$$x(x - 3) = 0$$

to see that by substituting $x = 0$ or $x = 3$ then the equation is satisfied. In this way, the solution of the equation involved a calculation as usual, to calculate the left hand side by substituting numeric values for the variable, to check that the equation is satisfied. The idea that the product of two factors can only be zero if one of the factors is zero was considered, but the method of factorization in general was not emphasized because the teachers were aware of the students' difficulties with algebraic manipulation. In general, the teachers focused on the use of the formula that they believed would work in all cases.

As in their previous handling of linear equations, the students focused on the specific operations that they used to solve the equations, rather than any general principles. The emphasis on the use of the formula in preference to other methods was reported by a 15-year old in interview, saying:

> *"When I looked at many of those [quadratic equations], I thought of the quadratic formula, I don't know why. It could be wrong, I don't know, but that's what I've thought. Because the other teacher that I had, she has always said like 'look, when you see this [a quadratic equation], you have to think of the formula'. (...) I remember that the teacher said that the formula needs a squared [unknown] with the [coefficient]* ***a****, then [coefficient]* ***b*** *together with the unknown, and then a number by itself."*

In general, even after these teachers had offered practice in solving specific quadratic equations, we will see that the overall impression formed by most students is the dominance of the formula.

Tasks with quadratic equations

The data used to investigate the students' conceptions of quadratic equations came from two instruments, an equation-solving task, with four equations:

$$3l^2 - l = 0\,;\; r^2 - r = 2\,;\; a^2 - 2a - 3 = 0\,;\; m^2 = 9$$

and a questionnaire that included two quadratic equations:

$$t^2 - 2t = 0\,;\; (y-3)(y-2) = 0\,.$$

The questionnaire also included a request to respond to the solution of the final quadratic equation as given by an imaginary student "John":

> To solve the equation $(x-3)(x-2)=0$ for real numbers, John answered in a single line that:
>
> "$x = 3$ or $x = 2$."
>
> Is his answer correct? Analyse and comment on John's answer.

Figure 2: John's Problem (question 8 of the questionnaire).

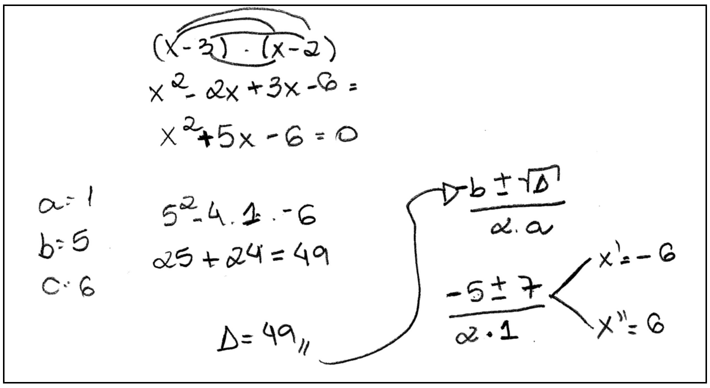
Interviews with selected students gave additional personal comments on how they interpreted the tasks and their solutions, which gave insights into how they might be thinking. The next section considers data regarding the six equations and "John's problem".

DATA COLLECTION

A total of 68 students gave their answers to the equation-solving task and 77 students responded to the questionnaire, due to absences on the day each instrument was administered. From an analysis of all the instruments, our main findings are that the students mainly interpreted an equation as a calculation, building on their previous experience working with numbers. For instance, when asked, "what is an equation?" in the questionnaire, 36 out of 77 students (47%) answered that *"it is a calculation in mathematics"* or some equivalent response. Less than half the students mentioned the unknown. Instead, the responses often focused on the equals sign interpreted as a signal to perform a calculation consistent with their earlier experience of the use of the equals sign in calculations in arithmetic.

Specifically, in the equation-solving task and the two quadratics in the questionnaire, *not one student completed the square or used factorization*, not even in the case of equations $t^2 - 2t = 0$ or $3l^2 - l = 0$. Fifteen students (22%) solved the equation $m^2 = 9$ to find the solution but only one (of three students who used the formula) considered the negative solution. In total, 18 students (23%) used the formula in at least one of the six equations from either instrument. Apart from the use of the formula and the solution of $m^2 = 9$ by transforming it to $m = \sqrt{9}$, *all other methods involved an erroneous strategy* to translate the quadratic into some kind of linear equation.

Further evidence that the students took the formula as the "right" way to solve quadratics arises in the responses to "John's Problem" (Figure 2). Thirty students out of 77 (39%) claimed that his solution was correct. Three (4%) mentioned the formula saying things like *"He must have used the quadratic formula in his mind."* Eleven students (14%) declared that *"John didn't solve the equation"* essentially *"because he did not use the formula."* Four students (5%) used the formula to solve the equation and compared the result with John's solution. One of these used the formula incorrectly and obtained different values from John, insisting that John was wrong (Figure 3).



"Ah! I don't know, but I think that John is wrong and I think that my way is right; I said my way, not my results, ok?"

Figure 3: A student's use of quadratic formula and his comments about it.

Apart from the use of the quadratic formula, every written strategy used by the students attempted to relate back to their earlier experiences with linear equations, by somehow converting the equation into a linear form. Nine students (13%) simply replaced m^2, r^2 or a^2 respectively by *m*, *r* and *a*, and then solved the equation as if it were linear. Others used the exponent of the squared term to square its coefficient (Figure 4), and nine students (13%) replaced m^2 by 2*m*, often with an intermediate calculation that suggests m^2 represents 'two lots of *m*' or '*m* and *m*', which becomes 2*m* (Figure 5).

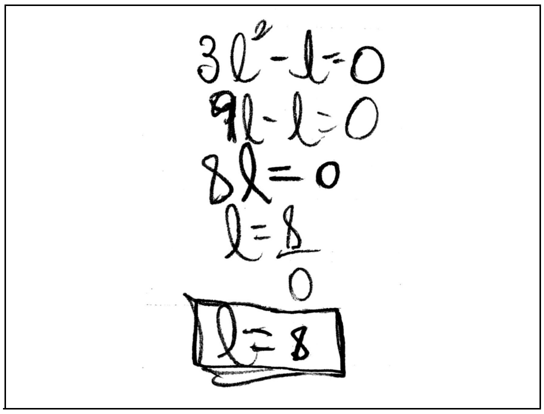


Figure 4: Multiplying the coefficient by the power

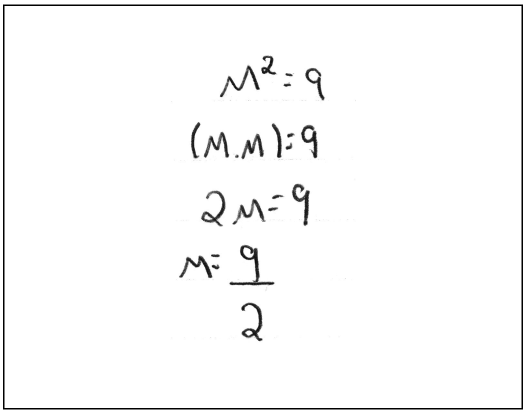


Figure 5: m^2 taken as the same as $2m$

In solving the equation $m^2 = 9$, several students responded as shown in Figure 6. In interview one of these students explained, *"the power two passes to the other side as a square root."* In this explanation, the student makes it clear that there is a movement of the exponent and a transformation in the power for a square root. In the interview, neither this student nor any other mentioned the possibility of another (negative) root.

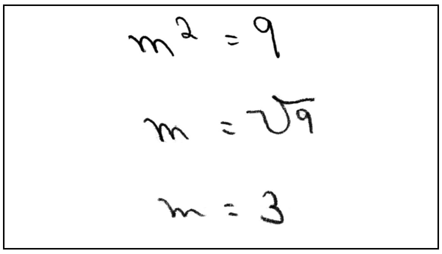


Figure 6: Passing the exponent to the other side as a square root

Satisfactory responses to "John's problem" (Figure 2) may also be regarded as involving movement of symbols, this time to "put" numerical values for the variable "into" the equation. Four students (5%) (three in the questionnaire and one during interview) said that John is right *"because putting* $x = 3$ *or* $x = 2$ *gives the number zero"*, while another two substituted each value in the equation (Figure 7).

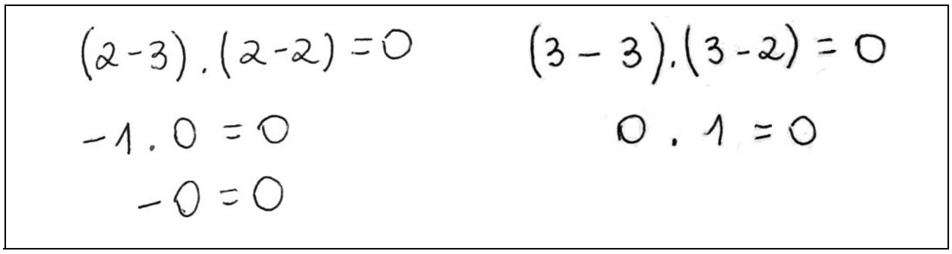


Figure 7: Replacing values for *x* in the equation.

One of those performing the substitution explained in interview:

> *Student: To see if the answer is right, I have put 3 here [in the place of x] to see what result I would get, and then another calculation with 2.*

Interviewer: Why have you put 3 in place of x, and then 2 in place of x?

Student: Because here it says that x is equal to 3 so, if x is 3, then I replace the number to see what I get.

Interviewer: And what happens if the result is the same as the one in the equation?

Student: If it is zero, then x is 3.

Not one student said that John's answer is correct by referring to the principle that guarantees that when a product is zero, one of the factors must be zero. Instead, some responses explicitly focused on *the need to carry out the calculation* to test whether the solution could be adjudged correct:

"*If he guesses that, as it equals zero, x should be 3 or 2, it is wrong. But maybe, he is very clever, calculated in his mind, and supposed that this is the answer.*"

or

"*I don't know, but I think it is wrong because he didn't do the calculation, he just put the results that were by the side of x.*"

DATA ANALYSIS

The data show that, although all the teachers said that at some stage they had showed methods other than the use of the formula to solve some quadratic equations, none of these methods were evident in the students' responses for either the questionnaire or the equation-solving task. It was not expected that students would complete the square to solve a quadratic equation, since the teachers gave it a low priority, and also because of the complexity it involves. However, it was expected that for equations $t^2 - 2t = 0$ and $3l^2 - l = 0$ students would factorize the expression and use the algebraic principle that if a product is zero, than one of the factors is zero.

This did not happen. There is a range of possible reasons for this. The teachers strongly emphasized the use of the formula as a general method to solve *any* type of quadratic; it was the last method to be taught and practiced and was therefore fresher in the mind. Meanwhile, in general, the other methods are more complicated. To get an answer, it is quicker and easier to quote the formula rather than go through the steps to complete the square. The method of factorizing to get a product equal to zero only works in simple cases and involves subtle algebraic manipulation. The main exception to this general principle is when the equation is given as a product, say $(x-3)(x-2)=0$, where the use of the formula requires an initial algebraic manipulation to get it into the form $ax^2 + bx + c = 0$.

Even the procedural use of the formula to solve the equation proved to be too difficult for most students. Of the 18 students who tried to use the formula, only seven were successful in solving at least one of the six equations. In general, even the use of the formula was unsuccessful.

Overall, the data reveals that, in building on an essentially procedural approach to linear equations, a small number of students had some success using the formula, a few added an additional procedural embodiment to solve $x^2 = k$ by shifting the power of 2 to the other side where it became a square root sign, $x = \sqrt{k}$. All other attempted solutions were either incomplete or used erroneous methods to rewrite the equation as a linear equation.

This experiment reveals the degeneration of procedural embodiments amongst this group of students as their procedural rules fall apart.

ORGANIZING A THEORETICAL FRAMEWORK

It may appear that the data collected in this study relates to the APOS theory of Dubinsky (1994), but only in the sense that the students' actions and processes were not encapsulated as objects, rather they were manipulated as 'pseudo-structural' entities in the sense of Sfard and Linchevski (1994). The data in this study goes beyond a single failure to encapsulate processes flexibly in linear equations to reveal a further deterioration at the next stage when solving quadratic equations.

The embodied theory of Lakoff (1987) also has some relevance in two distinct ways: the *conceptual* embodiment of an equation *as a balance* supports simple linear equations but fails to extend to more subtle cases where negative quantities do not fit the simple model. Instead the students build on a *procedural* embodiment that relates to how they think about their actions in terms of mentally moving symbols around and adding a touch of magic to get the right answer, such as changing signs, putting symbols underneath, or switching powers to roots. This method of procedural embodiment involves using an array of different rules in different contexts that for many students prove to be complicated and liable to error.

Theories of process-object encapsulation only operate here in the sense that the required encapsulation does not occur and the students operate in what Sfard terms a 'pseudo-structural' manner. In this study, virtually all the students operated in this way. Thus formulating the development in terms of APOS theory only operates in a reduced form in which Actions may possibly be seen as Processes, but not Objects. For this reason we see the importance in seeing the symbolic world of arithmetic and algebra having a *spectrum* of interpretations. At one end of the spectrum is a flexible compressed form of mathematics in which algebraic expressions can be mentally compressed and manipulated fluently as mental concepts. At the other is a more complicated procedural form of mathematics in which the student only operates using what Skemp called 'rules without reason'.

The distinction between instrumental understanding and relational understanding (Skemp, 1976) and between conceptual and procedural learning (Hiebert & Carpenter, 1992) has long established this phenomenon. Dubinsky and his colleagues (for instance, Dubinsky & Harel, 1992) show that college students often reach, at best, a process level of meaning for the function concept rather than as an object at a higher level (e.g. in a space of functions). Students who are measured on successive standards throughout their school lives are likely to focus on what they need to *do* to get through examinations, rather than to understand what the concepts are about, intimating that procedural operations may be desired to pass the test.

In arithmetic, Gray and Tall (1994) formulated the 'proceptual divide' in which some children continued to use procedural counting methods which limited their performance while others used the symbolism more flexibly and were able to develop ways of generating new relationships from known facts. In our study with linear and quadratic equations, we have a corresponding phenomenon in algebra where most of these students remain fixed in procedural methods of solution.

In a world where 'success' is measured in terms of knowing what to do in examinations, this data suggests that simply teaching students what to do and getting them to practice techniques may give unstable knowledge structures at one stage that fail even more seriously at the next.

The data is consistent with a broader interpretation of embodiment and symbolism as represented in the three-world model of mathematical development. There are two distinct forms of embodiment, one is the embodiment of perception, as involved in the conceptual embodiment of figures in geometry and mental thought experiments such as imagining an equation as a balance between the two sides, the other is the embodiment of action, as seen in the operations of arithmetic becoming number concepts and the possibility of the operations being conceived less flexibly as procedural embodiment shifting symbols.

The formal aspects observed in this study do not relate to the explicit use of the laws of arithmetic to manipulate symbols. They do not even relate to the general solution of equations by 'doing the same thing to both sides'. Instead specific rules are used based on the sensori-motor shifting of symbols, with additional aspects to get the correct answer such as:

1) for addition or subtraction: change sides, change signs:

e.g. $3x - 2 = 2x + 8$ becomes $3x = 2x + 8 + 2$.

2) for multiplication or division: change sides, put it underneath, or move it from the bottom and put it on the other side on top:

e.g. $2x = 4$ becomes $x = \frac{4}{2}$.

Those procedural embodiments are now met-befores in these students work when they move to solve quadratic equations. As they are used to such kind of embodiments, they develop two further procedural embodiments, both of which are inadequate:

3) for a square power: move it to the other side and change it to a root

e.g. $x^2 = 9$ becomes $x = \sqrt{9}$, revealing only one root.

4) to modify the square power of the unknown to see the equation in linear form:

e.g. i $3l^2 – l=0$ becomes $9l–l=0$ (by squaring the coeffiction) or $3l–l=0$ (by simply forgetting the power), both of which are erroneous.

The data reveals the severe limitations in this procedural approach. The analysis of the relationship between practical embodiments (such as a balance to model an equation) and symbolic algebra reveal the problems of short-term learning experiences that work with simple examples but can fail in long-term development of more sophisticated meaning.

DISCUSSION AND CONCLUSIONS

This paper has investigated how an understanding of linear equations based on procedural embodiments affects students' work with quadratic equations.

We find that, in the case of linear equations, the participants in this study did not build on embodied models, such as a balance, they do not encapsulate processes as mental objects, nor do they use formal principles such as 'do the same thing to both sides'. Instead of a conceptual embodiment underlying the symbolism of solving equations, such as a balance, they develop a procedural embodiment of symbol shifting in the mind and on paper to 'move towards' getting an answer. The procedural embodiments in linear equations involving 'change sides, change sign' and 'pass the coefficient over the other side and put it beneath' are extended with a new procedural embodiment to 'move the power over the other side and change it to a root'. In using the latter procedure, they invariably found only the positive root.

The teachers in this study are aware of the general nature of student difficulty, so they re-adjusted their goals to teach those aspects of linear and quadratic equations that, to them, seem to be the easiest and the most general. This is a natural goal to seek a level of success appropriate for students who are already finding algebra difficult. An approach that seeks positive advances is a widely used tactic to encourage students to succeed. However, this positive view needs to be balanced by the negative side: the met-befores that cause the students frustration, then anxiety, leading to a switch to the goal of seeking success in being able to carry out the procedures necessary to pass the examination. Here we have data that shows that such a strategy enabled a small number of students to be able to solve specific quadratic equations, but it did not help in general to encourage students to construct flexible meanings in algebra.

In considering the cognitive development of arithmetic, Gray and Tall (1994) formulated the notion of 'the proceptual divide' between those students who successfully developed flexible relationships between symbols operating dually as process or concept and those who remained locked in the use of lengthy counting procedures. Those who develop a flexible *proceptual* knowledge structure in arithmetic have a powerful generative engine to derive new facts from known facts while those who operate in a *procedural* manner have longer sequences of operations to perform that make arithmetic even more difficult for those who are already struggling. This research reveals a continuation of the proceptual divide into algebra where most of the students concerned do not develop a flexible proceptual use of symbolism that would make algebraic manipulation fluent and simple, instead they use procedural methods that have little conceptual meaning that is fragile and error-prone. We hypothesize that the difficulties that occur widely in algebra are an extension of the proceptual divide between those who develop flexible proceptual meaning in arithmetic and those who remain focused on lengthy procedural operations. Proceptual flexibility is the foundation of algebra as generalized arithmetic and gives meaning to the flexible manipulation of algebraic symbolism. Without a meaningful flexibility in arithmetic, an approach that focuses on procedural learning may lead, as here, to the use of procedural embodiments shifting symbols in a manner that may be fragile and prone to error.

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