THE UNIVERSITY OF WARWICK

What we have met before: why individual students, mathematicians & math educators see things differently

David Tall
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Set-befores & Met-befores

A ‘**set-before**’ is a mental ability that we are all born with, which make take a little time to mature as our brains make connections in early life.

The three major set-befores in mathematical thinking are

Recognition, Repetition, Language

A ‘**met-before**’ is a personal mental structure in our brain *now* as a result of experiences met before.

Many different met-befores are possible, depending on experiences available in our society at the time.

2+2 is 4 **after 2 comes 3** **addition makes bigger**

take-away makes smaller **multiplication makes bigger**

all expressions (such as 2+3, 22/7, 3.48x23.4) have answers.

The terms ‘set-before’ and ‘met-before’ which work better in English than in some other languages started out as a joke.

The term ‘metaphor’ is often used to represent how we interpret one knowledge structure in terms of another.

I wanted a simple word to use when talking to children.

When they use their earlier knowledge to interpret new ideas I could ask them how their thinking related to what was **met before**.

It was a joke: the word play met**A**phor, met**B**efore.

The joke *worked* well with teachers and children: What have you **met before** that causes you to think like this?

Set-Befores

The three major set-befores in mathematical thinking are
Recognition, Repetition, Language

Recognition + language allows classifying categories such as 'cat' and 'dog', triangle, square, rectangle, circle.

Repetition + language allows practising sequences of actions

… used in counting …
… column arithmetic …
… adding fractions …
… learning algorithms …

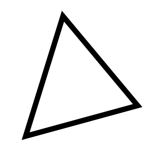
May be performed automatically without meaning

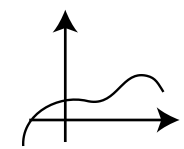
Symbols e.g. 3+2 may be compressed from process (addition) to concept (sum) to give flexible thinking (**procept**)

Building on Set-befores

In today's culture we have rich mathematical knowledge built over the centuries which we teach to our children.

- ***recognition*** leads to **embodiment** (in which we categorize and build knowledge structures about things we perceive and think about);
- ***repetition*** leads to **symbolism** through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (called procepts);
- ***language*** leads eventually to **axiomatic-formalism** (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.



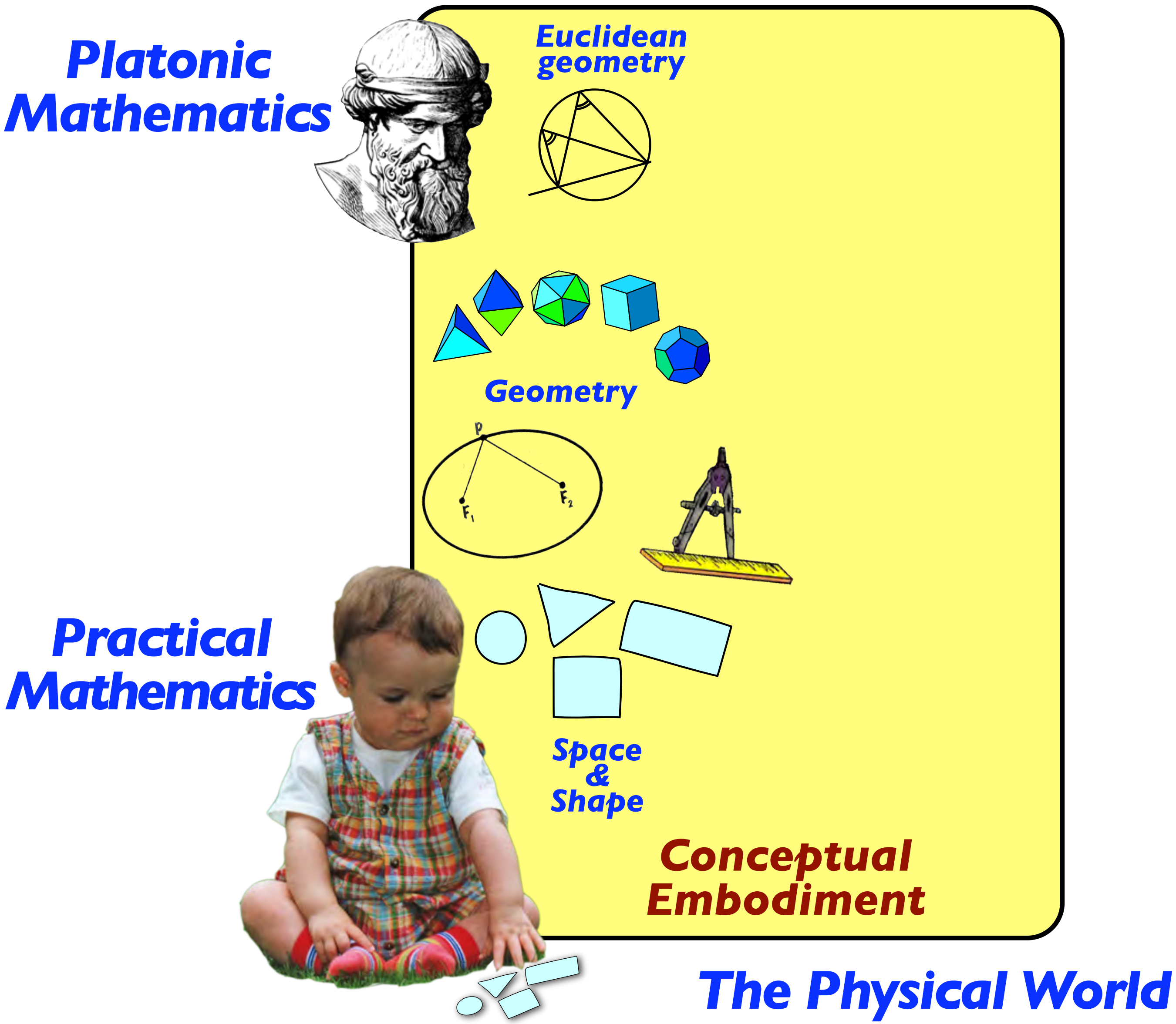


$3+4$

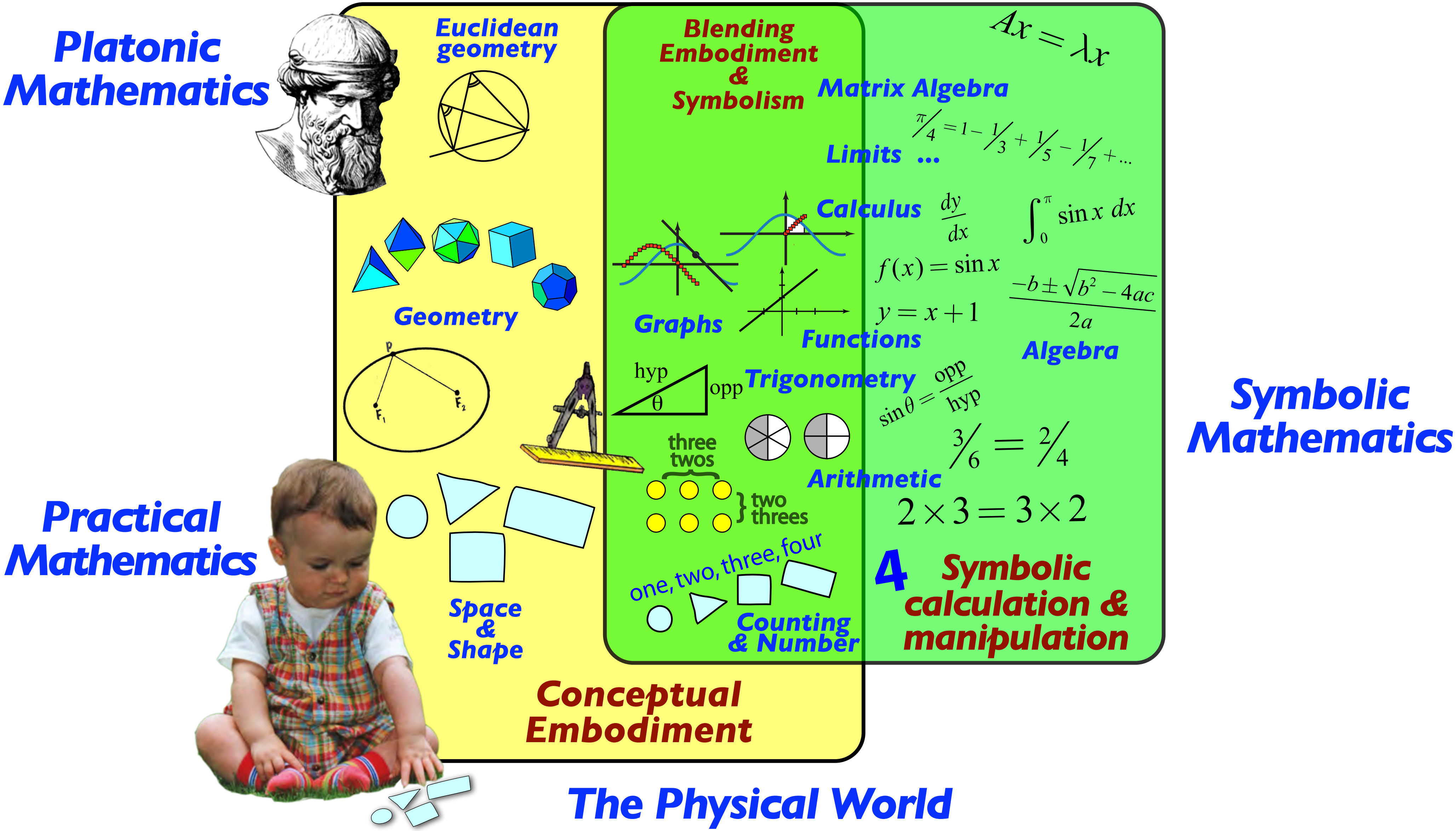
$\int \sin x \, dx$

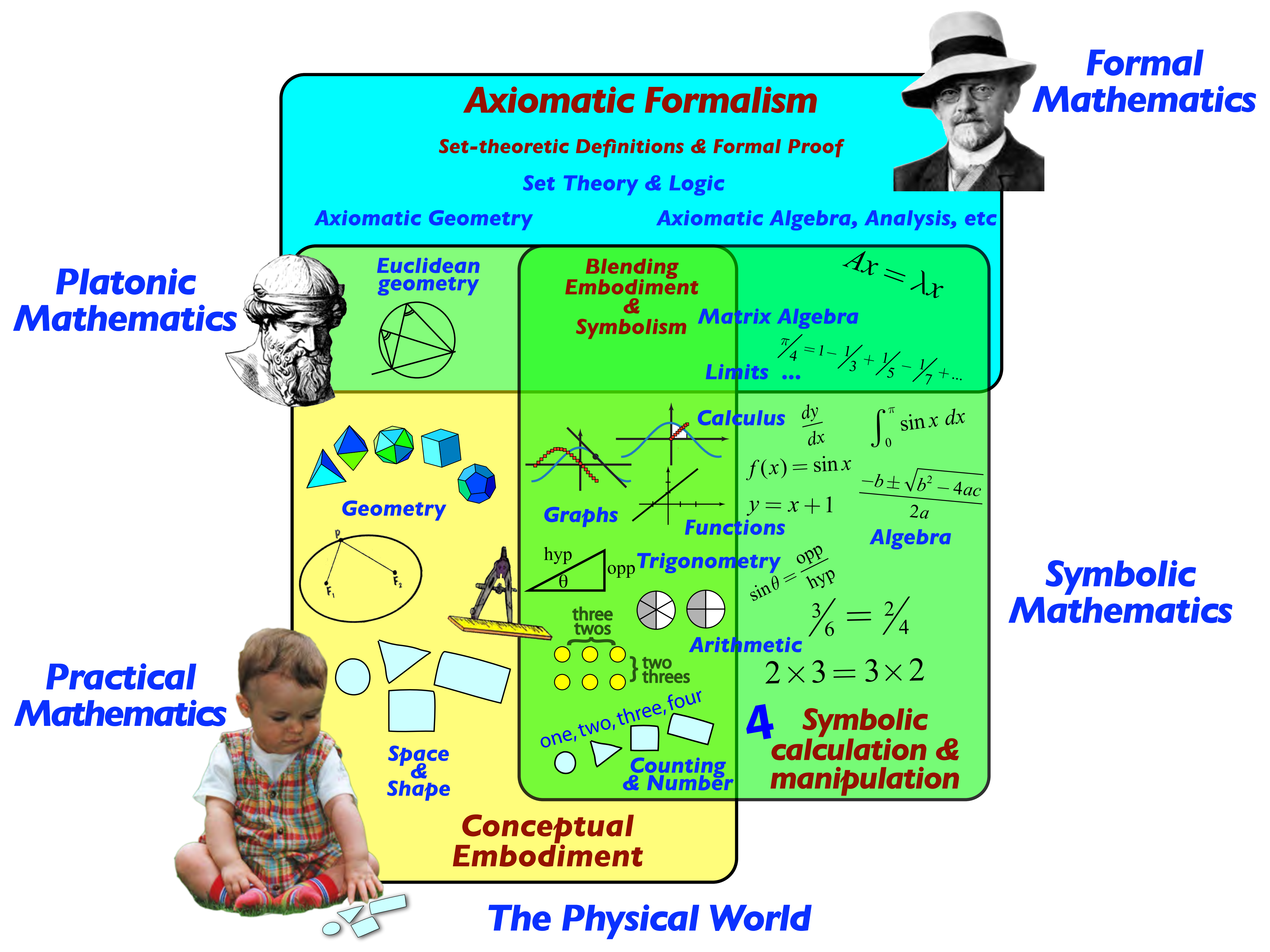
$\mathbb{R}$

$\aleph_0$



Platonic Mathematics
Euclidean geometry
Geometry
Practical Mathematics
Space & Shape
Conceptual Embodiment
The Physical World


Platonic Mathematics
Euclidean geometry
Blending Embodiment & Symbolism
$Ax = \lambda x$
Matrix Algebra
Limits ...
$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$
Calculus
$\frac{dy}{dx}$
$\int_0^\pi \sin x \, dx$
$f(x) = \sin x$
$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$y = x + 1$
Geometry
Graphs
Functions
Algebra
P
F_1
F_2
hyp
opp
θ
Trigonometry
$\sin\theta = \frac{opp}{hyp}$
Symbolic Mathematics
three twos
two threes
Arithmetic
$3/6 = 2/4$
$2 \times 3 = 3 \times 2$
Practical Mathematics
one, two, three, four
4
Symbolic calculation & manipulation
Space & Shape
Counting & Number
Conceptual Embodiment
The Physical World



Formal Mathematics
Axiomatic Formalism
Set-theoretic Definitions & Formal Proof
Set Theory & Logic
Axiomatic Geometry
Axiomatic Algebra, Analysis, etc
Platonic Mathematics
Euclidean geometry
Blending Embodiment & Symbolism
$Ax = \lambda x$
Matrix Algebra
$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$
Limits ...
Calculus
$\frac{dy}{dx}$
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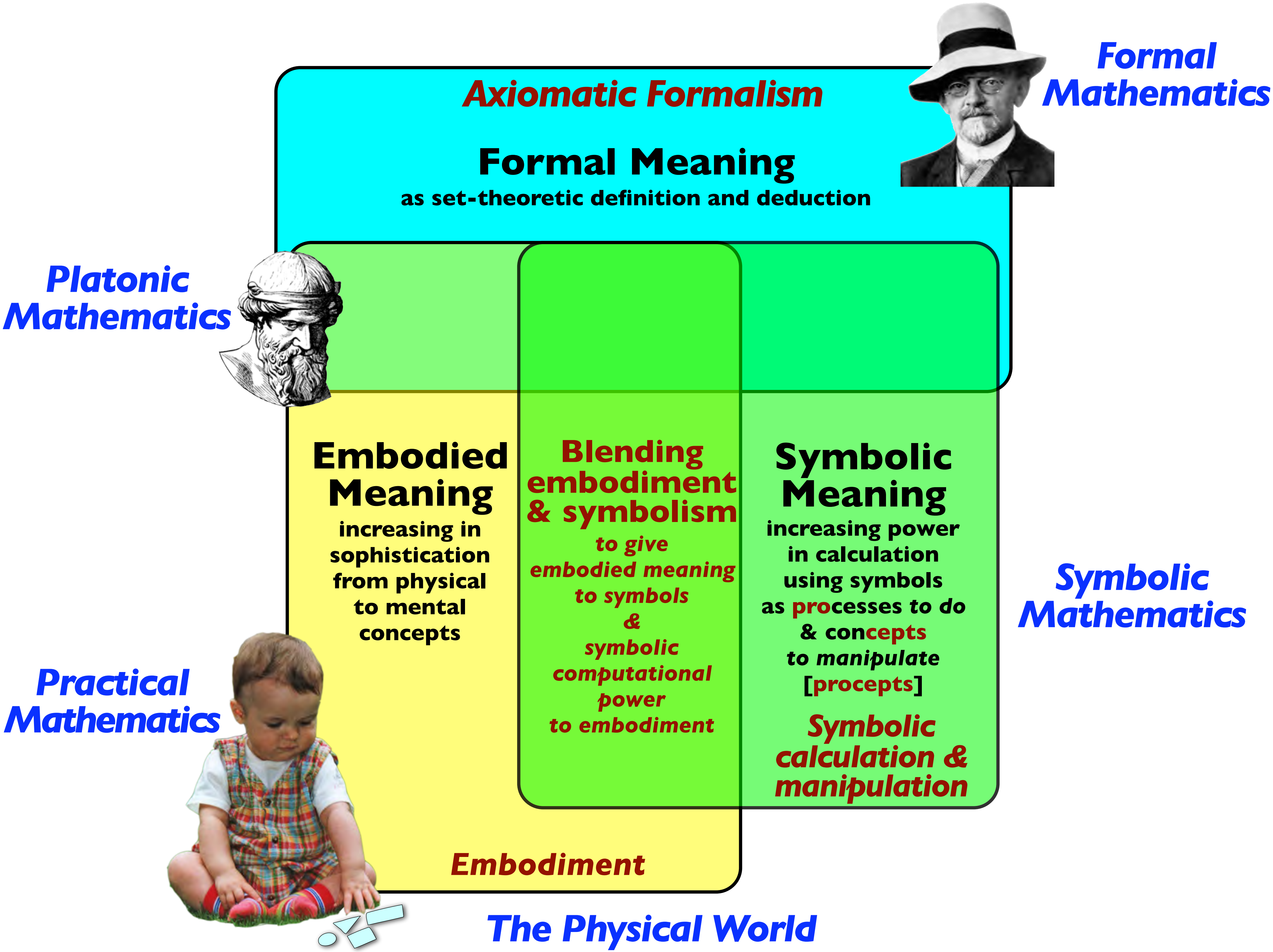


Formal Mathematics
Axiomatic Formalism
Formal Meaning
as set-theoretic definition and deduction
Platonic Mathematics
Embodied Meaning
increasing in sophistication from physical to mental concepts
Blending embodiment & symbolism
to give embodied meaning to symbols & symbolic computational power to embodiment
Symbolic Meaning
increasing power in calculation using symbols as processes to do & concepts to manipulate [procepts]
Symbolic calculation & manipulation
Symbolic Mathematics
Practical Mathematics
Embodiment
The Physical World

Mathematicians can live in the world of formal mathematics.

Children grow through the worlds of embodiment and symbolism.

Mathematics Educators try to understand how this happens.

Fundamentally we build on what we know based on:

- Our inherited brain structure (**set-before** our birth and maturing in early years)
- Knowledge structures built from experiences **met-before** in our lives.

Met-Befores

Current ideas based on experiences met before.

Examples:

Two and two makes four.

Addition makes bigger.

Multiplication makes bigger.

Take away makes smaller.

Every arithmetic expression 2+2, 3x4, 27÷9 has an answer.

Squares and Rectangles are different.

Different symbols eg △ and □ represent different things.

Met-Befores

Current ideas based on experiences met before.

Examples:

Two and two makes four. … works in later situations.

Addition makes bigger. … fails for negative numbers.

Multiplication makes bigger. … fails for fractions.

Take away makes smaller. … fails for negative numbers.

An algebraic expression 2x+1 does not have an 'answer'.

Later, *by definition*, a square is a rectangle.

Different symbols can represent the same thing.

Blending different conceptions of number

Successive number systems have properties that conflict

Counting Numbers

each number has a next with none between,
starts counting at 1, then 2, 3, ...
addition makes bigger, take-away smaller, multiplication bigger

Fractions

a fraction has many names: **$^1/_3$, $^4/_{12}$, $^7/_{21}$** *...*
there is no 'next' fraction
addition and multiplication involve new techniques
addition makes bigger, take-away smaller,
multiplication may be smaller

Integers

each number has a next with none between,
numbers can be positive or negative,
addition may get smaller, take-away may get larger,
multiplication of negatives gives a positive.

Implications for teaching

Transitions that involve unhelpful met-befores:

from counting to whole numbers

from whole numbers to fractions

from whole numbers to signed numbers

from arithmetic to algebra

from powers to fractional and negative powers

From finite arithmetic to the limit concept

from description to deductive definition

at many other transitions in development of concepts such as the function concept. (linear, quadratic, trig., log., exponential ...)

In each case, conflict between old knowledge (met-before) and new knowledge, can lead to procedural learning.

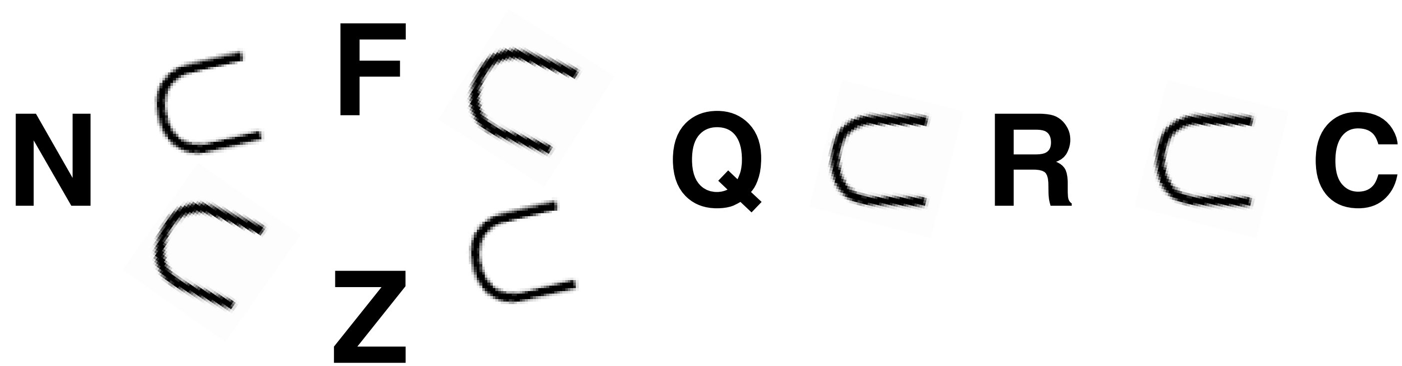
*From then on, **procedural learning** may be the only option!!!*

Increasing sophistication of Number Systems

Language grows more sophisticated as it blends together developing knowledge structures.

Blending occurs *between* and *within* different aspects of embodiment, symbolism and formalism.

Mathematicians usually view the number systems as an expanding system:



Cognitively the development is more usefully expressed in terms of blends.

Different knowledge structures for numbers

The properties change as the number system expands.

How many numbers between 2 and 3?

N	None
Q	Lots – a countable infinity
R	Lots more – an uncountable infinity
C	None (the complex numbers are not ordered)

A mathematician has all of these as met-befores

A learner has a succession of conflicting met-befores

From Arithmetic to Algebra

The transition from arithmetic to algebra is difficult for many.

The conceptual blend between a linear algebra equation and a physical balance works in simple cases for many children (Vlassis, 2002, Ed. Studies).

The blend breaks down with negatives and subtraction (Lima & Tall 2007, Ed. Studies).

Conjecture: there is no *single* embodiment that matches the flexibility of algebraic notation.

Students conceiving algebra as generalised arithmetic may find algebra simple.

Those who remain with inappropriate blends as met-befores may find it distressing and complicated.

Blending different conceptions of number

Real Numbers as a multi-blend

Formal

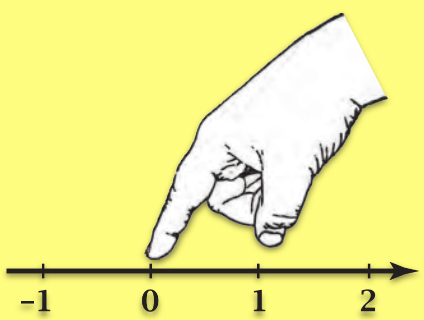
R

A complete ordered field as a ‘discrete set of elements’ satisfying specified axioms

A mathematician can have a formal view

Embodied

A ‘continuous line’ that can be traced with a finger



Symbolic

Numbers as decimal symbols that can be used for accurate calculation

$$\sqrt{2} = 1\cdot414213562...$$

A student builds on Embodiment & Symbolism

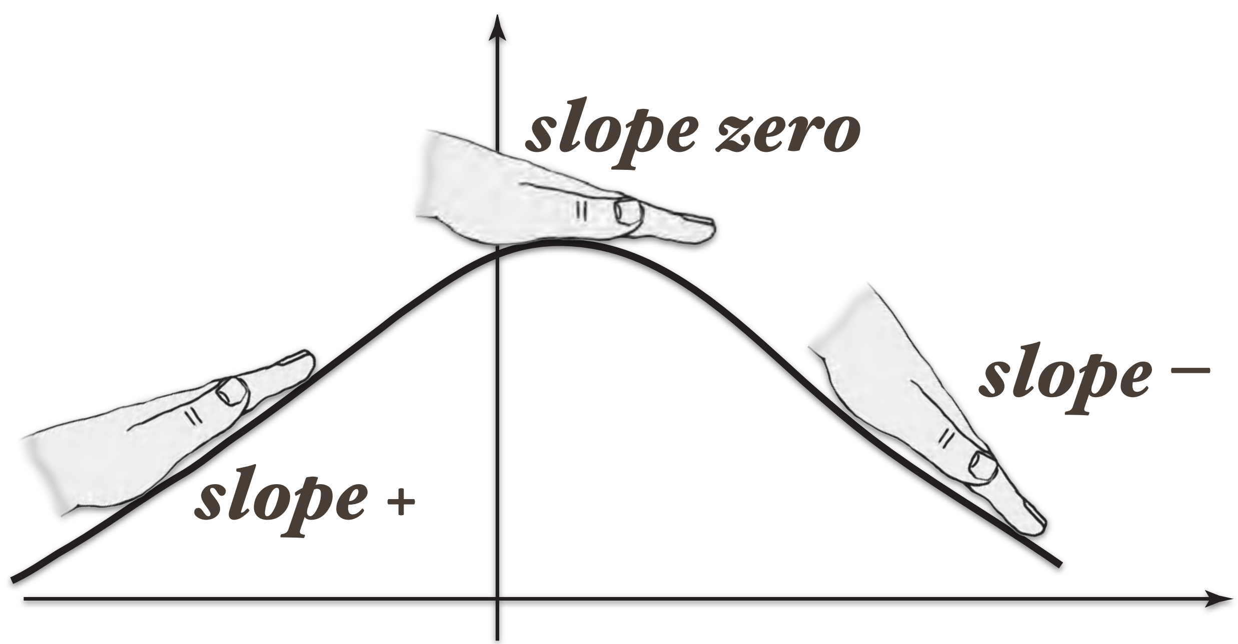
From Algebra to Calculus

The transition from algebra to calculus is seen by mathematicians as being based on the limit concept.

For mathematicians, the limit concept is a met-before.

For students it is not.

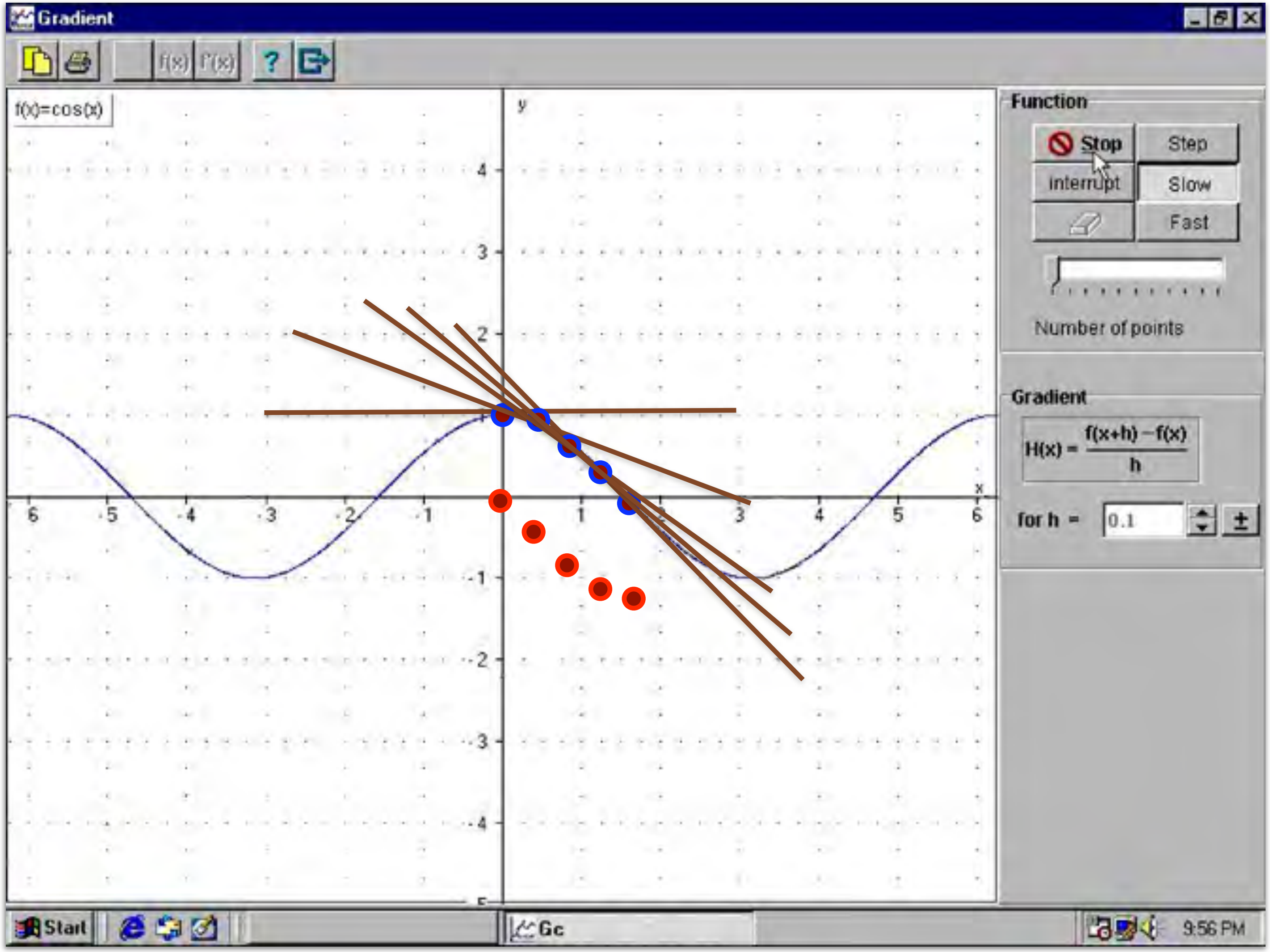
A student can see the changing steepness of the graph and embody it with physical action to sense the changing slope.



Calculus

Local straightness is *embodied:*

You can see *why* the derivative of cos is *minus* sine



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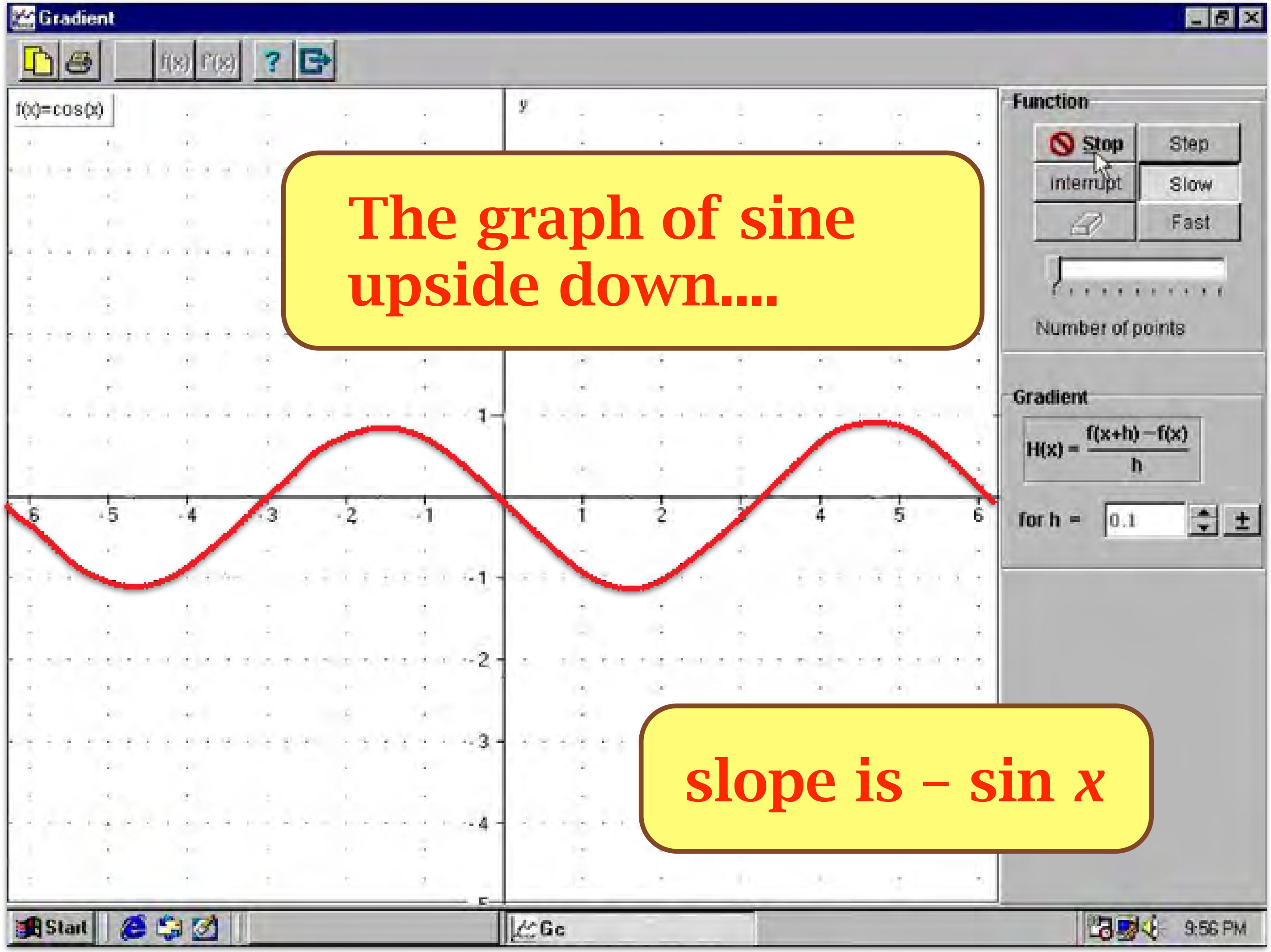
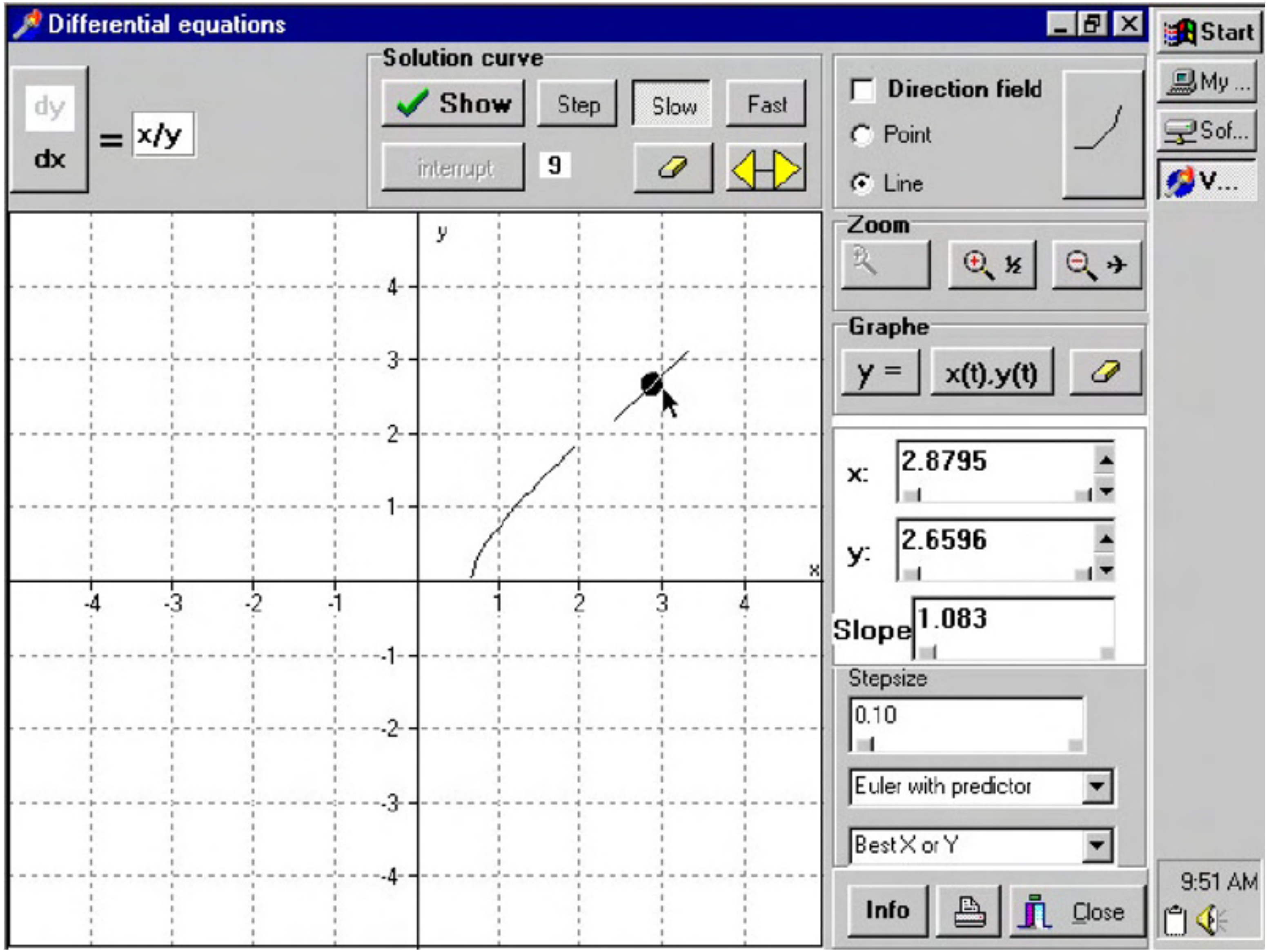


Calculus

Local straightness is *embodied:*

E.g. makes sense of differential equations ...



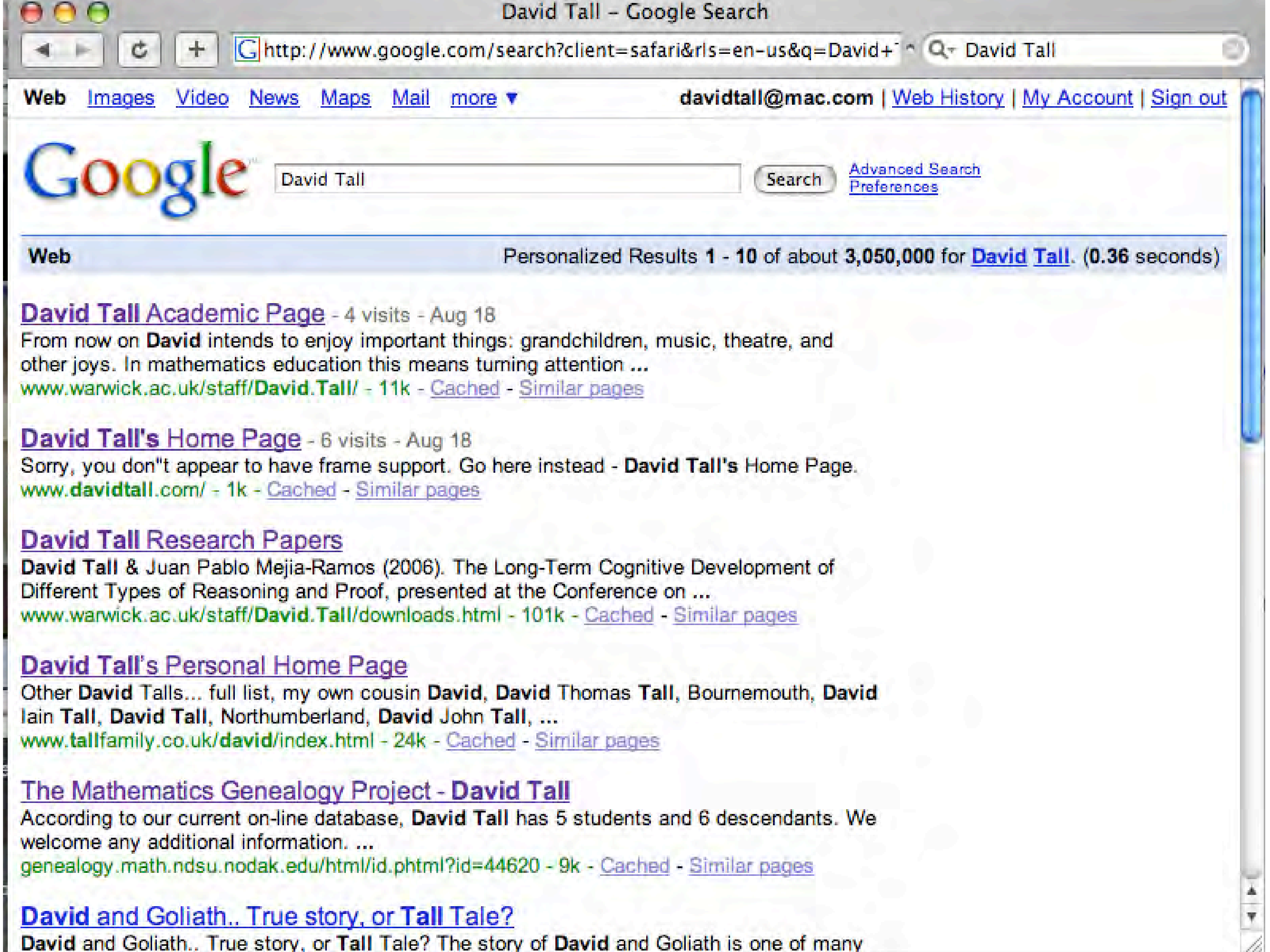
Reflections

As we learn, our interpretations are based on blending met-befores, which may cause conflict.

Learners who focus on the powerful connections between blends may develop power and flexibility, those who sense unresolved conflict may develop anxiety.

Mathematicians have more sophisticated met-befores and may propose curriculum design that may not be appropriate for learners.

It is the job of Mathematics Educators (who could be Mathematicians) to understand what is going on and help learners make sense of more sophisticated ideas.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David Tall - Professor in Mathematical Thinking

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See below for more information.

- information on several **research themes** with links to relevant papers:
  - cognitive development | concept image | cognitive units | cognitive roots | generic organisers
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News: A continual list of information on updates to focus on the most recent additions, including new papers, information on books including **Fermat's Last Theorem** (3rd Edition with Ian Stewart) and Intelligence, Learning and Understanding: **A Tribute to Richard Skemp** (ed. with Michael Thomas).

David retired in September 2006 and is now Emeritus Professor of Mathematical Thinking at Warwick

The downloads below include all my papers available in electronic form. Any problems: e-mail david.tall@warwick.ac.uk.
Go directly to papers published in 1975, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 2000, 01, 02, 03, 04, 05, 06, 07
The same papers are also organised under themes or, where appropriate, with research students.

See also: **Draft papers & articles under review.** UPDATED | **Selected Lectures** | **Curriculum Vitae**

2007~	Rosana Nogueira de Lima and David Tall. Procedural embodiment and magic in linear equations. Accepted for publication in *Educational Studies in Mathematics*. NEW
2007~	Eddie Gray & David Tall (2006). NEW Abstraction as a natural process of mental compression. Submitted to *Mathematics Education Research Journal*. [Reflections and developments of our work together.]
2007e	Setting Lesson Study within a long-term framework of learning. Presented at *APEC Conference on Lesson Study* in Thailand, August 2007.
2007d	Embodiment, Symbolism, Argumentation and Proof, Keynote presented at the *Conference on Reading, Writing and Argumentation* at National Changhua Normal University, Taiwan, May 2007.
2007c	Teachers as Mentors to encourage both power and simplicity in active mathematical learning. *Plenary at The Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing*, 17–19 March 2007, Abu Dhabi. [Overheads]
2007b	David Tall (2007). Embodiment, Symbolism and Formalism in Undergraduate Mathematics Education, Plenary at *10th Conference of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education*, Feb 22–27, 2007, San Diego, California, USA. [Overheads]
2007a	David Tall (2006). Developing a Theory of Mathematical Growth. To appear in *International Reviews on Mathematical Education* (ZDM). NEW
2006g	David Tall (2006).Encouraging Mathematical Thinking that has both power and simplicity. *Plenary*