LONG-TERM DEVELOPMENT OF MATHEMATICAL THINKING AND LESSON STUDY

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In recent years, international comparisons have revealed wide differences in mathematical performance around the world, generating a general concern for improving mathematics teaching and learning. Yet, while teachers strive to improve performances on tests, there is a growing realization that in many cases this leads to the teaching of procedures to increase test scores rather than encouraging a flexible understanding of mathematics for wider use. The development of comparisons that focus on a broader range of competencies, such as PISA, reveal continuing international differences.

In Japan, where students have consistently performed well on international assessments, teachers have collaboratively developed ways to foster students' flexible understanding of mathematics. Lesson study has been used in Japan for over a century to improve the preparation of a lesson or sequence of lessons, to predict how the children will react, to have a substantial number of observers watch the lesson, to review and improve the lesson plan in cycles to share with others. The lessons are based on a problem-solving approach in which teachers educate children to think for themselves, both within the lesson and over the longer term in a carefully designed curriculum.

What matters most is not just the effectiveness of an individual lesson but the long-term development of learners over time. Here we formulate a framework for the long-term development of mathematical thinking which has the potential to enhance understanding of lesson study and to improve its effectiveness. The framework is based on how humans learn to think mathematically. It begins with the individual's perceptions of, and actions on, objects in the world, on the one hand classifying objects as figures (as in geometry) and on the other practicing actions (such as counting) to develop mathematical concepts (such as number). Over the longer-term through increasingly sophisticated development of mathematical thinking, the child learns to use mathematical concepts to solve increasingly subtle problems and build eventually build towards advanced theoretical frameworks of pure mathematics defined axiomatically and proved formally.

INTRODUCTION

As our society becomes more dependent on technology and employment shifts from manual to intellectual activity, there is an increasing need to raise the quality of teaching and learning mathematics around the world. At present there are widely

Written Kenilworth, UK, September 8–14, 2007. Thanks are given to Patsy Wang-Iverson for her helpful suggestions.

diverging performances in different countries and a widespread desire to raise levels of performance in mathematics. The Japanese form of *lesson study* offers one way in which the nature of mathematical learning and thinking may be improved. It consists of a collaborative effort by many teachers in a three-part design: *preparation* (kyozai kenkyu), *lesson* (kenkyu jyogyu) and *review* (jyogyu kentoukai).

Preparation involves the design of a well-constructed lesson (usually as part of a sequence of lessons) that encourages pupil participation as an integral part of building mathematical ideas for themselves. The lessons are carefully planned and great thought is devoted to predicting how the pupils may react. When the *lesson* is presented, it is observed by a number of observers who then participate in the *review* to discuss ways of refining and improving the lesson. The cycle of lesson and review may then be repeated to eventually produce a refined lesson that can be modified with confidence by a wide range of teachers to meet the needs of their students and to encourage effective learning.

Lesson study in mathematics goes beyond the design of individual lessons to the development of a long-term teaching approach dedicated to understanding the nature of mathematical thinking and how this can be improved for the benefit of all students.

Lesson study has become a widely-used and highly-refined methodology, not only in Japan, but in also around the world, in particular in the APEC (Asia-Pacific Economic Cooperation) economies. As it moves into different economies, local versions of lesson study may develop, using the overall structure to its best effect in different locations.

Fundamentally the purpose of Lesson Study in mathematics is to improve not only the performance of students on examinations, but to improve the nature of their *mathematical thinking*, including not only the ability to perform routine tasks accurately and efficiently, but also to develop the abilities and attitudes to solve novel problems by thinking mathematically in new situations.

LONG-TERM DEVELOPMENTS IN MATHEMATICAL THINKING

Our purpose in this paper is to look at the long-term development of mathematical thinking over the years of mathematical growth as a child progresses through school and beyond. In the end, it is not just the quality of a sequence of lessons that matters, it is the cumulative growth of mathematical thinking over time, encompassing the full range of increasingly sophisticated mathematical knowledge and the willingness to think mathematically to solve new problems in new situations.

To build a long-term framework requires an understanding of the fundamental nature of the growth of mathematical thinking that can be shared by teachers throughout the grades in school and on to university and beyond. As the experience of the child grows, he or she learns new ideas and puts them together in increasingly sophisticated ways, developing new ways of working that are more flexible and more comprehensive than before. Our first task is to identify fundamental ways in which this growth occurs.

Van Hiele levels of development

Van Hiele (1986) proposed a framework that formulated a sequence of successive levels of thinking in geometry. Each one builds upon the previous one, but is more sophisticated and involves significant changes in meaning, so that the language used at the next stage is subtly different from the previous one, as the concepts become re-organised and more sophisticated.

At the first level of operation, a triangle is a figure with three sides, recognised by its visual shape. A good example is the ‘give way’ traffic signal seen on the road in some countries to tell vehicles to give way to other vehicles.

Figure 1: Give Way

This sign has three (equal) sides, but its corners are rounded. Nevertheless, it fits the general idea of a ‘triangle’. There are also a range of other shapes that are given names: triangles, squares, rectangles, hexagons, octagons, circles, and so on, each with its own special characteristics.

At the second level of operation, the notion of triangle is refined by focusing on its specific properties—it is a figure constructed with three straight lines, so that the line segments make a three-sided figure. At this level a triangle can have many shapes, but they are all have the same simple property of three straight sides. A square has four equal sides and all its angles are right angles; a rectangle has opposite sides equal and all its angles are right angles. While many adults will have the sophistication to see that a square is a special case of a rectangle, van Hiele observed that young children generally see squares and rectangles as being distinct: squares have *all* the sides equal, rectangles have only *opposite* sides equal.

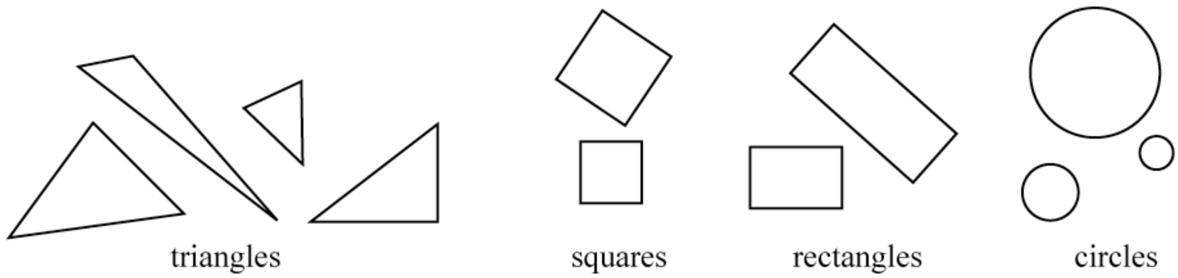


Figure 2: triangles, squares, rectangles, circles

At this level, properties occur simultaneously in figures, for instance an isosceles triangle has two equal sides and two equal angles and there is a symmetry down the middle. When these are seen as a total entity by the young child, these properties are all integral to the figure and occur *simultaneously*. At this stage no one property is a consequence of others.

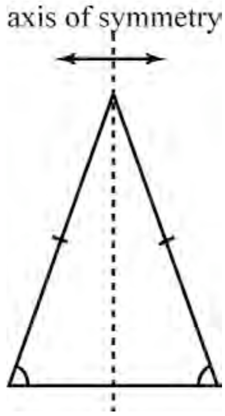


Figure 3: A triangle with both equal sides and equal angles

At the third level, the child begins to investigate the properties of triangles further, for example, seeing that the angles of a triangle always add up to 180° and that *if* a triangle has two equal sides, *then* that triangle has two equal angles. Definitions are refined. A square has all its sides equal and at least one angle a right angle; a rectangle has opposite sides equal in pairs, and at least one angle a right angle. The definition can now be so worded that a square is a special case of a rectangle. In general we now have not distinct classes of figure, but *hierarchies* of figures: quadrilaterals include parallelograms, which include rectangles, which include squares. Meanwhile all squares are rhombuses; they are the only rhombuses which are also rectangles. Trapezoids (with just one pair of parallel sides) are quadrilaterals, but are separate from parallelograms.

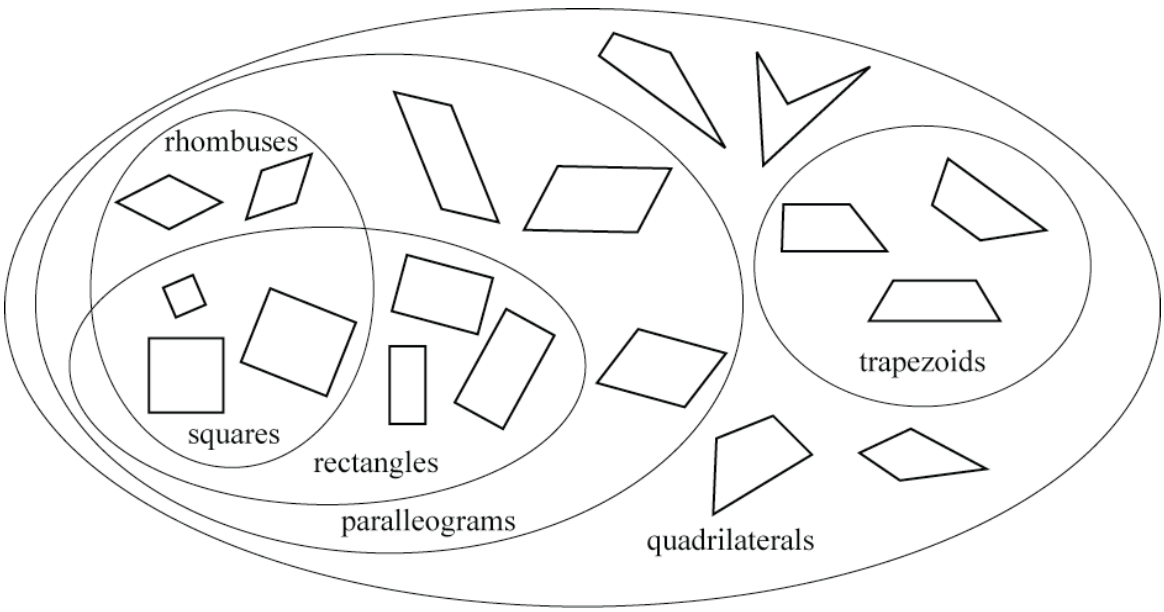


Figure 4: Squares *are* rectangles, which are parallelograms, which are quadrilaterals

At a fourth level, figures are imagined as *platonic* objects, where a line segment has length but no width and a point has position but no size. Their properties are organised into the deductive system of Euclidean geometry.

There is a further level of rigour in which new geometries become possible, including projective geometry, spherical geometry, elliptic geometry and hyperbolic geometry; new geometric systems may also be defined only in terms of lists of axioms from which their other properties may be deduced by formal proof.

The van Hiele levels have influenced the teaching of geometry in many countries around the world. However, the precise assignment of an individual to a particular level has proved to be problematic. Gutiérrez, Jaime & Fortuny (1991) reported that a student could respond at several different levels on different questions and proposed that it is more sensible to report the percentage responses of each student at successive levels rather than assign a student to a specific level. The van Hiele framework remains highly relevant in lesson study as a template for overall development from perception through description, definition and on to Euclidean proof.

It is interesting to note that, historically, the first four levels up to and including the theory of Euclidean geometry occurred over two thousand years ago, and it was over eighteen hundred years before Descartes introduced Cartesian coordinates that eventually linked geometrical form to modern algebraic symbolism. In many of our schools today, we teach geometry and algebra in parallel with connections between them, but this does not mean that the ideas of geometry and algebra necessarily develop in parallel. It would be perfectly possible, as it was in Greek times, to study geometry without any corresponding links to modern algebra.

There are articles extending Van Hiele levels from geometry to other areas including those by Freudenthal (1973), Hoffer (1983) and Isoda (1996a). Van Hiele himself has written about levels in arithmetic and algebra. He observed 'a change in level' from the act of counting to the concept of number, with reference to the Piagetian test in which a child is shown five dolls and five umbrellas equally spaced out. The child agrees there is the same number of dolls as umbrellas and each doll has one umbrella. But when the umbrellas are moved together, the child says there are more dolls and there is no longer an umbrella for each doll. Van Hiele wrote:

> Being able to count does not include that the child understands that a set of umbrellas has a constant cardinal number. (Van Hiele, 2002, p.37)

He concluded that there is a change in level from being able to count to understanding the concept of number.

On the other hand, he did not consider there was any corresponding change in level from arithmetic to algebra:

> The transition from arithmetic to algebra cannot be considered the transition to a new level. Letters can be used to indicate variables, but with variables children are acquainted already.

> Letters can be used to indicate an unknown quantity, but this too is not new.
> (Van Hiele, 2002, p.39)

He continued by saying that it is possible to have a different level in approach to algebra if it is formulated in terms of axioms and definitions:

> It is possible to introduce a theoretical level of algebra in secondary school but meeting such a case is exceptionable. In 1987 I encountered a secondary school in New York where algebra was started with groups. The group was defined by axioms and definitions and no examples of numbers and addition or multiplication were given. Elements were just elements, no more and the operator was just given by the necessary axioms and indicated by *. This seemed to me the most direct way to cause instrumental learning.
>
> The normal instruction of algebra is just a continuation of arithmetic; no problems are expected unless the children are overloaded with information. (Van Hiele, 2002, p.39)

However, learners *do* have great problems with algebra:

> For some, audits and root canals hurt less than algebra. Brian White hated it. It made Julie Beall cry. Tim Broneck got an F-minus. Tina Casale failed seven times. And Mollie Burrows just never saw the point. This is not a collection of wayward students, of unproductive losers in life. They are regular people […] with jobs and families, hobbies and homes. And a common nightmare in their past. (Deb Kollar, *Sacramento Bee* (California), December 11, 2000.)

So here we have a problem: Van Hiele does not see a change in level from arithmetic to algebra, despite the difficulties encountered by children, but he *does* see a change in level from counting to number and also between algebra as taught in school and algebra as formulated as axioms in the theory of groups. The resolution of this apparent conflict lies in what he means by a *change in level*. His theory is based on ideas of the thesis of Koning (1948), which was, in turn, founded on the theory of Piaget. In 1955, Van Hiele wrote:

> You can say somebody has reached a higher level of thinking when a new order of thinking enables him, with regard to certain operations, to apply these operations on new objects. The attainment of the new level cannot be effected by teaching, but still, by a suitable choice of exercises the teacher can create a situation for the pupil favorable to the attainment of the higher level of thinking. (Van Hiele, 1955,
> translated into English in Van Hiele, 2002)

Note that here he talks of applying certain *existing* operations on *new objects*. Piaget goes further than this: he asserts that the *operations themselve*s ***become*** *new objects*. This idea of 'encapsulating processes as mental objects' has been formulated and studied by many authors. It offers a change in level of thinking in symbolism that complements van Hiele levels in geometry.

Compression from actions to thinkable concepts

A distinction may be made between the way we think about actions that we *do* and concepts that we *think about*. This is often the basis of a separation in curriculum building between 'skills' (to do) and 'concepts' (to think about). To emphasise the aspect of thinking about concepts, Gray and Tall (2007) used the term 'thinkable concept'. The term is tautological for a concept is necessarily something we can think about. Nevertheless, by using the term 'thinkable concept', an emphasis is placed on the way in which we think about it and this proves to be helpful in highlighting the flexible use of mathematical symbols as operations to do and concepts to think about.

In arithmetic and algebra, there is a change in level of thought that arises from a shift in thinking as *operations* at one level become *mental objects* at another. Not only does counting become number, addition also becomes sum, repeated addition becomes product, differentiation becomes derivative, and so on. The idea of an *operation* that takes place in time becoming a *thinkable concept* that exists outside of a particular time and place is an ongoing theme in a wide range of theories (Dubinsky & McDonald, 2001; Sfard, 1991; Gray & Tall, 1994). This idea also has its origins in the works of Piaget (Piaget & Inhelder, 1985, p. 49) who remarked that 'actions and operations become thematized objects of thought or assimilation.'

Our current interpretation (based on Gray & Tall, 1994) of the relationship between mathematical operations and mathematical concepts sees operations such as counting being compressed over time into concepts such as number. This happens throughout the mathematical curriculum, where the operation of *addition* becomes compressed into the concept of *sum*, *repeated addition* becomes *product*, *sharing* becomes *fraction*, an operation of *evaluation* (double the number and add one) becomes an algebraic *expression* ($2x+1$), *differentiation* gives the *derivative*, *integration* gives the *integral*.

Traditional teaching often focuses on giving children practice with a specific procedure—such as column addition, column subtraction, column multiplication and long division—until it can be performed efficiently and accurately, and is then applied in range of problems. Such an approach may be effective, but if learned by rote without meaning, may also be fragile and lead to errors.

A good example of this is the witty observation of Robin Foster (1993) who questioned whether 'practice makes imperfect'. His considered the routine introduction of column subtraction beginning with examples like $39-24$ which do not require carrying. The child correctly writes

$$\begin{array}{r} 39\ - \\ \underline{24} \\ \underline{15} \end{array}$$

The child then practises similar examples and gets them all right and appears to be succeeding.

Then the child is then given new problems such as $33-29$ and writes

$$\begin{array}{r} 33\ - \\ \underline{29} \\ \underline{16} \end{array}$$

The result is clearly incorrect. It is a common error in column subtraction. However, the error may not have originated at this stage. The child may be simply repeating a procedure practiced earlier: calculating the 'difference' between the numbers in the tens column and the numbers in the units column. With only experience of whole numbers, the 'difference' between 3 and 9 is 6 and the 'difference' between 9 and 3 is also 6. So the difference between the 3 and the 9 in the units column is 6. In this way it is possible for a child to get a correct result using a method that works in one context but fails in another. Teaching procedures without meaning is fragile and may fail in more sophisticated contexts.

The Japanese approach developed through lesson study does not limit the child to a single procedure. It encourages children to suggest a range of different ways to carry out a particular operation and to choose a method that is, if possible, more efficient, more accurate and more meaningful. This is part of a spectrum of different ways of compressing an operation into a thinkable concept. Four successively sophisticated stages may be identified:

(i) A single step-by-step procedure to carry out the operation (*procedure*);

(ii) Several different procedures, to choose the most efficient (*multi-procedure*);

(iii) The realisation that the different procedures may involve different sequences of steps, but they all achieve 'the same effect' (an overall *process*);

(iv) The effect is considered as a concept in itself (a thinkable *concept*).

We use the term 'stage' here to distinguish it from the notion of 'level' in the sense of van Hiele. The four stages increase in sophistication, but that does not mean that they are encountered in sequence by each child. A procedural approach may begin with stage (i) and perhaps progress to stage (ii), both of which are essentially procedural, taking place in time. In lesson study, the tendency is to begin at stage (ii), with the learners encouraged to consider a range of different procedures to carry out the required operation. At stage (iii) different procedures having the same effect are considered 'equivalent', and at stage (iv), the whole operation may be symbolised and considered as a thinkable concept that may be used flexibly in different ways.

The full change from stage (i) to stage (iv) *is a single change in level* in the sense of van Hiele and Piaget, from thinking of an operation *as a step-by-step procedure* to thinking of it *as a mental object* in its own right. It is only when the thinker can conceive of the operation as a thinkable concept in the mind at a single point in time that the change has occurred fully from operation to thinkable concept. Gray & Tall (1994) introduced the term *procept* to speak of a thinkable concept that uses symbols to represent the operation as a *pro*cess and as a thinkable con*cept*. A procept is not just a single symbol such as 4+2, it is a family of equivalent symbols representing procedures with the same effect. Thus the thinkable concept of the number 6 can also be represented flexibly as 4+2, 2+4, 3+3, 7–1.

This change in level does not necessarily occur in all learners. Some may remain at a procedural level, either with a single procedure or having several procedures from which they can choose the most efficient. Such procedures are carried out in time so that multi-step problems require the individual to coordinate several procedures one after another. This is far more demanding than operating with symbols as thinkable concepts independent of time. Procedural operations can cope with simple routines but may not lead to a full range of mathematical thinking to solve new problems in novel situations.

On the other hand, if the change in level does occur in a learner, stage (iv) procepts as thinkable concepts are now mental objects on which new operations may be performed. For instance, when counting is compressed into number, then numbers themselves can be operated on, for instance, by addition, becoming the concept of sum; then the sum may be repeated to give the concept of multiplication, and so on. This leads to successive compressions of operations into thinkable concepts throughout arithmetic and algebra (Tall et al, 2001).

Fractions

The stages from operation as procedure (i) to thinkable concept (iv) occur in many contexts in mathematics. For example, fractions occur through sharing operations, first dividing an object or a collection of objects into a specified number of equal parts, then selecting a specific number. The fraction $^3/_4$ denotes the division into 4 equal parts and selecting 3 (stage (i)). In Japanese, this retains the language of the process and is called '4 bun-no 3' which says '(divide into) 4 (equal parts) and take 3'. In English it is called 'three quarters', and in American English 'three fourths'. By the same token, $^6/_8$ is 'divide into 8 equal parts and take 6', to give 'six eighths'. The English words 'quarters' and 'eighths' act as *nouns* to name the size of the part.

As operations, the fractions $^3/_4$ and $^6/_8$ are quite different. 'Three quarters' has 3 parts (quarters) and 'six eighths' has 6 parts (eighths). The Japanese fraction names '4 bun-no 3' and '8 bun-no 6' likewise represent different operations of sharing. However, what is essential is that the total quantity is the same in each case.

The fractions $^3/_4$ and $^6/_8$ are called 'equivalent' fractions: a form of words that suggests that they are equivalent *but not the same*. (stage (ii)). But when they are seen to 'have the same effect', in terms of *quantity* rather than in terms of the number of pieces, the shift is made to stage (iii). When the fraction is marked as a point on the number line, the points representing $^3/_4$ and $^6/_8$ are *one and the same*. At this final stage (iv), the symbols $^3/_4$ and $^6/_8$ are now seen as different ways of writing *the same* thinkable concept, marked as a single point on the number line as a rational number.

At this final level, the symbol $^6/_8$ can mean 6 eighths (of a particular quantity). Adding 6 eighths and 3 eighths gives 9 eighths. If eighths are seen as quantities, there is no reason to limit our thoughts initially only to fractions less than one. Splitting three chocolate bars into eighths gives 24 eighths. Sharing them between 12 children gives each child 2 eighths, the same quantity as one quarter.

Adding two thirds and one half as fractions requires the fractional parts to be the same size. Two thirds is the same quantity as 4 sixths and one half is 3 sixths; we may add 4 sixths and 3 sixths in the same way that we deal with whole number addition to obtain 7 sixths. Addition of fractions becomes simple at stage (iv).

Multiplication is also easier to interpret at this level. It requires taking a fraction of a fraction, such as a third of a fifth being a fifteenth, or two thirds of a fifth being two fifteenths. As procedures, these operations become increasingly complicated, but as thinkable concepts at stage (iv), a learner with flexible knowledge of whole number arithmetic may build a more sophisticated but simple approach to the arithmetic of fractions.

Functions

The same sequence of stages from procedure to thinkable concept occurs with the concept of function. First we may meet procedures such as 'double the number and add 6' which involves a different sequence of operations in arithmetic from 'add 3 to the number and double the result'. Expressed symbolically, the first is $2x+6$, the second is $(x+3)\times 2$ which is often written as $2(x+3)$ because we have stylistic conventions for writing expressions. At stage (ii) these expressions have two distinct meanings but when they are seen as 'equivalent', and one can be replaced by the other, we reach stage (iii). If these are written as functions, $f(x)=2x+6$ and $g(x)=2(x+3)$, then at stage (iv) they are seen as being exactly the same function, with the same input-output relationship and the same graph.

Vectors

Another example is the notion of vector. A vector is initially introduced as a quantity with magnitude and direction. It can be conceived as an operation to shift a figure in the plane by that given magnitude and direction. At stage (i) it might be an action: push a figure (say a triangle) in a certain direction by a given magnitude (figure 5).

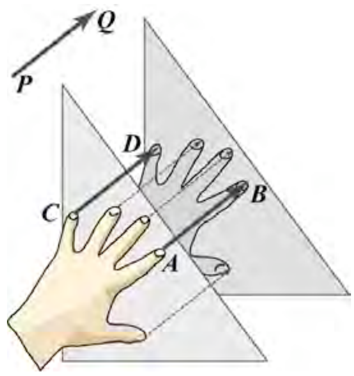


Figure 5: A translation by a given magnitude and direction

Here the first finger moves from A to B and the little finger from C to D. Both AB and CD as vectors have the same magnitude and direction as do the movements of any other point on the triangle. The translation can be represented by any one of these vectors. Indeed, a vector PQ which does not touch the triangle at all can also be used to represent the translation so long as it has the given magnitude and direction (stage (ii)). If we look at the vectors alone (figure 6), we find a whole collection of vectors, AB, CD, PQ etc, which are *equivalent* in the sense that they all have the same magnitude and direction (stage (iii)).

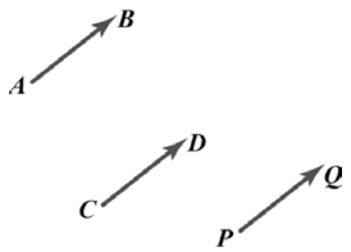


Figure 6: a set of equivalent vectors with the same magnitude and direction.

The final stage (iv) is a change in focus where we imagine a *single vector **u*** of the given magnitude and direction that may now be moved to start at any point in the plane. This moveable vector is a *free* vector. It has only magnitude and direction and can be moved to start at any point we choose.

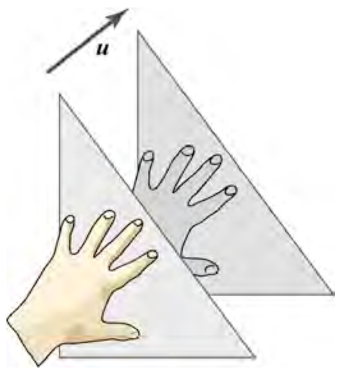


Figure 7: A free vector ***u***

Once again we reach a highly compressed level of thinking in which a single mental object—the free vector—operates at a level to replace the multitude of equivalent vectors in a compact and flexible manner.

Each of these examples shares the same framework of compression from a single operation represented in many *equivalent* ways to a single mental representative that operates in its own right as a flexible thinkable concept at a higher level.

What is interesting about this development is the way in which the notion of equivalence operates as a staging post in the middle of the development, before being compressed into a single flexible mental entity that can be used in a simpler manner. It happens often in mathematics. For instance adding –2 is a different operation from taking away +2 and these are usually introduced using special notations such as $+(^{-}2)$ to add $^{-}2$ and $-(^{+}2)$ to subtract $^{+}2$. However, one soon realizes that $^{+}5+(^{-}2)$ and $^{+}5-(^{+}2)$ are different operations with the same effect: both result in the operation $5-2$, which is 3 The compression of operations into flexible thinkable concepts often passes through an intermediate stage of equivalence before moving on to a simpler flexibility at a higher level.

As time passes, the learner is introduced to new contexts where a procept may become richer in meaning: the procept 6 in whole number arithmetic broadens to include $^{12}/_{2}$, $6^{1/2}\times 6^{1/2}$, $(-2)\times(-3)$ or $(\sqrt{5}+i)(\sqrt{5}-i)$ as the context shifts from whole numbers to fractions, rational powers, negative numbers or complex numbers.

We now see that the development of number concepts involves many cycles of development of new knowledge structures as the number system broadens in new ways. The child who has difficulties with these shifts in meaning is likely to find mathematics becoming increasingly complicated and difficult to understand. To build mathematical thinking in a comfortable way requires successive blending and compression of knowledge into thinkable concepts.

The Japanese Professor

Many Japanese teachers have a subtle way of encouraging their students to seek more sophisticated ways of doing mathematics. In Japanese the word for professor is Ha-Ka-Se (Ha is は, Ka is か and Se is せ), which means Doctor or Professor in Japanese. Each syllable has a separate meaning. The Japanese word ***ha***yai means fast, ***ka***ntan means both easy and understandable, and ***se***ikaku means accurate and logical. *Fast*, *easy* and *accurate* are useable words for elementary school children to compare various processes. Using Ha-Ka-Se in the every day classroom, especially in whole class work discussion, helps children become accustomed to comparing alternative procedures to seek those that are faster, easier, more accurate.

Stigler & Hiebert (1999) describe the features of a standard Japanese lesson (following a problem-solving approach) as follows:

> Teachers begin by presenting students with a mathematics problem employing principles they have not yet learned. They then work alone or in small groups to devise a solution. After a few minutes, students are called on to present their answers; the whole class works through the problems and solutions, uncovering the related mathematical concepts and reasoning.

Japanese teachers prefer this problem-solving approach because they want to develop children who can learn and think by themselves (in Japanese, Mizukara Manabi Mizukara Kangaeru). Japanese teachers try to help their students develop thinkable concepts and methods of working that encourage personal construction of more sophisticated ways of thinking. These principles are incorporated in the Curriculum Standards (1989) in Japan.

Embodiment and Symbolism

The discussion so far reveals two distinct long-term developments of mathematical thinking:

1) the Van Hiele levels of sophistication in geometry,

and

2) the cycles of compression of operations into thinkable concepts in arithmetic, algebra and symbolic calculus.

The first builds on our perceptions of real-world objects and our actions upon them, designed to seek out their *properties* and build increasingly sophisticated ways of working with them. Tall (2004) referred to this combination of human perception and action as *conceptual embodiment*. The second builds on actions that are symbolised as process or concept, which may remain at a procedural level or may become

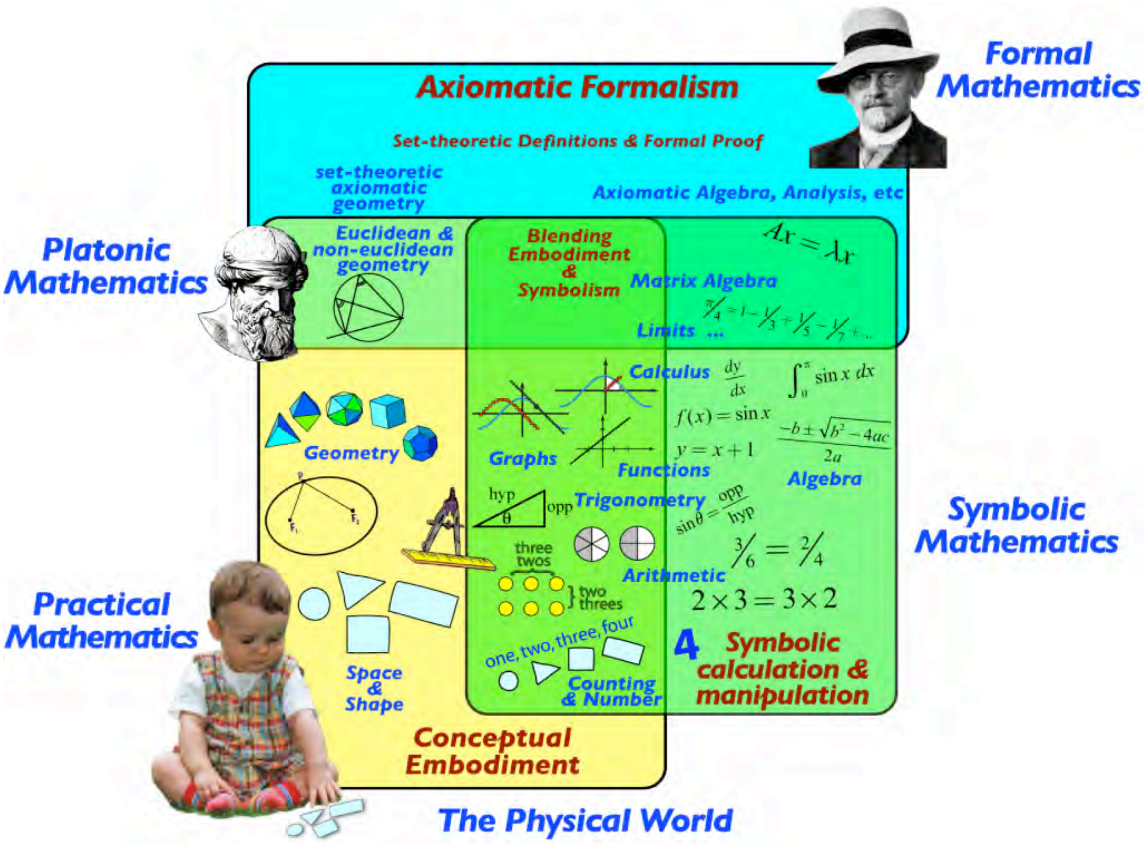


Figure 8: Three mental worlds of embodiment, symbolism and formalism

successively compressed as procepts that themselves may be operated upon at a higher level. This use of symbols as procepts is called *proceptual symbolism*.

The child in figure 8 is contemplating several objects. He has various possibilities, one of which involves exploring their properties focusing on the similarities and differences between them. Another is to perform an action on them as a collection without taking note of their individual properties, say by counting.

The first extends to the long-term development of conceptual embodiment, focusing on the individual nature of each object—a round circle, a triangle with three sides, a square, a rectangle. This begins an increasingly sophisticated development of ideas in shape and space, first as practical mathematics with ruler and compass and physical drawing, and on to study the properties of physical plane figures and solids, moving on to the more cerebral platonic conceptions of Euclidean geometry.

The second extends to a long-term development compressing operations into thinkable concepts that we can manipulate mentally in arithmetic and algebra.

Mathematical thinking blends together embodiment and symbolism. For instance, we may embody the notion of 2×3 either as 2 rows each with 3 discs or as 3 columns, each with 2 discs to see that 2×3 = 3×2. When we share objects into fractions, we may see that dividing into 6 and selecting 3 to get $^3/_6$ gives the same quantity as dividing into 4 and taking 2 to give $^2/_4$, so $^3/_6 = {}^2/_4$. By blending together embodiment and symbolism we give meaning to the properties of arithmetic, such as the commutative property, the associative property, the distributive property and so on.

(At this stage, these are *observations* of properties that appear to be quite general, not *laws* in the sense of something that is declared to be a rule that must be obeyed.)

Trigonometry begins as an embodied concept, calculating a ratio (opposite over hypotenuse), which is then compressed as a thinkable concept: the sine of an angle. This may be compressed further when the value of the sine for all possible angles is conceived as a single thinkable concept: the sine function.

Algebra involves using letters to represent numbers and relationships such as $y = x + 1$ that can be embodied as a straight-line graph, moving on to more general functional relationships and their corresponding graphs. Solutions to equations can be found either by symbol manipulation or by embodying the equations as graphs and seeing where they intersect.

Calculus studies the rate at which functions change, which can be embodied by seeing how the steepness of the graph changes as a finger is traced along the graph, then the graph of the changing slope can be seen as a new graph of the rate of change, symbolised as the derivative. Likewise the area under the curve changes to give a new function, the integral. The ideas of changing 'slope' and changing 'area' can be *seen* and *embodied* in action and perception, before it is necessary to introduce any concept of limit. It is only when accurate computation is required that number calculations are performed as accurately as possible. and the limit concept becomes essential to compute the precise symbolic functions representing derivative and integral.

The embodiment of the derivative as the slope of the graph can be *seen* by magnifying the graph on a computer. As part of the graph is magnified, the part that fits in the window on the computer screen looks more and more like a straight line. By keeping the scale the same on the horizontal and vertical axes, under high magnification the graph (of a differentiable function) 'looks straight'. This idea can give embodied meaning to differential equations where first order differential equation gives an expression for the derivative which is the slope of the graph at the point under consideration, so a solution can be found by putting together short line segments that each have the derivative as their slope.

Meanwhile, algebra gets more sophisticated with transformations being represented geometrically in two and three dimensions, leading to symbolic representations as matrices that can be used not only for transformations, but as representing systems of linear equations.

In 1900, Hilbert gave a speech at the International Congress of Mathematicians that not only posed 23 problems that occupied mathematicians throughout the twentieth century, it also introduced the axiomatic method to the mathematical community. In this new formal approach, mathematical concepts should be properly defined in a set-theoretic context, formulating systems given by a set of axioms that may be used to deduce any properties of the system by formal mathematical proof. This gives a new method of building mathematical theory: the axiomatic formal world of pure mathematics.

The framework for mathematical thinking is now expanded to three distinct ways of thinking about mathematics: (conceptual) embodiment, (proceptual) symbolism and (axiomatic) formalism.

It is interesting to note that Van Hiele's example of a different level of operation in algebra involves a shift from algebra as met in school to algebra given by axioms and definitions. What is happening here is the shift from symbolic school algebra to axiomatic formalism.

The proposed framework therefore consists of three parts:

> a development of conceptual *embodiment* as occurs in the development from practical interactions with shapes in space to verbal definitions and the platonic mathematics of Euclidean geometry.
>
> a succession of cycles in which operations are symbolised and compressed into thinkable concepts to give the proceptual *symbolism* of arithmetic, algebra and symbolic calculus in school.
>
> the re-organization of mathematical thinking into axiomatic *formalism* based on axioms for systems, definitions for new concepts built on those axioms and formal proof to deduce other properties.

Tall (2004) referred to these different ways of operation not just as *modes* of thinking arranged in increasing *levels* of operation, but as different *worlds* of mathematics which grow more sophisticated as the learner matures. The term 'world' is used here, rather than 'mode of thinking' or some other term, to suggest distinct frames of reference, one based on embodiment, one on symbolism and one on formalism. Each frame of reference is essentially a different 'world' of mathematical thinking that develops in the individual over time while each maintains its own ways of thinking. The use of the term 'world' is the same sense in which different people may be part of the same organization, yet seem to live in different 'worlds', playing different roles within the organization. One may speak of a master and a servant living in 'different worlds' within the same house, or an administrator and a teacher in an educational institution. Each follows its own set of conventions and each works in its own way, but there is cooperation to make the system work as a whole.

The three worlds of mathematics: embodiment, symbolism, formalism work together to build the ever-growing system of mathematical knowledge and our human way of mathematical thinking. Each world grows in its own way: embodiment in a van Hiele-type sequence, symbolism through progressive process-object compression, and formalism through definition and deduction. In school, embodiment and symbolism are blended together, with embodiment giving human meaning and symbolism giving power of calculation and symbol manipulation. There is also a development of definition and deduction, but these definitions are usually based on mental embodiments of the concepts or on symbolic manipulations that are performed. It is only when the switch is made to formulate mathematic in terms of set-theoretic axioms and definitions that we fully enter the world of formalism.

ENCOURAGING CHILDREN TO LEARN TO THINK MATHEMATICALLY

Given the framework of embodiment, symbolism (and later formalism), we may ask how children actually learn to develop successive levels of increasingly sophisticated thinking. First they have natural abilities built in by their genetic inheritance. These include abilities such as a child recognising its mother's face and more generally, being able to recognise the same object seen from different directions. A related ability is to subtly categorise different objects in the same category. For instance, a child can distinguish between a 'dog' and a 'cat' even though there are many different kinds of dog and different kinds of cat with many similarities between dogs and cats. This subtle human facility proves to be essential to mathematical thinking as the child recognises different shapes as a 'triangle' or a 'circle'. It enables us to see a specific example as a representative of a more general case, not only in geometry, but also in arithmetic where a rectangular array of counters with 3 rows and 4 columns can be seen as 4×3 or as 3×4, representing not just the specific layout, but as a representative of any other numbers of rows and columns, revealing that multiplication is independent of the order. In this way, it is possible to *embody* properties by some kind of physical context that is seen, not only as a specific instance of something, but *as a representative of a general principle*.

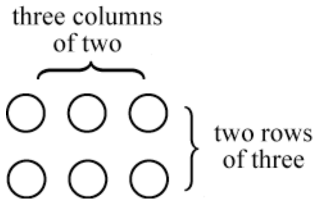


Figure 9: using embodiment to see that multiplication is independent of order

Another fundamental characteristic that we share with other species is the ability to practice a sequence of actions until we can perform them automatically without conscious thought. Even a dog can do this. This facility is the basis of learning procedures that are fundamental to mathematical thinking. However, there is a difference between procedures learnt without meaning that may break down when used in a new context, and procedures that are part of a richer structure of compressed mathematical knowledge with the flexibility to seek the solution of novel problems.

Met-Befores

A child must build on experiences already encountered in earlier life. These earlier experiences affect the way that we think *now*. What matters is not what we originally encountered, but how this previous experience affects our current thinking. Tall (2004) called a current structure in the brain, constructed from an earlier experience, a *met-before*. The term was derived originally from the idea of 'metaphor', which relates a current context to another one that it resembles in significant ways. The word ‘met-before’ was coined because it can be used in conversation with children to say, ‘what have you *met before* that causes you to think like that?’

Met-befores can have both positive and negative effects on new learning. The Japanese problem-solving approach of building on old knowledge is very focused on using old knowledge in new and better ways (see, for example, Isoda, 1996b). Sometimes this knowledge can be built on directly, but often a reconstruction is needed to work in the new situation, as is seen, for example, in the successive levels of van Hiele where ideas at a previous level conflict with a new level of thinking (Isoda, 1996a).

Met-befores need not be explicitly taught: they may be subtly conceived by the child as a result of his or her previous experiences. For instance, a child may never be taught explicitly that when two whole numbers are added, the result is bigger, when they are multiplied the result is usually bigger still, or that when a number is taken away, the result is always less. But these met-befores are widely found in children who have had experience of whole number arithmetic.

Isoda (2007) presented the problem of finding two numbers X and Y where

$$X \times Y = X - Y.$$

The problem was also presented in the form

$$\square \times \triangle = \square - \triangle,$$

that might be suitable for a younger child. Looking at the problem with the eyes of a child with experience only of whole number arithmetic, the left hand side starts with X and makes it *bigger* by multiplication (or the same when multiplying by 1), but on the right hand side, take away makes it *smaller*. There is absolutely no way a number can be made bigger and smaller at the same time to give the same result. So the equation has no solutions.

Experience with Japanese lesson study shows that many children familiar with zero have difficulty with the possible solution (0,0). In their earlier experience different symbols represent different things, so this met-before causes them to expect the square and triangle to have different values, so they cannot both be zero.

A small number of children who have begun to study the subtraction of fractions find the case $(1, {}^1/_2)$. Some who realise that it is possible to put fractions in the boxes may begin to find the sequence of solutions: $(1, {}^1/_2)$, $({}^1/_2, {}^1/_3)$, $({}^1/_3, {}^1/_4)$, …

Children at a later stage who are familiar with algebra may solve the equation

$$xy = x - y$$

by adding y to both sides, factorising the left hand side and dividing by $x+1$ to get

$$xy + y = x,$$
$$y(x+1) = x,$$
$$y = \frac{x}{x+1}.$$

So, for every x, except $x = -1$, there is a single solution for y. If $x = 1$, $y = \frac{1}{2}$; if $x = 2$, $y = \frac{2}{3}$, and so on.

Further stages follow. For instance, when irrationals are introduced, for $x = \sqrt{2}$, then

$$y = \frac{\sqrt{2}}{1+\sqrt{2}} = \frac{\sqrt{2}\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)} = 2 - \sqrt{2},$$

and later, for complex numbers, if $x = i$, then

$$y = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \tfrac{1}{2}(1-i).$$

At each introduction of new number systems, new solutions come to light. At each stage old experiences act as 'met-befores' that may cause obstacles to seeing possible solutions. The child's experience with whole numbers and the met-befores relating to 'multiplication makes bigger' and 'take away makes smaller' may be an obstacle to solving the equation for a child who has only encountered whole numbers.

There is a change from whole number arithmetic to fractions, where each whole number has a next number, for instance after 2 comes 3 with no others between, but fractions do not have a 'next' number and there are many fractions between 2 and 3. There is another change from positive counting numbers to negative numbers, where taking away a negative number gives not less, but *more* and multiplying two (non-zero) numbers always gives a positive value. There are further changes in shifting from fractions to real numbers with the introduction of irrational numbers that can never be precisely expressed as a fraction. And there are more changes in shifting from real numbers to complex numbers where now, the square of a number can sometimes be negative.

Long-term learning therefore requires more than simply building on previous experiences; at significant points in learning, old ideas that worked in a different situation (and continue to work there) need to be re-considered and modified to make sense in a new situation.

We do a disservice to learning if we believe that mathematics is simply a case of learning more and more subtle ideas building directly on what we learned before; increasing subtlety involves reconsidering one's old knowledge and modifying it to make sense of new situations.

Blending Knowledge Structures

When a learner meets new ideas in a new context, (s)he must build on former knowledge to make sense of the new ideas. The first action is to bring to mind what one already knows. This may recall ideas from different knowledge structures, which must be *blended* into a coherent whole.

Many concepts in mathematics are blends of different kinds of knowledge structure. Counting and measuring use numbers in different ways. A young child may use small blocks to put them in a line for counting, addition and takeaway.

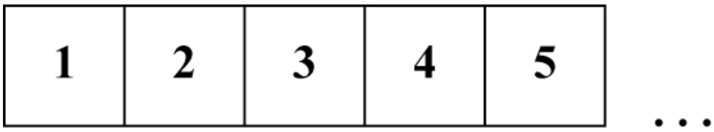


Figure 10: The number track with numbers as blocks

Blocks placed end-to-end give a *number track* (in the language of the English National Curriculum) where each number is counted on successive blocks. There is no zero on a number track. It is the *blocks* that are counted. There are no numbers 'between' the blocks, after 2 comes 3, then 4, then 5, and so on …

The *number line* starts with zero. Each successive number 1, 2, 3, 4, 5, … is represented by a point on the line, each a unit distance further to the right. Between each point on the number line there are many intermediate fractions. There is no 'next' real number, and between any two numbers on the number line, there are many more numbers.

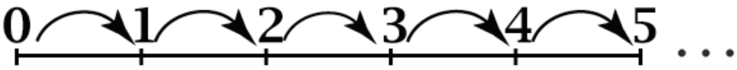


Figure 11: The number line with numbers as points

These differing embodiments—one involving discrete counting numbers, the other continuous measuring numbers—share a great deal in common. Two plus two always makes four, and addition, subtraction and multiplication work in the same way. But division operates differently. When measuring numbers are used to share 9 metres of cloth into 4 equal parts gives $2^{1}/_{4}$ metres for each share, but with counting numbers, dividing 9 children into 4 groups gives 2 in each group with one left over.

Blending different knowledge structures that have some features in common and some features that are different may occur in different ways in different individuals. Some may focus on the features that are relevant and shared, leading to powerful ways of thinking. Some may focus in part on the features that conflict and sense anxiety because they do not fit. In a given classroom, some children may feel the pleasure and power of ideas that fit together, others may be aware of other features that conflict and cause anxiety. Over time this can lead to a significant divergence between those that enjoy mathematics and succeed and those who begin to dislike mathematics and are more likely to fail.

DIFFERING CULTURAL APPROACHES TO MATHEMATICAL THINKING

Even though we claim that the underlying compression of knowledge into thinkable concepts is a common underpinning of mathematics in all cultures, building on embodiment, developing symbolism for counting and arithmetic in varying degrees, differing cultures have a wide range of ways of naming and using mathematical ideas.

In Japanese, the numbers one to ten are ichi (1), ni (2), san (3), shi (4), go (5), roku (6), shichi (7), hachi (8), kyu (9), ju (10). The numbers then follow a regular pattern with 11 as 'ju ichi' (ten one), 12 as 'ju ni' (ten two), and so on. Twenty is 'ni ju' (two ten), thirty is 'san ju' (three ten) and combinations of tens and units are said in the same way with 56 being 'go ju roku' (5 tens plus 6).

The number 100 is pronounced as 'hyaku' and counting above 100 involves extending the system to hundreds, tens and units in essentially the same way, so that, for instance, the number 256 is 'ni hyaku go ju roku' (2 hundred, 5 tens, 6).

This regularity of number names of small numbers in Japanese and other Asian languages makes learning number and arithmetic simpler than the irregular English names, including 'eleven, twelve, thirteen' that hinder the learning of place value and arithmetic. (Eleven is related to the embodiment of numbers using fingers, being the old English word for 'ei lief on' meaning 'one left over' and 'twelve' being 'twe lief' or 'two left', after counting ten fingers on our hands. It is the same root for the German words 'elf', 'zwölf' for 11, 12, before the greater regularity of dreizehn, vierzehn, for 13, 14, and so on.) There is considerable evidence that Japanese and Chinese children have a better grasp of place value at an earlier age than their Western counterparts (see, for example, Fuson & Kwon, 1991).

In addition to these different ways of representing numbers and other mathematical concepts, there are widely differing attitudes towards mathematics in different cultures. If the culture values respect for its elders and obedience by its juniors, then the teaching of mathematics may involve the transmission of procedural knowledge that must be practised diligently by its children. If the culture respects individuality and creativity, then this too may be reflected in the teaching of mathematics.

Though human cultures are different, the common underlying biological basis of mathematical thinking is shared. This common biological basis provides a framework that may shed light on the learning and teaching of mathematics over an individual's life-time. For example, if mathematical ideas fail to be compressed into thinkable concepts, then the ability to think mathematically may be severely compromised. Procedural learning in itself is not enough. Even if the procedures are performed efficiently and accurately, mathematical thinking requires a richly connected system of thinkable concepts to operate at its most powerful level, regardless of the particular culture in which the mathematics is being learnt.

Various studies have been carried out in countries around the world by doctoral students at Warwick University. These reveal that there is a widespread goal of 'raising standards' in mathematics learning, which are measured by tests that *could* promote conceptual long-term learning, but in practice, often produce short-term procedural learning that is likely to be less successful in developing long-term flexibility in understanding and in solving non-routine problems.

Procedural conceptions of fraction

Md Ali (2006) studied the teaching of fraction in Malaysia in a curriculum designed to raise the standards for all children, including the development of conceptual learning. However, he found that fractions were taught by classroom recitation with the teacher prompting the students to learn the procedures. More than one procedure might be given, for instance, calculating 'two-fifths of twenty-five' in two distinct ways. The first works out a fifth of twenty-five, which is five, then multiplies by two, to get ten. The second multiplies two times twenty-five to get fifty and divides by five, also to get ten. The children were certainly effective in performing routine calculations, but had difficulty in dealing with novel problems. In terms of the framework, they had reached a stage where they could choose the most efficient way of operation from more than one procedure, perhaps even seeing that the different operations have the same effect, but they did not seem to move to a higher level to develop the flexible use of fractions as thinkable concepts.

Procedural embodiments in algebra

Working with a group of committed teachers in Brazil, Rosana Nogueira de Lima (de Lima & Tall, 2006) found that teachers were concerned that their students found algebra difficult and so they concentrated on techniques that would enable them to solve routine equations on examinations. The students were taught to solve linear equations by using the principle of 'doing the same thing to both sides', but for most of them 'adding the same thing to both sides' became a meaningless rule to shift the term to the other side and 'change the sign'. For instance, $3x+2=8$ is solved by shifting the 2 to the other side and changing its sign, to get $3x=8-2$, so $3x=6$. Then a new rule was introduced to simplify this by 'moving the 3 over the other side', this time 'putting it underneath' to give the solution:

$$x=\frac{6}{3}$$

Such an activity gives what may be termed a *procedural embodiment*, remembering a sequence of actions to perform, rather than a *conceptual embodiment* that gives a coherent meaning to the underlying concepts. This method worked for a few more able students who could carry out the procedure accurately but without meaning. However, for most of the students, the procedure is fragile and many make mistakes, such as changing sides for the 3 in $3x=6$ with the additional 'changing signs' to get

$$x=\frac{6}{-3}$$

In solving quadratics, aware of the students' growing difficulties, the teachers taught them to use the formula. This worked for equations in the standard form $ax^2 + bx + c = 0$, but failed for the simpler equation $(x-2)(x-3)=0$ where the students were *given* the possible roots 2 and 3 and asked whether these were correct. Most wrote nothing in response, and most those that did write something attempted to multiply out the brackets and solve the equation using the formula.

Complications in the function concept

In a study in Turkey, Bayazit (2006) followed the teaching of two secondary teachers introducing the concept of function. While one had a simple approach in which every lesson focused on the meaning of the definition of function, the other, aware of the student difficulty, taught the students how to cope with specific problems.

The approach used by teacher Ahmet referred time and again to the simple idea that a function involves two sets A and B and, for each element x in A, there is precisely one corresponding element y in B which is called $f(x)$. For instance, Ahmet introduced the constant function $f(x) = c$ as the *simplest* of functions that mapped *every* value of x in A onto the single element c in B.

The other teacher, Burak, taught the 'vertical line test' as a specific test for functions, practising examples to get it right. He knew the students had difficulty with the constant function because the function did not 'vary with x'; his solution was to explain that a function did not need to have x in the formula.

While Ahmet patiently referred the students to the power of the definition, Burak taught many separate techniques to answer problems in tests. For instance, in handling the notion of inverse function, Ahmet used the definition to show that a function could only have an inverse if each element $y \in B$ had a *unique* element $x \in A$ to map onto it as $y = f(x)$. He emphasised the precise conditions under which a function could have an inverse and then linked this idea to many examples.

Meanwhile, Burak told the students first how to find the inverse function in a simple case: to find the inverse of $y = 2x + 3$, one would express x in terms of y by subtracting 3 from both sides and dividing by 2 to get $x = \frac{y-3}{2}$, then interchange x and y to get the inverse function as $y = \frac{x-3}{2}$.

He would often indicate to students that an examination or test required a particular way of learning:

> If you want to succeed in those exams you have to learn how to cope.
>
> Do not forget simplification. It is crucial, especially [in] a multiple-choice test.

When the students took their examinations, Ahmet's flexible approach was far more successful than the detailed approach with may rules proposed by Burak.

EXAMPLES OF LESSON STUDY IN ACTION

Lesson study develops a model teaching approach to prepare lessons in a manner that encourages learners to use problem-solving methods to consider various ways of interpreting mathematical ideas, to make connections and to choose ideas that enable them to build more sophisticated ways of thinking mathematically (see, for example, Isoda 2007, Okubo 2007 and Takahashi 2007).

Typical videos of lesson studies are to be found on the University of Tsukuba website, including four lessons delivered in Elementary School at grades two, three, five and six[1]. In the grade three lesson, based on the problem-solving approach, Hideyuki Muramoto presents one of a sequence of lessons in which he introduces the concept of multiplying a single digit number by a double-digit number in a third grade lesson. One way might be to teach a standard procedure for long multiplication. Mr Muramoto follows a lesson study approach of encouraging the children to explore a number of different ways of using their knowledge of single digit arithmetic to develop more sophisticated ways of multiplying bigger numbers.

Beginning multi-digit multiplication (Grade 3)

In a previous lesson, Mr Muramoto had given each child a piece of paper with three rows of 20 discs. He posed the problem 'how many discs are there?' and suggested considering various ways to calculate the total. The task was to break each row of twenty discs into smaller parts so that each part could be multiplied by three. Some children suggested breaking the twenty into four lots of five to multiply each five by three to get 15 and then to add the four 15s together to get 60 (Figure 12).

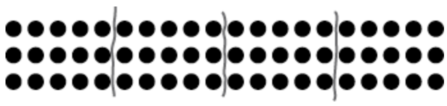


Figure 12: Twenty broken into fives to calculate twenty times three using smaller quantities

Other children suggested breaking the twenty into two tens, to calculate ten times 3 as 30, and adding 30 plus 30 to get 60. Others suggested breaking the row of 20 into 9 plus 2 plus 9, to get 9 times 3 is 27, 2 times 3 is 6, the other 9 times 3 is 27, so 27+6+27 is again 60.

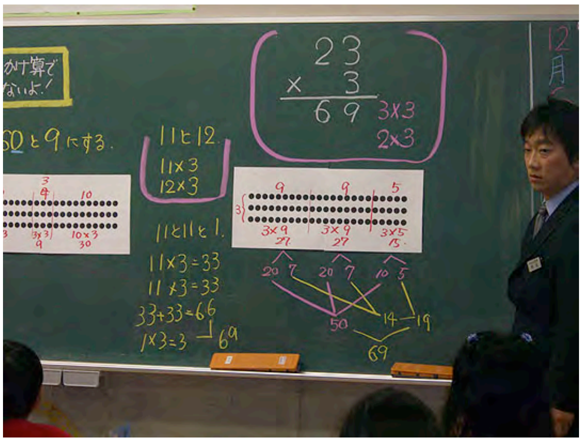
The research lesson (figure 13) extended this situation to the problem of calculating 23 times 3. Each learner had a piece of paper with three rows of 23 and the class was asked to work on the problem for a few minutes. Many children broke the 23 into 10+10+3, so that 23 times 3 is 30+30+9, which is 69, or into 20+3 to multiply by 3 to get 60+9, which is 69. After the previous lesson, none of the children suggested breaking 23 into 5+5+5+5+3. One suggested breaking 23 into 9+9+5, which involved

[1] Videos may be found at http://www.criced.tsukuba.ac.jp/math/apec/apec2007/index.html.en#video nd also on the website http://apec.pbwiki.com/Classroom%20Innovations%20through%20Lesson%20Study

Sharing ideas

Explaining to the class

Displaying ideas on the board



The teacher summarises

Figure 13: Lesson study activities

more complicated calculations, others suggested breaking 23 into 11+12 or 11+11+1. Having seen that some children were using the algorithm in a vertical two-column format, the teacher encouraged a child to explain this and then discussed with the class how this related to the other methods.

Although Mr Muramoto did not himself use the term 'Ha-Ka-Se' in the lesson, the idea of 'Ha-Ka-Se' is involved as it becomes clear that the solution breaking into 10s and 1s is faster, easier, and less likely to produce a mistake. He also used the greater complication of other methods to focus on the value of performing the multiplication in standard vertical form.

Using the framework of embodiment and symbolism, the lesson has the following format:

1. Embody the problem (here the product 23×3);
2. Find several different ways of calculation (here 23×3 is three lots of 10+10+3 or three lots of 9+9+5) where the embodiment gives meaning to the symbolism;
3. See the flexibility, that all of these are the same;
4. See that the standard algorithm is the most efficient.

Here, embodiment is used to give meaning while symbolism enables compression to an efficient symbolic algorithm.

It is not to be expected that all children in the class will cope with all the procedures suggested. Some will find the idea of breaking up 23 into 9+5+9 and performing the necessary calculations quite challenging. Others may see the full range of methods as equivalent ways of carrying out the calculation, with some more appropriate than others. However, the main aim of building the method of long multiplication as the most suitable algorithm gives each child an opportunity to make sense of the algorithm as the *best* of several ways rather than the *only* way to perform the operation.

Other lessons studied at the same time used a problem appropriate to the development of the children to enable them either to build new ideas or to use their old ideas in a problem-solving situation.

A Novel Problem in Grade Two: Candies on a Plate

A grade two lesson given by Takao Seiyama involved a novel problem that used the children's knowledge of geometric shapes and enabled them to play with relationships between numbers. Mr Seiyama's plan included the following elements:

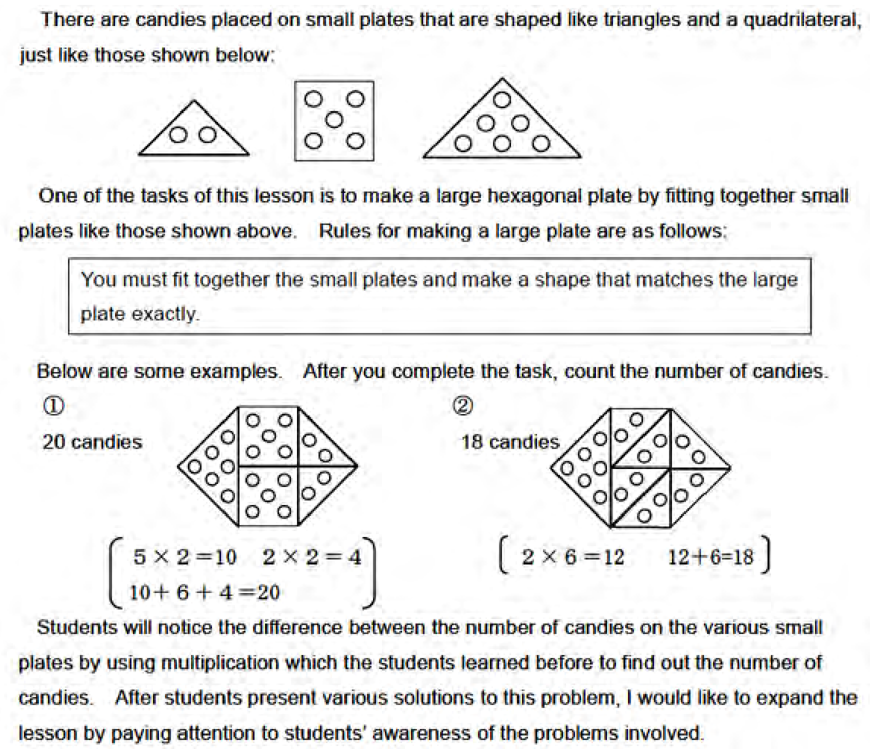
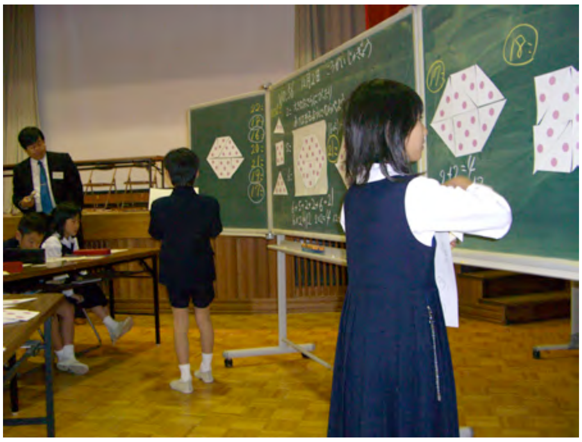
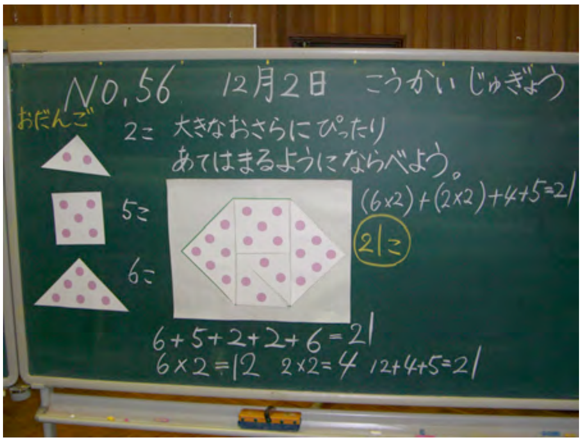


Figure 14: A problem-solving situation

This problem has a collection of triangles and rectangles that can be used to make up a particular hexagon that has a useful shape but is not regular. The children can experiment and report how they can put the shapes together to get different numbers of candies. This leads towards a shared analysis of the possibilities, developing ideas of proof based on a physical embodiment with only a small number of variations. The lesson encourages mathematical thinking to solve a novel problem, as opposed to the previous case that develops the specific technique of multi-digit multiplication.



Sharing ideas

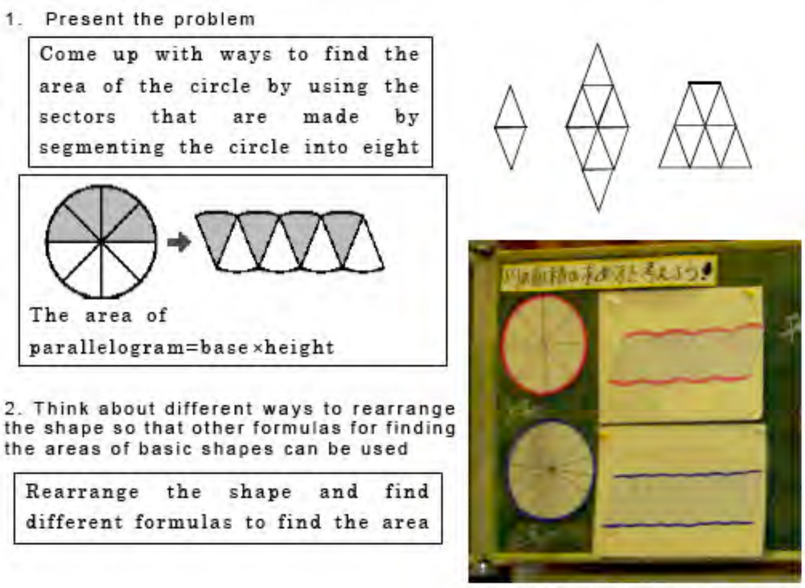


Displaying calculations

Figure 15: How many candies on the plate?

The Area of a Circle: Grade 5

The grade five lesson given by Yasuhiro Hosomizu to find the area of a circle is the seventh of a sequence of ten lessons calculating the areas of various shapes, including squares, rectangles, parallelograms and triangles. In the previous lesson, the class discussed various ways of calculating the area of a circle: by fitting it inside a larger square, by counting squares, and by cutting the circle into equal pieces to fit them together in various ways to get shapes for which area formulas are already known (figure 16).



dividing into 8 and 16 pieces

Figure 16: ideas for the area of a circle

The lesson began with a recapitulation of the previous lesson as individual children recalled different ways of finding the area of a circle. Each child had been given a cardboard circle cut into either 8 or 16 equal sectors and all were asked to re-arrange them to get a new figure whose area they could calculate. The suggestions put forward included not only rearranging the sectors to give an approximate parallelogram, but also other shapes including triangles and trapezoids.

Early on, one child considering the shape like a parallelogram explained that the pieces were not precisely triangles, but the more pieces, 8, 16, 32 etc, then the closer the figure would be to a parallelogram whose width is half the circumference and height equals the radius. This gave the area as half the circumference times radius. Using earlier knowledge that the diameter is 2 pi times the radius, the formula is $(2 \times \text{pi} \times \text{radius}) / 2 \times \text{radius}$ which another child translated into $\text{pi} \times \text{radius} \times \text{radius}$. Other shapes constructed had different ways of expressing the formula, allowing a discussion of different ways of writing a formula to express the same quantity.

A major point brought out by the teacher was the simple but subtle idea that the language used to describe the area switched from words involving 'width and height' appropriate for a parallelogram to 'diameter' and 'radius' that are appropriate for a circle.

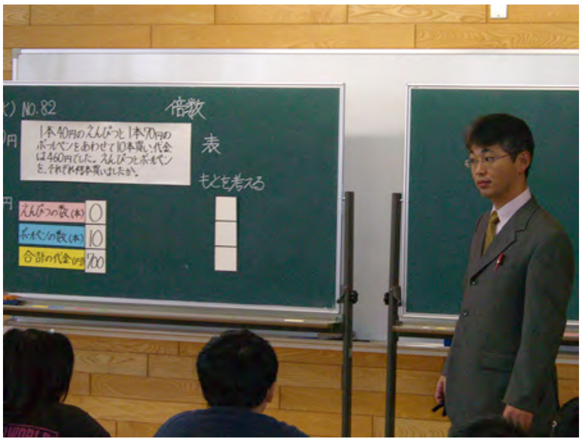
It will be many years before the children encounter the limit concept, so it may not be always clear how they make sense of the area. The aim here is to build on their perception that if a figure is cut up and reorganised without overlap, the area remains the same.

Working with different shapes, the idea of Ha-Ka-Se appeared again: to ask which of the various different ways was the easiest, the quickest, the most understandable. The discussion would continue in the next lesson, but it was already becoming apparent that the approximate parallelogram was a good candidate out of the several possibilities found.

Thinking Systematically – Grade Six

The fourth lesson, given by Atsutomo Morii, involved a single problem where children were told that a combination of pencils at 40 yen and pens at 70 yen cost 460 yen for 10 items, so how many pens and pencils were bought? The teacher was aware that there are quick ways of solving the problem, for instance, 10 pencils would cost 400 yen, so, to make up the total of 460 yen, an extra 60 yen was available to pay for pens instead of pencils, giving 2 pens and 8 pencils. However, the teacher's main purpose was to introduce the use of tables to solve the problem by encouraging the children to make suggestions and take up the idea of a table when it was proposed by one of the children.

He had already prepared vertical strips of paper that could be used to record 3 numbers: the number of pens, the number of pencils (making 10 in all) and the price. As children filled in their strips and stuck them to the board, others moved them to build up the whole table (figure 17). During the activity, the teacher paid attention to children who were struggling. Other children developed quick ways of building up the table, column by column, starting the first column with 0 pencils and 10 pens

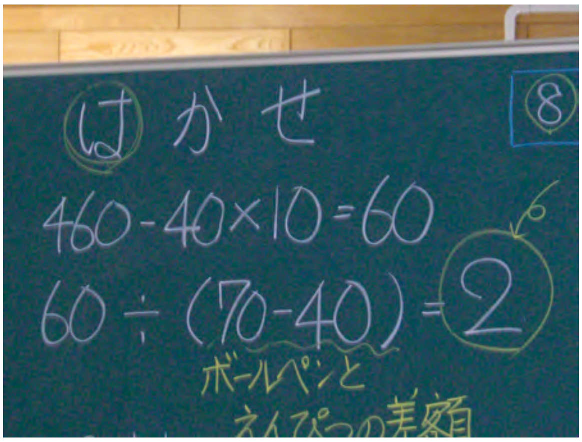


Considering a table layout

Helping those who were struggling

Building the columns

Organising the full table



A special solution

The end of the class

Figure 17: Thinking systematically using a table

costing 700 yen, as suggested by the teacher, but then instead of working out each column by addng the cost of pencils and pens, they found that increasing the pencils by 1 and decreasing the pens by 1 caused the cost to decrease by 30 yen each time, so that it was easy to fill the rows horizontally by successively taking 30 yen off the previous total.

Later a child suggested the quick solution mentioned above, starting with 10 pencils costing 400 yen and exchanging 2 pencils for pens to account for the extra 60 yen to get the given total of 460. The teacher held a conversation with the child and wrote up the calculations on the board as $460 - 40 \times 10 = 60$, then $60 \div (70 - 40) = 2$. This writing in general form reflects the longer-term intention to formulate general arithmetic expressions and shift later to the use of algebra.

The children also used Ha-Ka-Se to observe that the table method is accurate and meaningful, but perhaps a bit slow. (This was part of the long-tem intention to use this experience to justify the shift to general algebraic symbolism.)

By the end, the teach had a complete board layout from left to right, starting with the problem, laying out the table, and making other remarks, giving a complete record of the lesson. This is characteristic of lesson study in which the writing on the board is built up from left to right to tell the story of the lesson.

Each of the four lessons shows lesson study addressing ways of encouraging mathematical thinking. Two of them (multi-digit multiplication and area of a circle) use Ha-Ka-Se to build new ideas using ways of working that are fast, easy and accurate. One (thinking systematically) uses Ha-Ka-Se to prepare for the need to shift from arithmetic to algebra by demonstrating the successful, but long way of using a numeric table. One (candies on a plate) develops mathematical thinking in a novel situation, where the lesson is more focused on the development of problem-solving attitude than the building of specific mathematical knowledge to be used in the future.

THE ROLE OF LESSON STUDY FOR LONG-TERM DEVELOPMENT

Lesson study om Japanese elementary schools has produced significant improvements in the learning of mathematics, even in large large classes of 40 children, by using a carefully developed lesson designed to encourage the learner's participation. We believe it can benefit further by reflecting on different ways of teaching in different cultures, to seek ways in which mathematical thinking can be developed over the long-term. We see the framework not only helping in the analysis of the success of differing approaches in different cultures, but also in developing more focused ways of long-term construction of mathematical knowledge. While it seems evident that those who focus on the efficient and accurate use of procedures before attempting to solve more general problems have a certain success in examinations, long-term success needs to take into account the blending and compression of knowledge appropriate for higher levels of sophistication in mathematical thinking.

Here we have formulated mathematical thinking developing over time. It develops through successive levels of sophistication in embodiment, building from perception, through description, practical construction, definition, deduction and on to Euclidean geometry. In parallel, the symbolism of arithmetic, then algebra and symbolic calculus begin with actions such as counting that are symbolised and compressed into concepts such as number. Symbolism involves a sequence of successive cycles in which operations are compressed into flexible procepts operating dually as flexible processes and concepts. The child must build on experiences met before, but these are blended together in the biological brain in ways that are in part coherent and in part in conflict. Mathematics is therefore a vehicle for extreme power and pleasure when the best ideas are fitted together, but a source of anxiety when blends are in conflict. As a source of anxiety, it may block higher level thinking. At a lower level, it may be possible to learn to reproduce routine algorithms which may have some success but, on their own, they my fall short of the sophistication required to develop flexible mathematical thinking.

The long-term framework proposed may help consider which ways of teaching and learning are a better fit for the developing human brain. This may even question generally accepted ways of teaching mathematics. For instance, should we teach only 'proper' fractions first which leads to the met-before that a fraction is 'less than one.' Should we even distinguish between 'proper' and 'improper' fractions when both can arise naturally in well-chosen problems.

How is it best to introduce algebra in the long run? The embodiment of an equation as a balance has some advantages, with solutions arising through 'doing the same thing to both sides', but the balance only makes embodied sense when the quantities are *positive*, so it can act as an obstacle in understanding more sophisticated algebraic ideas involving positive, negative, or even complex quantities. Would it not be more sensible to seek the flexible meaning of symbolism as thinkable concepts and use the greater power of arithmetic computation and symbolic manipulation?

Do we need to begin the study of calculus with the limit concept? This is a met-before shared by most experts who see it as the foundation of calculus, yet the idea of rate of change can be *seen* in terms of the embodied changing steepness of the graph. Would it not be more helpful to *begin* with embodiment in the calculus and use numerical calculations for nearby points to get a good numerical approximation and then symbolic methods to get an exact value?

The cycle of compression from operation to thinkable concept may be seen to involve different procedures at first, but then different procedures with the same effect are said to be equivalent, including equivalent fractions $^2/_4$, $^2/_4$, equivalent expressions $3(x+2)$, $3x+6$, equivalent vectors, and so on. At the next stage of compression, these become single thinkable concepts—rational number, algebraic expression, free vector—which may be written flexibly in different ways.

The Japanese Ha-Ka-Se seeks ways that are fast, easy, accurate, which are often applied to operations, rather than thinkable concepts. How do we help children move from stage (iii) of fast, efficient, easy operations to thinkable concepts at stage (iv) are the essence of flexible thinking? In introducing concepts such as fractions and rational numbers, the fractions are first considered to be different before some are seen to be *equivalent*, then equivalent fractions are re-thought as a single rational number. The same occurs throughout mathematics as adding negatives or subtracting positives are at first equivalent, then compressed into the simpler notion of subtraction, or the concept of vector moves from many equivalent vectors to a single free vector flexibly representing any and all of them at the same time.

As learners become more sophisticated, embodiment can help the shift from equivalence to flexible procepts, for *equivalent operations have the same effect*. Equivalent fractions give the same quantity later becoming the same rational number, equivalent expressions always evaluate to give the same input-output relationships later to represent the same function. In this way, shifting attention from the *operation* as a procedure in time to the *effect* of the operation as a mental idea can help in the shift from distinct but equivalent operations to single thinkable concepts.

Physical embodiment is essential in the early stages to give genuine meaning to thinkable concepts. But as the mathematics becomes more sophisticated, new embodiments may become increasingly complicated. On the other hand, the completion of a cycle of compression from operation to thinkable concept enables more flexible thinking that is essentially simpler. This suggests a long-term aim to complete cycles of compression from operation to thinkable concept (procept) to produce increasingly sophisticated thinking.

With the arrival of a new technological era, we need to think about the appropriate ways to support mathematical thinking in the new context. For instance, computers carry out algorithms and the results may be often be presented visually to enable the learner to reflect on the relationship between embodiment and symbolism.

What is certain is that in a time of technological change and economic globalisation, we need to use every resource in our power to improve the ways we organise our lives. Lesson study viewed within a framework of long-term growth of mathematical thinking is a powerful tool to improve how we learn to think mathematically.

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