THE SIMPLICITY, COMPLEXITY AND COMPLICATION OF THE FUNCTION CONCEPT

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In this paper we focus on students' understanding of the core concept of function, which is mathematically simple yet carries within it a rich complexity of mathematical ideas. We investigate the linguistic complexity that reveals itself through the mathematically simple notion of a constant function and the representational complexity involved in different representations. Results reveal a spectrum of performance in which a few students may be able to grasp the function concept as a rich highly connected cognitive unit, but others are overwhelmed by a complexity that is highly complicated with many separate aspects weakly connected together.

INTRODUCTION

To a mathematician, the notion of function is a model of *simplicity*. What could be simpler than the idea that 'we have two sets and each element in the first is linked to precisely one element in the second'? The definition is not only mathematically simple, for the mathematician it provides access to a huge *complexity* of mathematical ideas. Some students are able to build this subtle combination of simplicity and complexity. For others, however, the situation is quite different. As they respond to being introduced to the notion of function, they bring their implicit understandings of language and all their previous experiences to bear on the task. The result for them is a highly *complicated* array of personal meanings that both help and hinder their interpretation of the mathematical concept.

LITERATURE REVIEW

The complexity of the function concept has been a focus of attention for almost a generation. Vinner (1983) drew attention to the distinction between the *concept definition* that mathematicians use to define a mathematical concept and the *concept image* which people generate in their mind. He also showed that most students use their own *personal concept definition* for the notion of function, giving highly idiosyncratic meanings to the term. Subsequently, two distinct lines of enquiry have occurred in the literature, on the one hand the nature of function as process and mental object (Dubinsky, 1991; Breidenbach et al, 1992; Sfard, 1992), and the multiple representations of a function (Confrey, 1994; Kaput, 1992; Keller & Hirsch, 1998; Leinhardt et al., 1990). Others have since combined these two aspects into a broader perception of 'horizontal growth' (between representations) and 'vertical growth' (in compression from process to concept) (Beineke et al., 1992; Schoenfeld et al, 1993).

THEORETICAL FRAMEWORK

Our study focuses on the nature of the concept definition and its interpretation in a variety of representations as presented in the Turkish mathematical curriculum. We are particularly interested in the subtleties of meaning attributed to the (personal) concept definition, and the meanings inherent in the various representations.

Thompson (1994), questioned the meanings given to representations seemingly shared by the mathematics education community:

> …the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation… the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.
>
> (Thompson, 1994, p. 39)

In practice, the simple core concept (given essentially by the definition) proves cognitively elusive and students are introduced to examples in various forms:

- a verbal representation of a function in formal or colloquial (everyday) language,
- a set diagram (representing a function by two sets and arrows between them),
- a function box (representing of an input-output relationship),
- a set of ordered pairs (considered set-theoretically),
- a table of values (often computed using a formula or computer procedure),
- a graph (drawn by computer or by hand),
- a formula.

Each of these has its own peculiarities that contribute to the complication of the student's concept image. To describe the manner in which the human mind copes with these images, the literature has developed a language of *prototypes* (typical instances) and *exemplars* (more specific cases), the latter often being seen as *clusters* of examples. (See Makin & Ross (1999) for a detailed discussion). Our analysis suggests that different representations are presented and interpreted in subtly different ways. For instance, set diagrams are often introduced as prototypes to represent general ideas, whilst graphs and formulae are met successively in clusters (linear functions, quadratics, trigonometric, exponential, and so on). This presents a dilemma for the curriculum developer:

> The learner cannot construct the abstract concept of function without experiencing examples of the function concept in action, and they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept. (Bakar & Tall, 1992, p. 13)

BACKGROUND OF OUR STUDY

The purpose of this paper is to explore the nature of this subtle mathematical blend of simplicity and complexity and to focus on problems that it presents to many students, with particular reference to a specific curriculum for the function concept. The Turkish curriculum, in which our study is situated, begins with the following:

Formal Definition

Let A and B two sets other than empty set, and f is a relation defined from A to B. If

i) $\forall x \in A$, $\exists y \in B$ s.t. $(x,y) \in f$,

ii) $(x,y_1) \in f$ and $(x,y_2) \in f \Rightarrow y_1 = y_2$

Then the relation f from A to B is called a *function* and denoted by f: $A \rightarrow B$ or $A \xrightarrow{f} B$. A is called the domain and B is called the range.

This mathematically simple notion is followed by a four part *colloquial definition* to help the student focus on the essential properties of a function. These explain properties (i) and (ii) and add (iii) different elements in A can be related to the same element in B, (iv) some elements in B may not be related to an element of A.

The students are then given experience of functions in different representations as

set diagrams, *ordered pairs*, *graphs* and *formulae*,

(but not as function boxes or tables). Our research is to investigate and analyse the use of formal definition and the concept images that arise (in the sense of Tall & Vinner, 1981), with particular interest in the role of *language* and the *prototypes* and *exemplars* that arise in the students written and spoken responses.

METHODOLOGY

One hundred students from four different upper-secondary schools in Turkey were given a questionnaire which asked whether various graphs, equations, correspondences between two set diagrams, sets of ordered pairs are functions or not. Based on an analysis of the responses, eight students were chosen to represent a spectrum of performance for individual interviews. The interviews were semi-structured with certain questions asked of all students with a flexible continuation to take account of individual responses. In this paper, we focus on two specific areas in the interviews: responses to the mathematically simple case of a constant function and responses to a range of questions relating to graphs (with marked domains), set diagrams, ordered pairs, and formulae.

DATA ANALYSIS

The constant function

Mathematically, the simplest function is a constant function that assigns to every element of the domain a single element in the range. Cognitively, however, this example is far from simple. It is a singular case violating strong links in the concept image (such as the idea that f(x) 'depends on' or 'varies with' x).

The following questions were asked to eight students from grade 9-10-11 (aged 15-16-17) in four different upper-secondary schools in Turkey:

I will show you some expressions. Can you tell me whether they are functions or not?

They were then, in succession, shown cards with the following written statements:

$y = 4$,
$y = 4$ (for all values of x),
$y = 4$ (for $x \geq 2$).

The subtle differences in these three expressions greatly affect the student responses so that there is a wide range of responses. Six of these are as follows:

	$y = 4$	**$y = 4$ (for all values of x)**	**$y = 4$ (for $x \geq 2$)**
Eren	*Yes*: This is $f(x) = y$ ... it can be a constant or a variable, this goes to a constant, thus it is a function.	*Yes*: It is a function which is not dependent on x.	*No*: not a function, there isn't anything for values less than 2 ... [When the domain was mentioned, he changed his response to *yes*.]
Can	*Yes:* It always goes to 4 in the range.	*No:* There are elements left in the range …[see below]	*Yes:* It goes [is assigned] for less than two. [see below]
Aysun	*Yes*: It is a constant function ... without x … dependent on y … $y = ax+b$ … without x it is constant.	*No*: There isn't any value for x…I can't see x here…there isn't any x on the other side of y.	*Yes*: I have to see two sets, this is function. In the domain, elements greater than 2 are assigned to 4.
Murat	*No*: not a function … I can't say the reason…I don't know.	*Yes*: function…because it says "for all values of x".	*Yes*: function, since 4 is greater than 2.
Irem	*No*: there should be an equation, but there isn't … there should be an unknown.	*Yes*: this may be a function … because it says for all values of x, thus there is y being a correspondent of x.	*Yes*: it can be function, then there was an equation, thus a function.
Deniz	*No*: there should be f in the front.	*Yes/Not Sure*: we used to put y instead of x … but we find x then.	*Yes/Not Sure*: function … x greater than 2, then I am not sure.

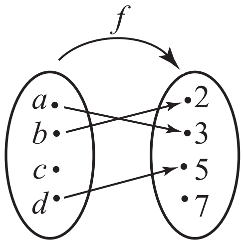
Table 1: Student responses to the various formulations of a constant function

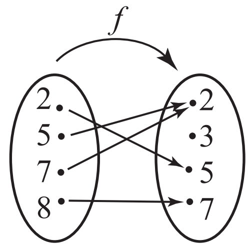
None of the students responded *yes* to all three examples, with the possible exception of Eren, who gave a simple definitional response to the first two examples, but was troubled by the third because some elements were not assigned. The interviewer drew his attention to the notion of domain and he immediately changed his response to declaring it to be a function using the colloquial definition. Student Can responded to the first part using the colloquial definition but drew a graph with only positive values of x in the second part, becoming confused. In the third he gestures to the values of x greater than two and says 'values less than 2', a possible slip of the tongue. Aysun correctly identifies the first example, but sees it as a special case of a linear function, is thrown off course by the absence of an 'x' in the formula in the second case, but correctly identifies the third using the colloquial definition. The other students (represented here by Deniz, Murat, Irem) respond in a variety of ways that reveal a range of complications. They respond to each separate example by noting certain aspects in isolation without at any stage referring to any form of function definition.

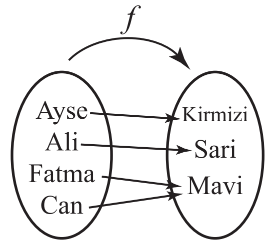
Different representations of functions

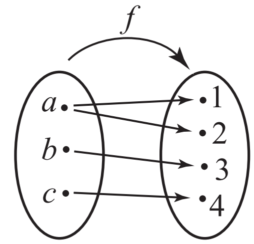
Students were given different representations of functions in isolation. They were given the following items and asked whether they were functions or not.

Set diagrams:

a) 

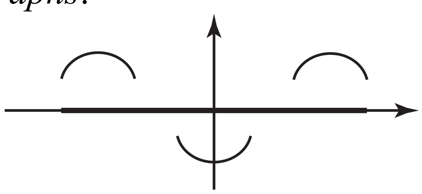
b) 

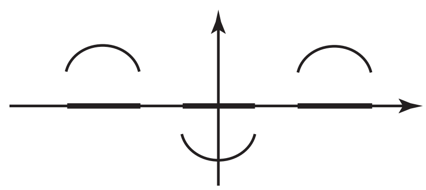
c) 

d) 

Set of ordered pairs: f: $\{1,2,3,7,9\} \to$ R $f = \{(1,3), (2,5), (3,2), (7,-1), (9,1)\}$

Graphs:





[*On the original, the thicker parts on the axis were marked red and denoted the domain*.]

Formulas:

$$\text{(a) } f:\boldsymbol{R} \to \boldsymbol{R}, \quad f(x) = \sqrt{x} \qquad \text{(b) } f:\boldsymbol{R} \to \boldsymbol{R}, \quad f(x) = {}^{1}\!/_{x}.$$

In the interviews, students were asked to explain why they consider these as functions or not. This gave the responses of eight students on four representations. These were then analysed first by considering each representation in turn, then by considering and comparing the performance of individual students.

Set diagrams were answered by seven students in terms of the colloquial definition. Four of these were successful and seemed to use the colloquial definition as their personal concept definition. For instance, Can said:

> (a) is not a function because there is an element left in the domain … (b) is function because for every element in the domain there is an element in the range … (d) not a function because there are two values for a (in the domain). (Can)

Three others used the colloquial definition, but were unsuccessful. For instance, Sena was confused about the properties of a function, saying ‘one can go to two’.

Ordered pairs again evoked the colloquial definition, this time from six students. Three of them successfully checked the definitional properties using the colloquial definition as their personal concept definition:

> Here there shouldn’t be anything left, 1,2,3,7,9 all of them are used and used once, thus I considered this as a function and all of the elements here are reals. (Eren)

> Function, because every element in the domain goes to an element in the range. (Can)

> All these in the domain are joined with different elements in the range. (Murat)

Graphs were more likely to provoke recall of examples rather than the definition:

> Function…there used to be graphs like that in the tests. (Fatma)
>
> If we join these (three pieces) then it looks like a function. (Irem)

The vertical line test was used by three students (Eren, Can, Aysun) who were able to focus on the role of the coloured domain for each graph. The other five students' responses were highly complicated and based on recalled exemplars, and selected details such as absence of numbers on the axes, as in the case of Sena.

Formulae also evoked examples rather than the definition. Only Aysun used the definition (in a colloquial form) to respond successfully to the question. Eren and Can were able to consider the domain of the formulas, but the other five student responses were complicated and did not use the core properties of function at all.

This first part of the analysis suggests that the responses to set diagrams and sets of ordered pairs act like prototypes that are less complicated than the responses to graphs and equations which are seen more as clusters of exemplars.

The second part of the analysis considered the performance of each individual student. Responses indicated that there is a spectrum of performance for students which we illustrate in terms of four categories which may overlap.

In the first category are two students (Eren, Can) who use their personal concept definitions to give a greater unity as they pass from one context to another, not only for set diagrams and ordered pairs, but also for graphs and equations.

In the second category is Aysun who used the colloquial definition of function for set diagrams and formulas and the vertical line test for graphs. She was unsuccessful with the function properties for ordered pairs.

In the third category are the two students (Fatma and Murat) who use their personal concept definitions for one of the two prototypes (set diagrams or ordered pairs). Their responses to the other representations were complicated combinations of exemplars and selected details without any reference to any definitional properties.

In the fourth category are the other three students who could not focus on the core concept for any of the representations. They produced complicated responses affected by the subtle differences in the particular representations without any reference to the definitional properties at all.

CONCLUSION AND REFLECTIONS

In this paper, we have considered how students deal with aspects of the linguistic and representational complexity of the function concept. Figure 1 illustrates our observations, with the positions of two starred items (function box and tables) inferred from other research (Crowley and Tall, 1999; McGowen & DeMarois & Tall, 2000). The data in this paper is drawn from a curriculum in Turkey, which begins by mentioning the formal definition, interprets it in a colloquial form, then

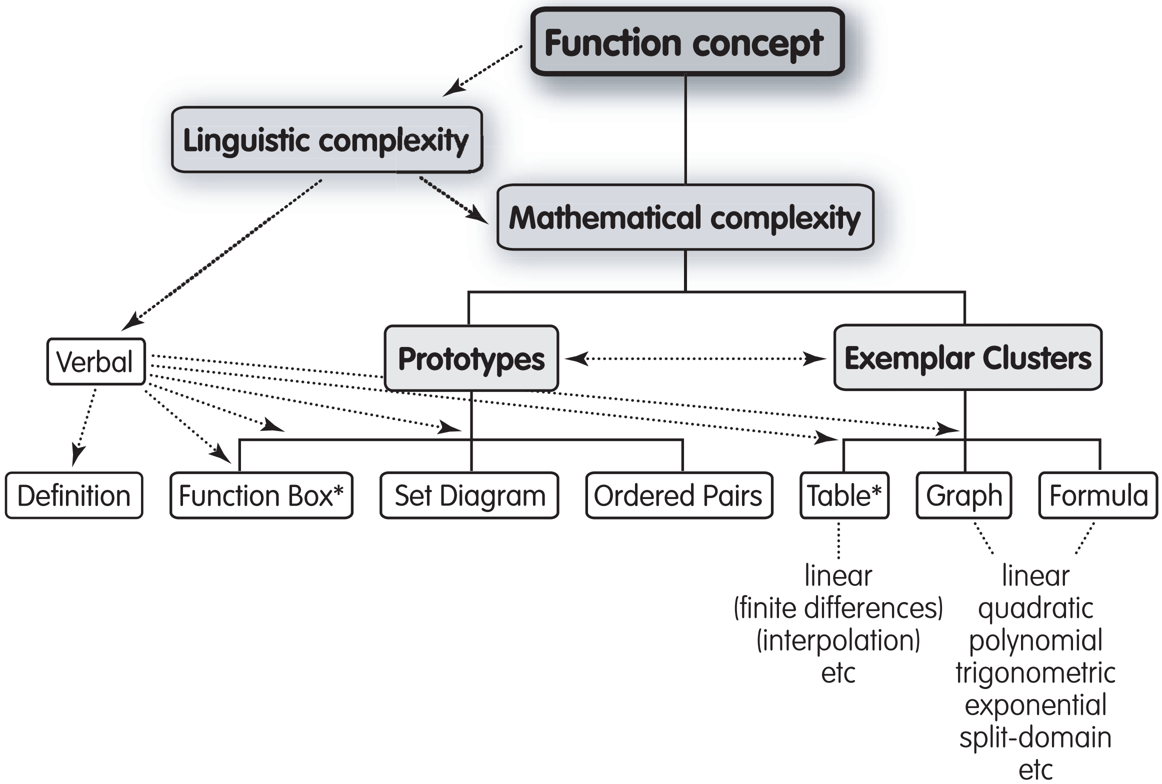


Figure 1: The complexity and complication of the function concept

focuses on set diagrams, ordered pairs, graphs and formulas. The case of a constant function in several variant expressions reveals that the simplicity of the core function concepts eludes most students. The empirical data suggests that a small number (represented by Eren and Can) generally use the core function concept definition in a colloquial form to link ideas across a range of different representational forms into a rich cognitive unit. Errors that they make from time to time are usually amenable to reconsideration through discussion. Essentially, for them, the core concept has a simplicity that can be applied in a variety of contexts.

The other students do not yet have the sophistication to see this simplicity. Some of them focus on the core concept in the form of a colloquial definition for the prototypical forms of set diagrams and ordered pairs. However, graphs and formulas are—at best—seen more in terms of exemplars, relating to types of functions they have met or, in the case of the less successful students, in terms of almost arbitrary aspects of the examples that they happen to focus upon.

References

Bakar, M. N. & Tall, D. O. (1992). Students' Mental Prototypes for Functions and Graphs. *International Journal of Mathematics Education in Science and Technology*, 23 (1), 39–50.

Beineke, J., Hawks, J. & Schwingendorf, K. (1992). Horizontal and Vertical Growth of Students' Conception of Function. In G. Harel, & E. Dubinsky (Eds.), *The Concept of Function* (pp. 133–149). MAA.

Breidenbach, D., Dubinsky, E., Hawks, J. & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics*, 23 (3), 247–285.

Confrey, J. (1994). Six Approaches to Transformation of Function Using Multi-Representational Software. In J.F. Matos and J.P. da Pedro (Eds.), *Proceedings of the 18th Conference of PME*, University of Lisbon, Portugal, 2, 217–224.

Crowley, L. & Tall, D. O. (1999). The Roles of Cognitive Units, Connections and Procedures in Achieving Goals in College Algebra. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME*, Haifa, Israel, 2, 225– 232.

Kaput, J. (1992). Technology and Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 515–556). New York: MacMillan.

Keller, B. A. & Hirsch, C. R. (1998). Student Preferences for Representations of Functions. *International Journal of Mathematics Education in Science and Technology*, 29, (1), 1–17.

Leinhardt, G., Stein, M. K. & Zaslavsky, O. (1990). Functions, Graphs, and Graphing: Tasks, Learning and Teaching. *Review of Educational Research*, 60 (1), 1–64.

Makin, V. S. & Ross, H. B. (1999). Prototype versus Exemplar Models in Cognition. In R. J. Sternberg (Eds.), *The Nature of Cognition* (pp. 205–243). MIT Press: MA.

McGowen, M., DeMarois, P. & Tall, D. O. (2000). Using the Function Machines as a Cognitive Root. *Proceedings of 22nd Conference of PME-NA*, Tuscon, 247–254.

Schoenfeld, A., Smith, J. & Arcavi, A. (1993). Learning: The Microgenetic Analysis of One Student's Evolving Understanding of a Complex Subject Matter Domain. In R. Glaser (Ed.), *Advances in Instructional Psychology*, 4 (pp. 55–175). Hillsdale, N.J.: Erlbaum,.

Sfard, A. (1992). Operational Origins of Mathematical Objects and the Quandary of Reification – The Case of Function. In G. Harel, & E. Dubinsky, (Eds.), *The Concept of Function* (pp. 59–84). MAA,.

Tall, D. O. & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limit and Continuity. *Educational Studies in Mathematics*, 12, 151–169.

Thompson, P. W. (1994). Students, Functions, and the Undergraduate Curriculum. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, I, CBMS Issues in Mathematics Education*, 4 (pp. 21–44).

Vinner, S. (1983). Concept Definition Concept Image and the Notion of Function. *International Journal for Mathematics Education in Science and Technology*, 14, (3), 293–305.