

Infinity – the never-ending struggle¹

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Infinity has fascinated mankind since time immemorial. Zeno revealed that, whether we consider space and time to be infinitely divisible or consisting of tiny indivisible atoms, in both cases paradoxes appear. Despite this uncomfortable problem, practical mathematicians continued to use a range of infinitesimal and indivisible methods of calculation through to the 17th century development of the calculus and beyond. At the beginning of the 19th century, infinitesimal methods were still widely used.

Dedekind's construction of the real numbers suggested that the real line consists only of rationals and irrationals with no room for infinitesimals. He began with the set Q of rational numbers and proceeded to construct a set R of 'cuts' of the set Q which consist of two subsets A, B where every element of A is less than every element in B . He showed that these cuts were of two types. The first type corresponded to a rational number r with rational numbers less than r in A and rational numbers greater than r in B . (In this case the rational number r could be in either A or in B .) The second type did not have a rational number sitting between A and B . He showed that the set of cuts formed a system with elements of the first type corresponding to rational numbers and elements of the second type corresponding to irrational numbers. This construction 'completed' the real line by adding irrational numbers to 'fill in the gaps' between the rational numbers. In such a number line, there is 'no room' for infinitesimal quantities.

The arithmetization of analysis by Riemann confirmed this view that no number α on the real line could be 'arbitrarily small', for if $0 < \alpha < r$ for all positive real numbers r , then $\frac{1}{2}\alpha$ is positive and even smaller than α . Infinitesimals therefore did not fit into the real number system.

When Cantor constructed the concept of infinite cardinal and ordinal numbers, he developed a remarkable extension of counting finite sets to define the cardinal number of an infinite set with an operation of addition corresponding to the union of two sets and multiplication corresponding to the Cartesian product of two sets. Two infinite sets are said to have 'the same cardinal number' when they can be put in one-one correspondence. This was not without its difficulties. For instance, in the infinite case, a set and a proper subset could now have the same cardinal number, which contradicts finite experience and continues to cause confusion in those learning the theory today. The arithmetic of cardinals also has no use for infinitesimals because infinite cardinals do not have multiplicative inverses.

By the beginning of the twentieth century, infinitesimal ideas were theoretically under attack, but they still continued to flourish in the practical world of engineering and science, often as a 'façon de parler', representing not a fixed infinitesimal quantity, but a variable that could become 'arbitrarily small'.

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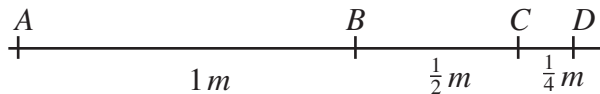
Matters became interesting in the mid-twentieth century when Abraham Robinson introduced his theory of ‘non-standard analysis’; infinitesimals were formulated on a logical basis. However, this too was not without its critics for it invoked the axiom of choice to assert that such entities existed without being able to give a specific finite construction.

To this day the debate continues. Although the infinite cardinals are generally accepted by the mathematical community, there are mathematicians who fully embrace the theory of infinitesimals in non-standard analysis, those who deny their existence and assert the pre-eminence of standard analysis, and an even greater number who do not agonise over the foundational problems and simply get on using mathematics for practical purposes.

In this volume of papers we consider the contribution of the psychology of mathematical thinking to the debate. It proves to shed considerable light on vexed questions surrounding the concept of infinity, for it reveals the human basis for the understanding of infinity and infinitesimals, not only in students, but in the minds of mathematicians themselves.

Initial pioneering work of Efraim Fischbein (1978) revealed the conflicting nature of intuitions of infinity widespread in our students. His empirical research ranged over the potential infinity of the limiting process and the actual infinity of cardinal number theory. He found that students’ intuitive conceptions of limiting processes tended to focus more on the infinity of the process than on the finite value of the limit. For instance, students in grades 8 and 9 were asked the following question:

- (i) Given a segment $AB = 1m$. Let us add to AB a segment $BC = \frac{1}{2}m$. Let us continue in the same way adding segments of $\frac{1}{4}m$, $\frac{1}{8}m$, etc. Will this process of adding segments come to an end?



- (ii) Let us consider question (i). What will be the sum of the segments $AB+BC+CD+\dots$ etc. ?

He found that 84% of students surveyed thought that the process in (i) would never end 14% thought it would. In (ii) only 6% thought the sum of the segments would be 2, 17% thought it would be less than 2 and 51% thought it would be infinity (Fischbein, 1978, p. 68). This ‘never-ending struggle’ with the potential infinity of the process proved to offer a serious cognitive obstacle to students’ understanding of the limit concept.

Fischbein identified similar paradoxical conflicts in the intuition of cardinal infinity. He argued that

- (1) our intellectual schemes are genuinely built on our practical, real life experiences and therefore propositions like “the whole may be equivalent to its parts” contradict our usual, mental schemes;
- (2). our intuitive interpretation of infinity is that of pure potentiality and this interpretation naturally lead to the conclusion that all infinite sets have the same (infinity) number of elements.

Consequently, Fischbein anticipated that students' responses to various questions concerning the comparison of infinite quantities would fall into two opposite categories: "Infinite" answers (e.g., "all infinite sets are equivalent") and "Finite" answers (e.g., "a proper subset of a given set is not equivalent to the set"). These hypotheses were confirmed in a series of studies (Fischbein, Tirosh and Hess, 1979; Fischbein, Tirosh and Melamed, 1981). He concluded that individuals' conceptions of infinity are 'labile and self-contradictory'.

Tall (1980) made a step forward in clarifying the issues by noting that, while counting and measuring concepts were consistent in the finite case, they had distinctly different properties in the infinite case. The infinite extension of counting gives the theory of infinite cardinals while the infinite extension of measuring concepts gave rise to a different form of infinity which he termed 'measuring infinity', arising in a range of theoretical contexts including non-standard analysis.

In this set of papers we follow the distinction between measuring infinity and cardinal infinity by focussing on the difference between the infinities and infinitesimals in the calculus on the one hand and the cardinal infinities introduced by Cantor on the other. Our study begins with the historical development of these two separate strands, considers the different ways in which they are extrapolated from finite experience to various theories of infinity, reviews literature in empirical studies on students' conceptions and considers different curricular approaches appropriate for limits in the calculus and for infinite cardinals in formal mathematics.

HISTORICAL DEVELOPMENT

In the first of our two historical studies, Israel Kleiner considers the role of infinity and infinitesimals in the development of the calculus. He characterizes the evolution of three major elements of calculus: a set of rules or algorithms (a "calculus"), a theory to explain why the rules work, and applications of the theory and of the rules to fundamental problems in science. He describes three major periods in the development of calculus: the *naïve* period (in the 17th century), the *formal* (in the 18th century) and the *critical* (in the 19th century), and proceeds to discuss four stages in the historical development of the related mathematical ideas: *discovery* (or invention), *use*, *understanding*, and *justification*. Kleiner then draws some implications from the historical account relevant to the teaching and learning of calculus, for instance, "To begin a calculus course with a definition of limit may be logically constructive but pedagogically destructive. In general, rigor for rigor's sake will defeat the student." He also raises some critical issues, for example, "We can teach calculus without function, as Newton, Leibniz, and their immediate successors have shown. Should we?" The last part of Kleiner's article is devoted to a description of the historical development of the non-standard analysis of Robinson, and to the related, unavoidable didactical issue: "Should we teach calculus via the method of non-standard analysis?"

In the second paper on the historical development of cardinal infinity, Hans Niels Jahnke considers the epistemological and didactic views of the genesis of Cantor's cardinal and ordinal infinities. He presents the story of the development of Cantorian set theory from both mathematical and epistemological perspectives. He goes on to describe and reflect upon the various arguments that Cantor himself used to defend and legitimate

his innovative, intuitively disturbing ideas, and to attempt to convince his contemporaries that the transfinite numbers are coherent generalizations of finite notions. Based on this evidence, Jahnke draws three major epistemological and didactic lessons from the Cantorian story:

- (1) The essentially new dimension in Cantor's approach to infinity is the creation of a notion of infinity fundamentally divorced from the idea of continuity.
- (2) Cantor's theory is a generalization of finite expressions, yet, many of his ideas are so strongly opposed to common sense that they may lead to the perception of the infinite as artificial and unnatural.
- (3) Cantor's discussion of the nature of mathematical generalizations and the importance of the freedom of pure mathematics has implications in mathematics instruction.

He concludes by putting the case that reference should be made to Cantor's commitment to his fundamental theoretical vision and to the important contribution of such an approach to the development of innovative mathematical ideas.

CONCEPTUAL DEVELOPMENT

After these papers on historical and epistemological considerations we turn our attention to the conceptual development of infinity. In the first article in this section, David Tall considers how the finite human mind contemplates infinite concepts. He distinguishes between 'natural' concepts of infinity that arise through extending finite experiences to the infinite case and 'formal' concepts of infinity framed in modern axiomatic approaches. He details how the axiomatic method provides a context in which selected finite properties can be formulated to give corresponding axiomatic theories. Individual theories may then be coherent in themselves whilst having significantly different properties from other theories. For instance, non-standard analysis has a complete arithmetic in which infinite quantities have infinitesimal inverses but cardinal infinity does not. This reveals not one 'self-contradictory' concept of infinity, but several different formal concepts of infinity, each coherent within its own context. Thus prejudices which may arise through a natural focus on a particular infinite concept (such as cardinal infinity) must be seen in their own context and not used to denigrate concepts in other contexts (such as the notion of infinitesimal). He explains how thought experiences (built on concept images) may suggest theorems that may be proved by formal deduction (based on concept definitions). He also reveals that the reverse path from formalism to natural images can also occur through the proof of 'structure theorems' that have imagistic interpretations suitable for more sophisticated thought experiments. As an example, he shows how any ordered field F that contains the real numbers (defined in a purely axiomatic manner) satisfies a structure theorem that enables it to be represented visually as a number line with infinitesimal detail revealed by magnification. This shows that the apparent limitation of the visual number line to represent only real numbers is a restriction imposed by the manner in which it is interpreted. A more sophisticated interpretation can "see" infinitesimals on a number line. He uses these ideas to refocus the discussion on how the student can be assisted to come to terms with particular concepts of infinity in a various clearly defined contexts.

John Monaghan complements this viewpoint by considering a range of research studies on the views of infinity conceived by young people without experience of formal mathematical ideas. He begins with a description of two potential pitfalls that may affect research in young peoples' ideas of infinity. First, the real world is finite and there are no real referents for discourse on the infinite. Therefore, the researcher has to provide the context, and this involves using concepts that may not necessarily make sense to the child in the manner intended by the researcher. The second, essentially similar, pitfall involves the meanings of the language used when talking with the child about infinity.

Monaghan goes on to describe and discuss several studies on children's understanding of infinity, from the early research of Piaget & Inhelder (1956) and Taback (1975), a study of the contradictory nature of infinity (Fischbein, Tirosh & Hess, 1979) and studies of young children's conceptions of infinite numbers (Falk, Gassner, Ben Zoor & Ben Simon, 1986). He then describes the result of his own study, examining young people's conceptions of infinity (Monaghan, 1986), and several studies that investigate notions of number that children perceive as infinite (Fischbein, Jehiam & Cohen, 1995; Vinner & Kidron, 1985). This leads into a discussion of the impact of various contexts (numeric versus geometric, counting versus measuring, static versus dynamic) and how the nature of the task affects students' responses.

TEACHING APPROACHES

The next two articles turn to the teaching of infinities in different contexts. Joanna Mamona-Downs presents a didactical approach to the teaching of limiting processes of real sequences and Pessia Tsamir describes and analyses a research-based approach to the teaching of cardinal numbers.

Mamona-Downs suggests three didactical steps in the teaching of the concept of limit. In Stage 1, students are presented with several typical tasks aimed at uncovering their intuitions of limit processes of real sequences. She describes the tasks, the rationale for their presentations and the major issues to be debated and discussed in the classroom. She argues that such discussions expose the learners to opposing views of limits that may be used to attempt to develop a more coherent appreciation of the formal definition. In Stage 2, the formal definition of the limit of real sequences is introduced and analysed, discussing the role of each of the symbols and explaining the nature of the formal definition. Stage 3 is devoted to a careful evaluation of the intuitive beliefs evoked in Stage 1 in light of the formal definition presented in Stage 2. Mamona-Downs argues that such a discussion is designed to lead to the refinement of intuitions and a more complete understanding of the limit concept.

Pessia Tsamir focuses her study on student concepts of cardinal infinity. She demonstrates how research-based knowledge about students' inconsistent responses to different representations of infinite sets can be used to raise their awareness of contradictions in their own reasoning and to guide them toward using the one-to-one correspondence as the unique criterion for the comparison of infinite quantities. She begins by briefly summarizing the results of previous studies on students' responses to different representations of the comparison-of-infinite-sets tasks (e.g., Tsamir, 1999; Tsamir & Tirosh, 1999). The main result is that a student's decision as to whether two infinite sets have the same number of elements largely depends on the specific representation of the given infinite sets in the problem. The main part of the article

describes an empirical study using a research-based “It’s the Same Task” activity. Taking part in this activity led most of the participating students to realize that producing contradictory reactions to the same mathematical problem is problematic. Moreover, the vast majority of the participants avoided the contradictions inherent in an informal approach to cardinal infinity by using the one-to-one correspondence as the unique criterion for comparing infinite sets. Finally, Tsamir discusses her findings in the light of Fischbein’s recommendations for instruction of intuitively based conceptions.

CLOSING THOUGHTS

Our collection of articles on infinity closes with the last paper of Efraim Fischbein, who passed away during the early stages of preparation of this volume. It was left in final draft form and has been prepared for publication by the editors assisted by Tommy Dreyfus. It reveals Fischbein linking long-held philosophical views of infinity with more recent empirical evidence and theoretical analysis. The never-ending journey that he began in analysing conceptions of infinity over a quarter of a century ago is left for us to continue. We dedicate this publication to his memory.

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