Students' Concept Images for Period Doublings as Embodied Objects in Chaos Theory

Soo D. Chae and David Tall

Mathematics Education Research Centre
University of Warwick, U.K., CV4 7AL
S.D.Chae@warwick.ac.uk, David.Tall@warwick.ac.uk

This research was conducted using computers and oscillators at the University of Warwick, UK. Several kinds of concept images as metaphorical objects are found, including those related to an oscillator-generated image, and the theoretical bifurcation tree. Empirical evidence is presented to support the hypothesis that students' concept images consist of metaphorical objects in the form of graphic representations of the final orbits; the notion of period doubling is seen not symbolically, but in terms of one orbit bifurcating visually into a 'doubling' of the preceding orbit. Based on the empirical evidence we propose a cognitive development that occurs with many students.

1.1 Introduction

Modern computer software offers the possibility of an enactive interface which enables the student to experiment with mathematical situations combining information from various modes of representation including verbal, enactive, graphic, numeric, symbolic. Our interest here is in the various uses students make of available modes and how these contribute in combination to the construction of mathematical ideas. The research reported here concerns a working session in which students experiment with two technological environments which each provide an example of period doubling. The first (and more substantial) experiment uses software (logis) which draws the $x = f(x)$ iterations of the function $f(x) = \lambda x(1-x)$. As λ increases from 1 to 3 and beyond, what begins as a convergent process settling on a single point bifurcates to a cycle of length 2 at λ=3, and then later to cycles of length 4, 8, and so on. The investigation here is to estimate the numerical values of the points where bifurcation occurs and to test whether they are consistent with the sequence being a geometric sequence. The sequence converges to a point called the Feigenbaum point and the final task is to estimate the numerical value of this point. Our major interest is in the imagery of the mathematical concepts that are evoked by the students when they reflect on their activities during the experiment. In particular, how visual images are used in the development of the concept of period doubling. The second uses an oscillator with a monitor showing the X-Y plane representing the two voltages at different points in an electric circuit with oscillating current. Initially the graph is a single closed loop, but as the voltage is increased, this bifurcates into a period two orbit seen as a "double loop". The purpose of this experiment is for the student to gain experience of the period-doubling phenomenon in a practical setting.

A concept image is something non-verbal associated with the concept name by means of visual representations, mental pictures and a set of impressions or expressions that can subsequently be transformed into something verbal (Vinner, 1991). However, since not every concept has a visual image, it would be better to interpret that a concept image is anything associated with the concept. The phenomenon of period doubling reminds us of the idea of process-object encapsulation in the sense of Dubinsky (1991). Based on this view, we have investigated students' concept images of period doubling through their responses drawn in their mind's eye via post-test after computer and oscillator experiment using a methodology described in the following section.

2. Methodology

The subjects were students enrolled on the Experimental Mathematics course at the University of Warwick in 1999. Most were the first-year mathematics students including students from computer science, economics and engineering.

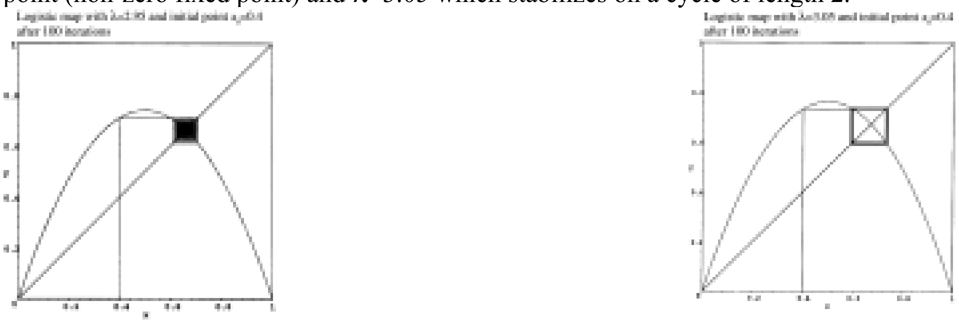
2.1 Software

The software *logis* is written in Maple V and is an updated version of *xlogis* originally written in C (Chae & Tall, 2000, to appear). The *logis* software draws the graph of a function $f(x) = \lambda x(1-x)$ on the closed unit interval [0, 1] and the line $y = x$. Iteration is performed by inputting the following four parameters.

- a value of λ for the function $f(x) = \lambda x(1-x)$;

- a starting point of iteration, x_0;
- total number of iterations, n;
- number of initial iterations you do not want to be displayed, m.

The last facility is most important because it allows the student to draw the later iterations by which time they see only the pictorial representation of the stabilizing cycle. This allows the final cycle itself to be seen as an *object* on the screen. Although successive pictures cannot be drawn simultaneously, this allows the student to draw successive states and imagine each state bifurcating into the next. Figure 1 shows two different values of λ, the first λ=2.95 when the iterations converge slowly to a single point (non-zero fixed point) and λ=3.05 which stabilizes on a cycle of length 2.



(a) convergence to one root before λ= 3 (λ=2.95) (b) period doubling after λ= 3 (λ=3.05)

Figure 1. Graphic representations of iterations of $f(x) = x$ using *logis*

Figure 2(a): homing in on a cycle of length 4 **2(b)**: Omitting initial part to see the stable cycle

2.2 The experiment using computers and oscillators

Students explored the phenomenon of period doublings using the oscillator and computer with instructions distributed before the class. After investigating the period-doubling phenomenon on on-screen, they simulated same phenomenon via the oscillator experiment using the Poincaré map. Their first activity involved experimenting with various starting points and increasing values of λ to observe period doubling at λ_0=3, then at a sequence of increasing values λ_1, λ_2, … which seem to be increasing and bounded above and therefore might approach a limit λ_∞ (the Feigenbaum point). To gain supporting evidence (though not formal proof) they were instructed to compare successive terms to see if the sequence seemed to converge geometrically and then use this information to get a numerical approximation to λ_∞. Out of twenty students, fourteen found a good approximation. Our research is focused on what mental conceptions the students used from the array of symbolic, numeric and graphic information available as a result of computer and oscillator experiment.

2.3 Post-test

After the experiment, we distributed a post-test to students, asking them to answer the questions and return the test within three days. The test comprised several questions designed to reveal the students' concept definition and concept image. For the paper, the following question relating to students' concept image for period doubling was selected:

- What first comes into your mind when you think about 'period doubling'? Please draw/give an example of a period doubling in your mind's eye.

3. Results and Discussion

Students' responses are classified into three categories: 1. Embodied object as a base object, 2. Process of change from one state to the next, 3. Encapsulated object.

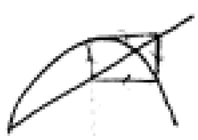
Embodied object as a **base object**		**Process of change**		**Encapsulated object**	
Oscillator-generated circle	1	Oscillator-generated image: from period 1 to period 2	4	ongoing bifurcation using oscillator-generated images	1
Oscillator-generated period 2 orbit	1				
Computer-generated image: period 2 orbit	1	Computer-generated image from 1 to 2	1	The bifurcation tree	1
period 4 orbit	4	From 2 to 4	3		
period 8 orbit	0	From 4 to 8	1		
		Previous experience	1		
Total	7		10	2	

One no-response is not included.

Table1: The classification of students' concept images for period doubling

Figures 3 and 4 show that most students' concept image for period doubling is based on graphic representations generated by the computer. Some of the responses (figure 3) reveal the visualisation as an embodied object (an orbit) and some as a process of change from one state to the next (figure 4).

Computer-generated image:



period 2 orbit:

1 Student

period 4 orbit : *4 students*

Figure 3: Thinking about period doubling in terms of a visual object

Computer-generated image
from period1 to period 2

(the period 2 picture is incomplete)

1 Student

From period 2 to period 4

3 Students

From period 4 to period 8

(1 Student)

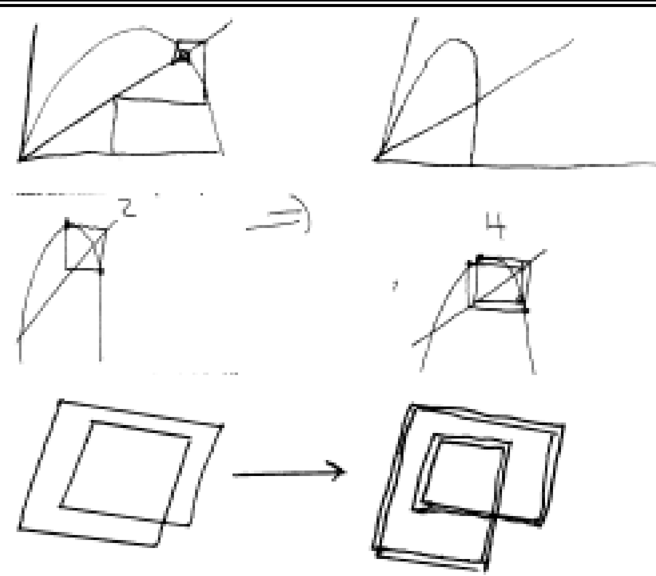


Figure 4: Thinking about period doubling as a process of change from one state to the next

The results of figures 3 and 4 may be classified as first constructing 'base objects' (the pictures of the 'final cycle') and then 'seeing' the process of change from one state to the next. This process of change is the process of period doubling. This process could be considered as the first stage of a process-object encapsulation to yield the *concept* of period doubling. Our purpose is to analyse what mental representations the students use for each stage of this construction.

Considering those who evoke an image from the oscillator, we see that four out of seven give the process from period 1 to period 2 and one gives only the period 2 picture. If we see the latter as the *result* of the period doubling (which is highly likely), then *five out of seven focus on the move from period 1 to period 2.* However, if we consider those evoking an image from the $x = f(x)$ computer iteration, we find that of the ten students involved, *only two focus on the move from period 1 to 2 and eight focus on a move at a later level*.

Why is this? If the students focus on the symbolic form of $x = f(x)$ iteration, then a shift from period 1 to 2 is a shift from the orbit $\{x_1\}$ to $\{x_1, x_2\}$ and from period 2 to 4 is from $\{x_1, x_2\}$ to $\{x_1, x_2, x_3, x_4\}$. From a symbolic viewpoint, the simplest period doubling is from $\{x_1\}$ to $\{x_1, x_2\}$. In the graphical representation, on the other hand, an orbit of period one is just a point and this does not *look* like an orbit. Hence it is plausible that on drawing a prototypical picture of period doubling, the individual is more likely to draw the change from period 2 to period 4 (or higher) rather than from period 1 to period 2. In the case of the oscillator, on the other hand, the shift from period 1 to period 2 already looks like a prototypical period doubling, and so is the natural example to draw.

This suggests strongly that the students are thinking of the *visual* representation in preference to the symbolic or numeric when imagining the process of period doubling. Another piece of evidence for this is that five out of ten do not draw a fully correct picture. Part of this freehand sketch of a period 4 orbit is quite difficult to do from memory. For example, some students actually draw two separate orbits rather than a single connected one (figure 5).

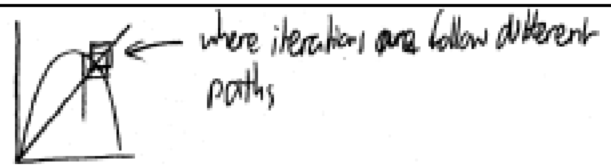


Figure 5: an 'orbit' drawn as two separate squares

Figure 6: Concept image using the bifurcation tree

Thus, although the students may imagine the orbit in their mind's eye, they may not be able to draw it accurately.

Only one student out of 20 offered an image based on the bifurcation tree (or Feigenbaum diagram) (Devaney, 1990) introduced by the supervisor in the class discussion before the experiment (figure 6).

The only other student sketched a picture based on the period doubling of a sine curve by relating one sine curve with another of period twice the length (figure 7). This image could have arisen from previous experience

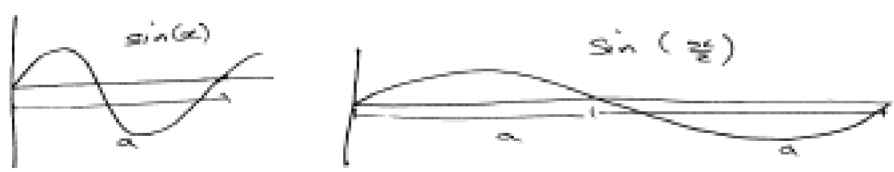


Figure 7: concept image based on sine curve

3.1 Theoretical formulation

We now formulate the results found in terms of processes and conceptual representations. First we see that the most elementary images that the students evoke are in the form of a single 'final' cycle. This relates to the picture seen by the student on the screen. This is not to say that they may not understand that a potentially infinite process of approaching the limit cycle is in progress, only that they refer to the different stages of bifurcation in terms of the visual image of the stabilized graphical orbits. A graphical visualisation is 'embodied' in the sense of Lakoff & Johnson (1999) or Lakoff & Nunez (2000), in that it is an object as perceived by the visual senses and can be traced physically by pointing, to give it a sensori-motor aspect. It is on such an embodied object that the thinking processes operate, imagining it to change dynamically from one stage to the next. We have termed such an object a 'base object' in the sense that the construction of period doubling is based on such objects.

The second stage is to relate one stage of the cycle to the next, say from period two to period four. This is a dynamic process in which period doubling takes place. It occurs as part of potentially infinite recursive iteration.

Given the wide reference to 'process-object theories' (Tall *et al.*, 2000), we may ask if the period doubling process is encapsulated as a period doubling concept. It turns out that there are some interesting features here. The behaviour near the point of bifurcation is not easy to discern from the computer screen alone. For $\lambda<3$ students can see convergence to a point and for $\lambda>3$ they can see an orbit of period 2. But what happens at $\lambda=3$ in the picture is not clear: does it converge slowly to a point, or to a tiny period two orbit? Chae and Tall (2000) reveal that this can cause an epistemological obstacle. Although convergence occurs at $\lambda=3$, it is so slow that some students (who do not check convergence symbolically) may sense that the process does *not* converge. Instead, as we have seen, many students have a mental image of the successive final cycles and are aware approximately *where* period doubling is. This limited knowledge, without fully knowing what happens at or near the point concerned, is sufficient to allow the student to be able to find a numerical approximation to it. We therefore hypothesise that the concept of period doubling occurs most of all only at an impressionistic embodied level seen onscreen and imagined visually, but not as a fully encapsulated mathematical object.

None of the students actually draw more than a single instance of the process in the x = f (x) iteration (we have already noted the difficulty of drawing the later pictures). One student evokes an oscillator image of the ongoing doubling process (figure 8). His diagram is indicative of the development, although his diagrams are in error.

Figure 8: Ongoing period doubling

The student giving a mental picture in terms of a bifurcation tree has an image of the total potential infinity of bifurcations (again in a visual sense) as one of the most successful students in the group. We, therefore, suggest that this is a possible next stage. Thus we see the concept of period doubling in this experiment being seen as a visual process onscreen and related to the corresponding numerical data. The initial process of convergence to a point and later to a cycle of length 2, 4, … is seen as an impressionistic embodied object that can be imagined visually. This may be seen to take part in a successive dynamic process of period doubling, which is sufficient to act as a basis for numerical exploration of the Feigenbaum limit of the bifurcation points. Thus the computer experiment has a powerful role to play in which visual imagery is an integral part of a symbolic and numeric process, coming to the fore and acting as a cognitive support for the concept of period doubling.

4. Conclusion

The data above show that 17 out of 20 students constructed a concept image from graphic representations generated by computers or oscillators. This finding supports the hypothesis that graphic representations play an important role in forming the concept image of period doubling. A natural cognitive sequence appears to be exhibited in which students see first the process of iteration, then the embodied visual image of the final cycle at each stage. This visual image is a base object that takes part in a process of period doubling bifurcation seen in terms of the imagined move from one stage to the next. The visual process is sufficient to act as a basis for finding numerical approximations to the points where period doubling occurs although there may be cognitive difficulties interpreting precisely what happens *at* the point of bifurcation. The most successful person in the group saw the total structure within the Feigenbaum diagram that represents the potentially infinite process of period doubling in a single visual representation. In this way we see the interactive computer software provides an environment for student exploration that overcomes the great difficulty of the symbolic theory by supporting it visually with dynamic computer images.

References

Chae, S. D. & Tall, D. O. (2000). Construction of Conceptual Knowledge: The Case of Computer-Aided Exploration of Period Doubling, *Proceedings of the British Society for Research into Learning Mathematics*, 2000

Devaney, R.L. (1990). *Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics*, Addison-Wesley Publishing Company.

Dubinsky, E. (1991) 'Reflective Abstraction' in D.O.Tall (Ed), *Advanced Mathematical Thinking*, Kluwer, 95-123.

Lakoff, G. & Johnson, M. (1999). *Philosophy in the Flesh*. New York: Basic Books.

Lakoff, G. & Nunez, R. (2000). *Where Mathematics Comes From.* New York: Basic Books.

Tall, D. O., Thomas, M., Davis, G., Gray, E. M., Simpson, A.(2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior*, 18 (2), 1–19.

Vinner, S. (1991). The Role of Definitions in the Teaching and Learning of Mathematics, in D.O.Tall (Ed.), *Advanced Mathematical Thinking*, Kluwer, 65-79.