Reflections on Early Algebra

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This paper is a contribution to a discussion on early algebra at the 25th Annual Conference of PME. It is predicated on a perceived need to frame early algebra within a wider theory of symbol development. I shall use an existing theory (Tall et al, 2000) to place the study of Carraher Schliemann and Brizuela (2001) within a broader framework. It will reveal the study to be situated in a preliminary (but vital) stage between arithmetic and algebra. Furthermore it will suggest theoretical and practical links to earlier arithmetic that need to be considered and reveal the reported activity as a small step for many of the children involved along the path to what I shall term 'evaluation algebra'. After this point, however, there is still much to do and the traditional discontinuities that occur in the transition to full 'manipulation algebra' still remain to be faced at a later stage.

Introduction

At this conference we celebrate 25 years of research meetings organised by the International Group for the Psychology of Mathematics Education. It is therefore fitting to place this response within this context. Carraher, Schliemann and Brizuela (2001) refer broadly to a range of previous research, partly to report observed difficulties and partly to respond to suggestions to 'bring out the algebraic character of arithmetic'. This means that, apart from using the advice to 'algebrafy' arithmetic, the 25 years of previous research is not used in any foundational way. My analysis of their paper therefore uses a global theory of developing symbolism to place the research in context.

Analysis

Carraher et al (2001) base their research in a 'typical' class of 9 year-olds and is implicitly an approach to teach 'algebra for all'. I begin, therefore, by looking at their data to see if they are actually reaching every child in the class and also to analyse precisely what kind of algebra the children appear to be learning.

The class is presented with a story in which two children start with the same unspecified amount of money on Sunday and spend and receive specific amounts on successive days. When the researcher Bárbara asks the class if they know how much money they have, 'the children state a unison "no"', but 'a few utter "*N*" and Talik states "*N*, it's for anything."' Thus we have some children who have already met the idea of using a letter to stand for a number and some who presumably have not. Our first piece of evidence is that, faced with adding 3 dollars to the initial unspecified amount, we are told that 'only three children do not write *N*+3 as a representation for the amounts on Monday.' What is missing is an analysis of what the children individually bring to the class from their previous development and why some children are more adept at algebraic thinking than others. This in turn requires a theory of longer-term development that is consonant with empirical evidence.

Tall et al (2000) present a theory of symbolic development arising from earlier work of Gray & Tall (1994) and many others. This reveals a *bifurcation* in performance in arithmetic between those who become entrenched in a procedural mode of counting to do arithmetic and those who develop *proceptual* thinking involving the flexible use of symbols as both process and concept. (This is not to be interpreted as a naïve prescription that the successful always get better and the less successful get worse. The case of Emily (Gray and Pitta, 1997) reveals a child growing from counting procedures to flexible number concepts by being given support using a calculator that carries out the procedures for her so that she can concentrate on the conceptual relationships.) However, the theory does intimate that what children bring to a given situation—depending on their preceding development—radically affects how and what they learn. It can have a profound effect on early algebra.

For instance, the English National Curriculum in England intended to use arithmetic problems such as the following as a precursor of algebra:

(1): $3+4=\square$, (2): $3+\square=7$, (3): $\square+3=7$.

Although these *look* like algebra, they are certainly not. Children perform them using their repertoire of methods of counting and deriving or knowing facts. Question (1) can be done by any counting method, (2) can be done by 'count-on' from 3 to find how many are counted to get to 7. Equation (3) is more subtle. If the child senses that the order of addition does not matter, the problem is essentially the same as (2); and can be solved by count-on from 3. If not, the child who counts has a far more difficult task to find out 'at what number do I start to count-on 3 to get 7?' This involves trying various starting points to count-up using a 'guess-and-test' strategy.

Foster (1994) used these three types of question in a study of 'typical' children in the first three years of an English Primary School. He found a significant spectrum of performance in the first year where the lower third were almost totally unable to respond to questions of types (2) and (3). By the third year the top two-thirds of the class obtained almost 100% correct responses but the lower third obtained 93% correct on type (1), 73% correct on type (2) and 53% on type (3). Seen in the light of procept theory, this suggests that the lower third operate more in a procedural than a flexible proceptual level. This would be consistent with the lower third of a class in Grade 3 in the USA including children who are more procedural than proceptual, which, in turn is consistent with difficulties with algebraic qualities of arithmetic exhibited by some children in this study. I would counsel, therefore, that in carrying this activity out in a classroom context, some children are already struggling and need special individual care. Even those who succeed in writing down the symbolism '*N*+3' are likely to be using it in a manner different from that observed by an expert.

When the symbols introduced into the work of Carraher *et al* are analysed, they are all of the form of an unknown quantity followed by successive numerical additions and subtractions, such as N+3–5. (On the web-site related to the paper, there are also considerations of equivalence of expressions such as N+3–5 and 3+N–5.) The children can, and will, make their own interpretations of the meaning of the symbolism.

The researcher Bárbara leads a discussion using 'the number line centred on N' to visualise the symbols N+3–3 in terms of shifts along the line starting from N. The paper describes how she 'writes a bracket under 3–3 and a zero below it […] and extends the notation to N+3–3 = N+0 = N.' This is the description of what she—the expert, in this case—sees. *But does each individual children see and think in this way?*

A wide array of literature reports children conceptualize the 'equals' sign as an *operation*, not as an equality between two expressions. It is here that procept theory helps. Bárbara and her co-researchers have the ability to switch between seeing the symbol N+3–3 as an expression for a single mental concept on the one hand and as a process of successive steps on the other. She can see the 'equality' of the two concepts. As the discussion unfolds, it is *Bárbara* who writes N+3–5 = N–2 and the *authors* of the paper who call 3–5 a 'sub-expression', claiming that Jenny 'writes a zero under it'. However, we have a reproduction of the work of Nathan, who writes not zero, but '=0'. An alternative, and more likely, explanation is that some (perhaps most) of the children are interpreting the symbols as *processes* to be performed rather than as *expressions*.

As *all* the formulae in the paper consist of an unknown N followed by number operations, the context allows the children to operate essentially in an *arithmetic* mode. They are not asked to *operate* directly on the unknown, rather this is the starting point from which arithmetic operations occur and can be the main focus of attention.

There is evidence that some children work with N as an unknown. For instance, 'Talik shows how this works if N=150.' This inhabits an intermediate stage that Thomas & Tall (2001) call *evaluation algebra* in which expressions are used to represent a general arithmetic operation (as, for instance, they do in a spreadsheet). This is an earlier stage than full-blown *manipulation algebra* where the symbols are freely manipulable entities as expressions and sub-expressions. In evaluation algebra, symbolic expressions are seen as *processes* of evaluation. Manipulation algebra sees them as *procepts* representing either process or manipulable concept.

Carraher *et al* (2001) ask in their title: 'can young children operate on unknowns?' The evidence they provide reveals that their approach has absolutely *no* operation *on* unknowns in the sense of symbol manipulation. There *is* evidence of *evaluation* by substitution (as a by-product rather than a direct focus of the activity). In general, the children's activity involves arithmetic operations on arithmetic symbols.

Is this a problem? Absolutely not. Some children are evidently becoming familiar with the use of a letter to represent a specific but (to them) unknown number. Thus at least one aspect of the development of algebra is beginning to take root. However, the journey through evaluation algebra and on to manipulation algebra is a long one and for many *but not all* children it will involve difficult cognitive reconstructions.

All the symbols used in the activities are read in the usual left-to-right direction in Western languages. There is still a long way to go for children to cope with expressions such as 2+3x where the product of 3 and x must be performed *before* 2 is added to it.

The work of Carraher *et al* is certainly a first step, but it needs to be explicitly aware of what individual children might bring to the task and where they might go later.

Concluding Remarks

I have suggested that the study of 'early algebra' needs to be seen not only as an activity in itself but also as part of a longer-term development. The activities need to be carefully analysed to see as clearly as possible what it is that the children have to build on and what it is that they are likely to be thinking. In the analysis presented here I have indicated conceptions that children may bring to the enterprise that may hinder or help them (for example, arithmetic as procedures or as flexible process-and-concept). I have analysed what some children might be doing with the symbols (operating with them as processes, rather than seeing them as expressions). I have emphasized the chosen limitations (symbolism read from left to right, starting with an unknown that may be left on its own, allowing a focus on the arithmetic operations that follow). There is the evidence that some children (eg Talik) have taken the first step into evaluation algebra by substituting a number for the unknown. However, it is less clear as to who sees the symbolism N+3–5 = N–2 as an arithmetic *process* and who see it as an *equality* of mental expressions (concepts).

A step has been taken by some (many?) of the children. It is a significant step. But it has implicit properties (reading an expression left to right, perhaps seeing the expression as 'a process to do' rather than 'a concept to manipulate', perhaps coping by working only at an arithmetic level). Such properties may become part of the child's mental structure that needs reconstructing at a later stage. The major cognitive obstacles of manipulation algebra that afflict many, but not all, children in the bifurcating spectrum of performance still remain to be addressed in the future. If the bifurcation we have observed continues to occur (and it seems to be very persistent), it may be that some may be profitably focused on evaluation algebra that has powerful uses in computer contexts whilst others develop proceptual flexibility required for manipulation algebra.

References

Carraher, D., Schliemann, A. & Brizuela, B., (2001). Can Young Students Operate On Unknowns? *Proceedings of the 25th Conference of the International Group of the Psychology of Mathematics Education*.

Foster, R., (1994). Counting on Success in Simple Arithmetic Tasks. *Proceedings of the Eighteenth Conference of P. M. E.*, Lisbon, Portugal, II 360–367.

Gray, E. M. & Tall, D. O., (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal of Research in Mathematics Education*, **26** (2), 115–141.

Gray, E. M. & Pitta, D., (1997). Changing Emily's Images, *Mathematics Teaching*, 161, 38–51

Tall, D., Gray, E., Ali, M., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M., Yusof, Y., (2000). Symbols and the Bifurcation between Procedural and Conceptual Thinking, *Canadian Journal of Science, Mathematics & Technology Education* 1, 80–104.

Thomas, M. O. J. & Tall, D. O., (2001). The Long-Term Cognitive Development of Symbolic Algebra (submitted for review, available from www.warwick.ac.uk/staff/David.Tall).