# Aspects of the construction of conceptual knowledge: the case of computer-aided exploration of period doubling

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*This research was conducted using computers focusing on establishing the connection between visual orbits and symbolic theory based on a concept definition. It is suggested that students who make a connection through visualisation via dynamic computer software have an advantage in understanding the concept of period doubling. The role of the supervisor proves valuable in helping students to construct conceptual knowledge by using appropriate directing questions during the experiment. It is proposed that we can help students to develop their conceptual knowledge by connecting visualisation and symbolism through computer-aided exploration guided by the supervisor and mentor.*

## Introduction

According to Eisenberg and Dreyfus (1991), students prefer to think algebraically rather than geometrically when they are solving problems, and the authors give several reasons for this in terms of social, curricular and epistemological factors. Sierpinska (1987) formulates the notion of geometrical obstacle as an epistemological obstacle caused by the graphic representation of an attractive fixed point. She argues that this seems to block the students' thinking by focusing on the convergence to the fixed point in an immediate, intuitive and global way, which obscures subtle ideas in the potential infinity of the symbolic process. She wonders whether it would be worth beginning the instruction by asking students to work out their own graphic representations of the iterations of functions using given definitions. In this paper we suggest that a powerful alternative is to begin with the students interacting with a dynamic graphic representation rather than generating their own static graphic representations. This is proposed because:

1. Obstacles may be caused by both the language and the graphic representations used in the initial introduction and this may subtly affect students' conceptions.
2. By guiding students to solve problems based on a concept definition using a graphic representation, the teacher as mentor can encourage them to generate their own links between visualisations and formal symbolism.

   > Experience modifies human beliefs. We learn from experience or, rather, we ought to learn from experience. To make the best possible use of experience is one of the great human tasks and to work for this task is the proper vocation of scientists. (G. Polya, 1954, p. 3)

3. Seeing a problem as a dynamic whole is different from seeing separate static pictures.

## 1. Procedural and Conceptual Knowledge

Hiebert and Lefevre (1986, p.3) state that the crucial characteristic of conceptual knowledge lies in the rich relationships it contains between specific pieces of

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information. It may be considered as a well-connected web of knowledge, accessing and detaching information flexibly. In contrast, procedural knowledge can be characterised as a form of sequential knowledge, constructed in a succession of steps.

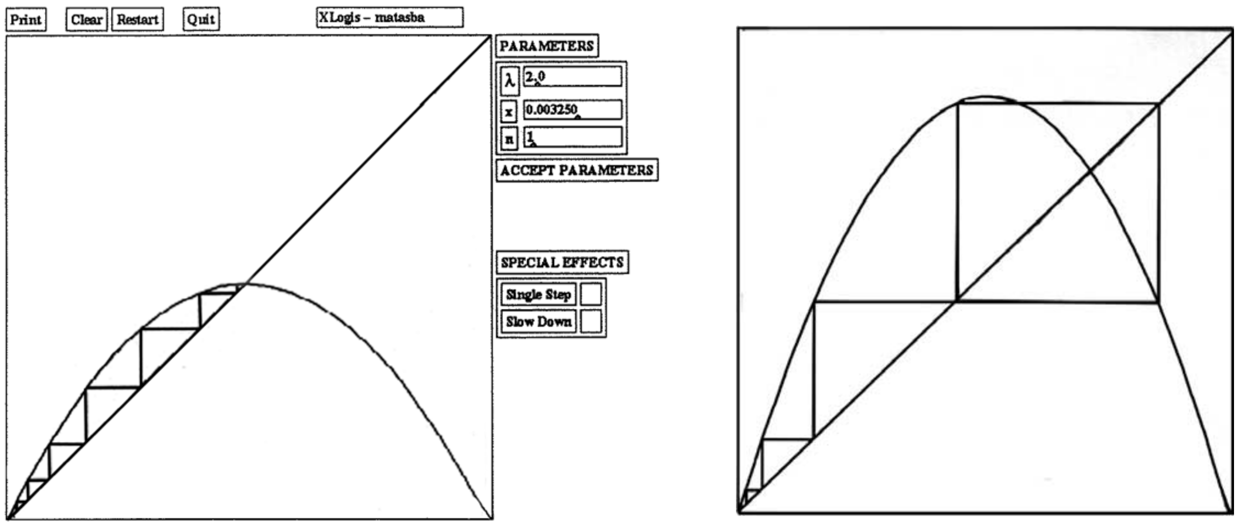
Heid (1988) showed that students in an experimental calculus class using a microcomputer as a tool for visualizing graphs and for manipulating symbolic procedures developed a broader conceptual understanding than students in a traditional class focusing mainly on symbolic procedures. She found that students gaining conceptual knowledge in this way were able to develop concepts further than those using procedural knowledge. Many other researchers (eg Tall, 1991) contend that students using interactive dynamic computer software gain a much better insight into mathematical concepts than those following a traditional curriculum.

In this study we therefore consider conceptual knowledge being constructed through visualisation using interactive graphical software. To investigate this idea, the present research was conducted using the framework outlined in the next section, focusing on the establishment of the connection between visual orbits of $x=f(x)$ iteration and symbolism of the orbits.

## 2. Research framework

### 2.1 Subjects

The study involved thirty first-year students enrolled in the Experimental Mathematics MA112 course at the University of Warwick. It is focused on the part of the course which involved experimentation using oscillators and computers and, in particular, on the use of the software *xlogis* (designed for Sun workstations for the Warwick Mathematics Department) to experiment with the period doubling of $x=f(x)$ iteration (figure 1). The first named author supervised three groups and one of these groups with seven students was selected for closer study.




(a) convergence to one root of $f(x) = x$ (b) period doubling

Figure 1. Graphic representations of a logistic map using *xlogis*

## 2.2 Instruments

Various forms of data were collected in the study. Students were given a *pre-requisite test* by the course tutor to focus their attention on the necessary preliminary knowledge and were assessed on a *written assignment* handed in three days after the session which reported their observations and inferences from the experiment. Soo D. Chae acted as supervisor and participant-observer, using *audio-tapes* and writing *field notes*. After the session, the students were given a *questionnaire* designed to identify their understanding of the basic concepts and the relationship between the visual and symbolic representations.

## 2.3 Pre-requisite test

This was designed to investigate students' awareness of the concept of "geometric convergence" which was given in terms of the following definition and the accompanying question:

> A sequence $(a_n)$ is said to *converge geometrically* if the ratio $(a_{n+2} - a_{n+1})/(a_{n+1} - a_n)$ converges to a limit $r$ with $0 \leq r < 1$ as $n$ goes to infinity. Write down an example of a sequence that converges geometrically.

**2.4 Field notes and audio-tapes** were used to investigate students' responses to what they saw on the computer screen and to analyse their questions and feelings towards graphical representations.

## 2.5 The computer experiment

The software *xlogis* is designed to enable students to control either a single step or the iteration as a whole by selecting special effects; it is intended to lead students to develop conceptual knowledge. The students were given the following tasks to experiment with the logistic map $f(x) = \lambda x(1-x)$.

1. Use *xlogis* to investigate what happens when $\lambda$ increases through the value 3.0.
2. Use *xlogis* to investigate the dynamics for $\lambda$ between 3 and the value $\lambda_1$ for which the period 4 orbit occurs. What happens when $\lambda$ goes through $\lambda_1$?
3. As you increase $\lambda$ beyond $\lambda_1$, you should see a sequence of period doubling bifurcations. Use *xlogis* to obtain estimates of the parameter values $\lambda_n$ for which the nth period doubling bifurcation occurs.

   What do you notice about the way the $\lambda_n$ converge? The parameter value $\lambda_\infty$ to which they converge is called the accumulation of period doublings. Try taking the ratios of successive differences. What does the result tell you? Can you think of a way of seeing this by drawing a graph?

## 2.6 The role of the supervisor

In order to improve effective experimentation, the supervisor assisted and responded to the group, providing support and explaining the phenomenon of period doubling. Sometimes the supervisor offered advice and provided directing questions to keep the students going if they were stuck. Three different types of questions were used: for *structuring*, *opening-up*, and *checking* (Ainley, 1988). For instance, an *opening-up question* responds to a student's request by asking the student to think more about it:

> Student: What is happening when the function cycles between two values?
> Supervisor: What comes to your mind when the function cycles between two values?

A *checking question*, on the other hand, would check what the student had just done:

> Student: The function seems to be hitting four points. So, is this lambda one?
> Supervisor: Are you sure it is hitting four points?

## 2.7 Students' self-written reports

Students were asked to write up their observations and answers as they proceeded, and then to summarise their mathematical ideas and arguments clearly and hand them in within three days. The students' reports on the mathematical questions posed during the experimentation provided a valuable source of data. According to Mason (1982), this kind of activity is valuable for helping students to reflect on what they have done and how they have done it. The supervisor graded reports using criteria that emphasised the quality of students' ideas without seeking perfect presentation.

# 3. Results and discussion

## 3.1. Students' concept image about fixed points and obstacles

One question on the questionnaire (figure 2) gave the student the definition of a fixed point and asked then to identify the fixed points of the iteration $x=f(x)$ in a picture. Despite the definition saying it is a point $x$ such that $f(x) = x$, the students were drawn to the centre of activity in the dynamic representation, which is the point where the line $y=x$ meets the curve $y=f(x)$. This is an example of Sierpinska's geometric obstacle. In this case it proved easily corrected through discussion.

## 3.2 Geometric convergence

| Case | Number (%) | Concept Image |
|---|---|---|
| A & C | 18 (60%) | A fixed point of *f* is where the graph of *f* intersects the diagonal |
| A & D | 1 ( 3%) | *[the correct response]* |
| A, C, & E | 1 (3%) | |
| A, B, C & E | 1 ( 3%) | D is not on the line $y=x$ |
| C | 2 ( 6%) | |
| E | 1 ( 3%) | |
| No response | 6 (20%) | |

Table 1 Concept images for fixed points

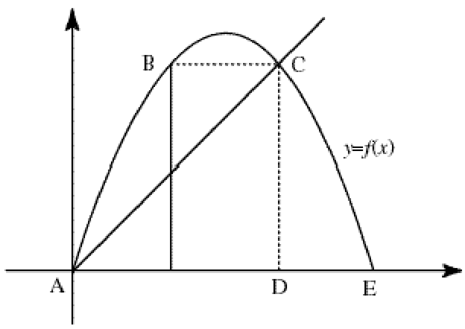



Figure 2. Which points are fixed points of the given function?

The students responded to the request for an example of geometric convergence. Of the seven representative students chosen, two did not reply, two gave incorrect responses ($a_n = 1/n$) and three gave correct examples ($a_n = 2^{-n}$, or $10^{-n}$, or $e^{-n}$).

### 3.3 Students' formulations of period doubling

The students' written reports were analysed to see if they were able to respond to the tasks given them (section 2.5). The seven students selected for observation during the experiment already included four who did not give a correct example of geometric convergence. Some were able to explain the general notion of period doubling using either a graphic argument, a symbolic argument, or a numerical argument. For example, one successful student considered the numerical sequence of period doubling bifurcations ($\lambda_0$=3, $\lambda_1$=3.449, $\lambda_2$=3.545, $\lambda_3$=3.565, $\lambda_4$=3.3.57) and observed:

> $\lambda_n$ seems to be converge in a geometrical way. The ratio ($\lambda_n - \lambda_{n-1}$)/ ($\lambda_{n-1} - \lambda_{n-2}$), is taken to see the way it converges.

This was the only student who was able to give some kind of proof of the convergence of $\lambda_n$ to $\lambda_\infty$. Five of the others were able to give some numeric, graphic or symbolic generalisation. Two did not give an explanation of convergence. Notice that these are precisely the two who gave no graphic response (table 2).

| | geometric convergence | Proof | Numeric | Graphic | Symbolic | Generalisation to $\lambda_\infty$ |
|---|---|---|---|---|---|---|
| S1 | yes | no | yes | yes | yes | yes |
| S2 | yes | yes | yes | yes | yes | yes |
| S3 | no | no | yes | yes | yes | yes |
| S4 | no | no | yes | yes | no | yes |
| S5 | yes | no | no | yes | yes | yes |
| S6 | no | no | no | no | no | no |
| S7 | no | no | yes | no | yes | no |

Table 2. Students written responses in various representational aspects

## 4. Summary

As a starting point for constructing conceptual knowledge, we discussed ideas related to conceptual obstacles caused by geometric representation and by students' concept images. Later, students were observed in an experimental mathematics class in order to study the impact of interactive graphic representation and the provision of guided questioning. Students were motivated when they saw the phenomenon of period doubling represented on computer screens and oscilloscopes. Two students who were unable to give examples of geometric convergence were nevertheless able to give some general explanation of the convergence of the values for period doubling. Two others were not. The five students in the sample that were able to give a general explanation of the convergence of period doubling were precisely those who were precisely the same students who were able to give a graphic explanation. This is consistent with the idea that the dynamic visual software and the directed mentoring of the students can be of value in building conceptual links, at least for those students able to give some meaning to the visual representation.

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