Function: Organizing Principle or Cognitive Root?

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The function concept is often used as an organizing principle for algebra and beyond. Here we consider its value as a cognitive root (a concept which serves as a basis for cognitive development). Current theories of multiple representations and theories of encapsulation of process as object are used to build a view of function in terms of different facets (representations) and different layers (of development via process and object). Results of interviews with three students in developmental algebra will be used to highlight the model and to discuss the value of the function concept as a cognitive foundation to growth in mathematical understanding.

Introduction

The function concept is often suggested as an organising principle in mathematics:

> We believe that function is the fundamental object of algebra and that it ought to be present in a variety of representations in algebra teaching and learning from the outset.
> (Yerushalmy & Schwartz,1993, p.41)

It has become a central concept in school and university curricula around the world. We agree that the function concept can be a powerful foundation for logical organisation, but we question its suitability as the basis for a cognitive development.

Tall (1992, p. 497) defined a *cognitive root* as a starting concept with the "dual role of being familiar to students and providing the basis for later mathematical development". He considered the function concept as a possible cognitive root, counselling that there were serious obstacles such as the encapsulation of function as a manipulable object (eg Dubinsky & Harel, 1992; Sfard, 1992) and the complexity of coordinating alternative representations (Cuoco, 1994). Here we consider these two dimensions—the links between various representational *facets* of the function concept, and the *layers* or levels of compression in process-object encapsulation (DeMarois & Tall, 1996). These are traced through a remedial college algebra course based on the function concept.

Framework

The facets studied will include the function *notation* (including the meaning of $f(x)$), the *colloquial* use of a function machine as input-output box, the standard *symbolic* (algebraic formulae), *numeric* (table) and *geometric* (graphic) facets, with the *written* and *verbal*. These will be represented as sectors of a disc (figure 1) in which movement towards the centre is seen as compression through the layers *pre-procedure, procedure, process, object,* and *procept*. Pre-procedure denotes that the student has not attained the procedural layer. Students at the procedure layer are dependent on carrying out a sequence of step-by-step actions. Students at the process layer can accept the existence

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of a process between input and output without needing to know the specific steps, and see two procedures with the same input-output as the same process. The object layer denotes the capacity to treat the idea as a manipulable mental object to which a process can be applied. The procept layer indicates the ability to move between process and mental object in a flexible way.

To allow each facet to be linked directly to any other, the picture should be seen as having individual slices (facets) that can be moved and connected in any way.

An alternative representation (figure 2) is used to show the direct links between selected facets, some of which may be non-existent or in one direction only for individual students.

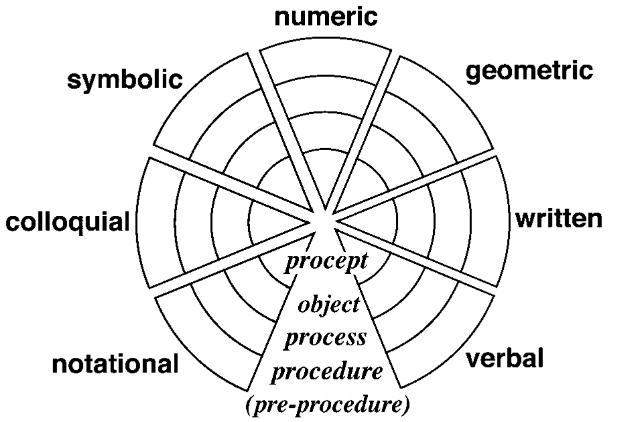

Figure 1: facets and layers of the function concept

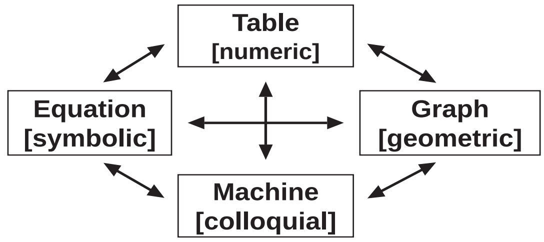

Figure 2: possible links between function facets

Student Conceptions of Function

DeMarois (1998) studied students taking a developmental algebra course at a community college. The students completed pre- and post-course function questionnaires and several participated in a post-course interview. Her, we focus on three students AF, BF and CM, where the first letter denotes the grade achieved (A, B, C) and the second denotes the gender (M or F). AF is a liberal arts student between 21 and 25 years of age. BF is a business student between 26 and 30, CM is a biology student over 30. AF had studied 1.5 years of algebra before college, BF and CM had taken 1 year. AF and BF were taking their first college mathematics course, CM had previously attended a basic mathematical skills course

. Function machines were used to analyse the colloquial facet. The majority of students displayed some understanding of function machines on the pre-test. In the individual post-course interviews, one question provided data on colloquial, verbal, numeric, and symbolic facets (figure 3).

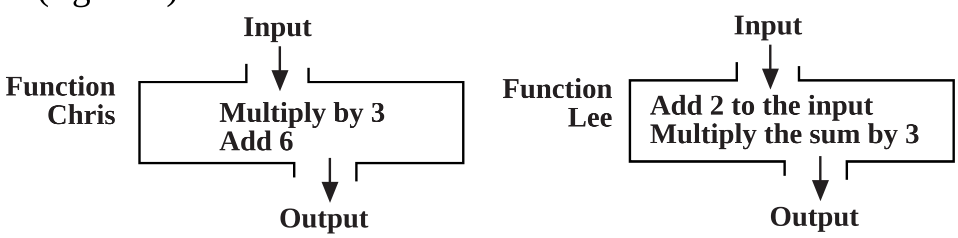

Figure 3: equivalent functions

Students were asked to write expressions for each function machine and asked whether the two function machines represented the same function (table 1).

	Chris	Lee	Are the functions Chris and Lee equal?
AF	$3x + 6$	$3(x + 2)$	Yes, if I distribute the 3 in Lee, I get the same function as Chris
BF	$x3 + 6$	$(x + 2)3$	Yeah, but different processes
CM	$3x + 6$	$x + 2(3x)$	No, you come up with the same answer, but they are different processes

Table 1: Function machines as procedure, process and mental object

The three responses show AF speaking in terms of a mental *object*, BF in terms of *process* and CM in terms of *procedure*. AF easily links the colloquial and algebraic facets. BF gives a literal translation of both function descriptions showing less flexibility moving from colloquial to algebraic. CM sees Chris and Lee as different procedures (in our terminology). He also gives a literal translation of the second function as "x+2 three times", revealing that he is less comfortable relating the colloquial facet to the algebraic.

Further research into links between symbolic, arithmetic, geometric and colloquial facets was performed by asking the students to respond to the following questions:

- given a specific equation, create a table, a graph, and a function machine;
- given a specific table, create an equation, a graph, and a function machine;
- given a specific function machine, create a table, a graph, and an equation; and,
- given a specific graph, create a table, an equation, and a function machine.

They were encouraged to create the other forms in any order they wished. Tables 2-4 display the results where "√" indicates a successful attempt and the numbers indicate the order in which the representations were created.

AF From ↓ to →	Equation (symbolic)	Table (numeric)	Function machine (colloquial)	Graph (geometric)
Equation		√ (1)	√ (2)	√ (3)
Table	√ (2)		√ (3)	√ (1)
Function machine	√ (1)	√ (2)		√ (3)
Graph	√ (2)	√ (1)	√ (3)	

Table 2: Creating representations: AF

Although AF was able to start with any representation and eventually get to any another the routes taken were not always direct (see figure 4). Given the equation, AF said:

> I am much more comfortable with the function machine and the table as opposed to creating a graph on my own. I'm not as comfortable doing a graph on my own.

Given the table, AF first created the graph, but went back to the table to create the equation. She used the graph to determine the type of equation but then used the table to determine the slope using finite differences:

> I'm trying to find the finite difference. I know from the graph it looks like it will be a line so I think it will be linear which I know is $y(x) = ax+b$. So for that I need the slope and the 0 input which I already have which is –3. It looks like the slope is 2 so I get $y(x) = 2x–3$.

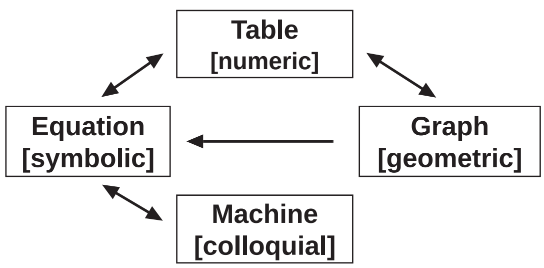

Figure 4: direct links between facets for AF

BF From ↓ to →	Equation (symbolic)	Table (numeric)	Function machine (colloquial)	Graph (geometric)
Equation		√ (2)	√ (1)	√ (3)
Table			√ (3)	√ (1)
Function machine	√ (1)	√ (2)		√ (3)
Graph		√ (1)		

Table 3: Creating representations: BF

BF proceeded as in table 3. She could start from equation or function machine and generate all other facets, but was only able to move between table and graph when starting from one or the other. She kept trying to generate equations or function machines using only one point. She was thus unable to find the slope and could not make other links to equation or function machine (figure 5).

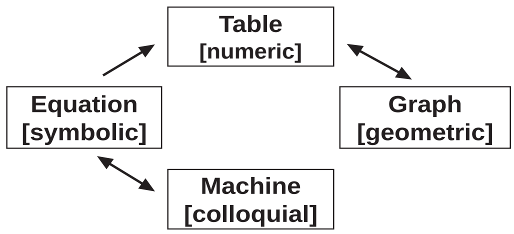

Figure 5: direct links between facets for BF

CM was also able to start from the equation or function machine and generate all other facets. Starting with a table he drew a graph, but could not cope the other way (table 4).

CM From ↓ to →	Equation (symbolic)	Table (numeric)	Function machine (colloquial)	Graph (geometric)
Equation		√ (1)	√ (3)	√ (2)
Table				√ (1)
Function machine	√ (2)	√ (1)		√ (3)
Graph				

Table 4: Creating representations: CM

He had a limited ability to pass directly from one facet to another (figure 6). He said:

> I'm not real sure on equation or function machine.
> *If you had to choose between the two, which would you prefer?*
> It doesn't matter. I don't like either. I really don't like anything that has to do with math.
> [The pained look on his face and the nervous body language speak volumes.]
> *You like tables?*
> Yeah. Tables are a little bit easier for me. I trust those more than having to figure out stuff.

Given a graph he drew a table outline and said:

> No. I can't do it.
> *You started to do a table.*
> Yeah, ummm. If I were to sit down and think about it for a while I probably could. That's the way a lot of math is to me. I just keep trying different ways until I hit upon one that works. To save my life I probably could, but I'm not real sure.

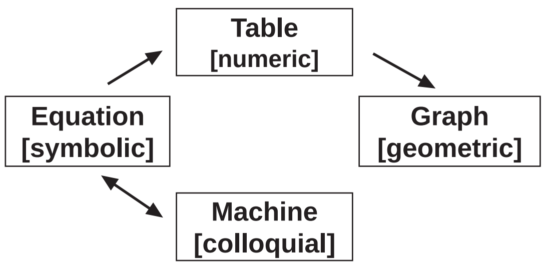

Figure 6: direct links between facets for CM

CM struggled throughout the course using inflexible procedures and limited connections between representations. He became frustrated and gave up easily, particularly where graphs were involved.

Overall, AF's performance on this series of questions was flawless. BF demonstrated good connections between symbolic and colloquial and between numeric and geometric, but only from the first of these pairs to the second. CM established a connection between symbolic and colloquial, but any connection to graphs was tenuous at best.

Student profiles

Visual profiles (Figure 6) of the concept images of function at the end of the course were created for each of the three students through analysing all the collected data. The shading indicates layers of each facet attained by the end of the course.

AF demonstrated knowledge during the interview that was at least equivalent to that displayed on the post-course survey. Her knowledge of the verbal facet matched her written facet since her verbal and written descriptions of function were identical. She was able to assimilate alternate definitions easily into her own concept image. AF did exhibit difficulty during the interview dealing with implicit equations as functions of one variable in terms of the other. She did not use the "uniqueness on the right" condition (Breidenbach et al., 1992, for example) in her selection of functions from a set of equations. She initially denied the constant function is a function, but later changed her mind. She displayed proceptual abilities working with both tables and function machines. She is easily able to think of them as functions (static objects) and as processes (dynamic objects). Her understanding of graphs was developing even as we conducted the interview. She did not need to know a specific procedure, recognizing each graph as representing a set of input-output pairs. She was not prototype-driven and although she did not initially seem to know how to apply the "uniqueness to the right" condition, after some instruction, she

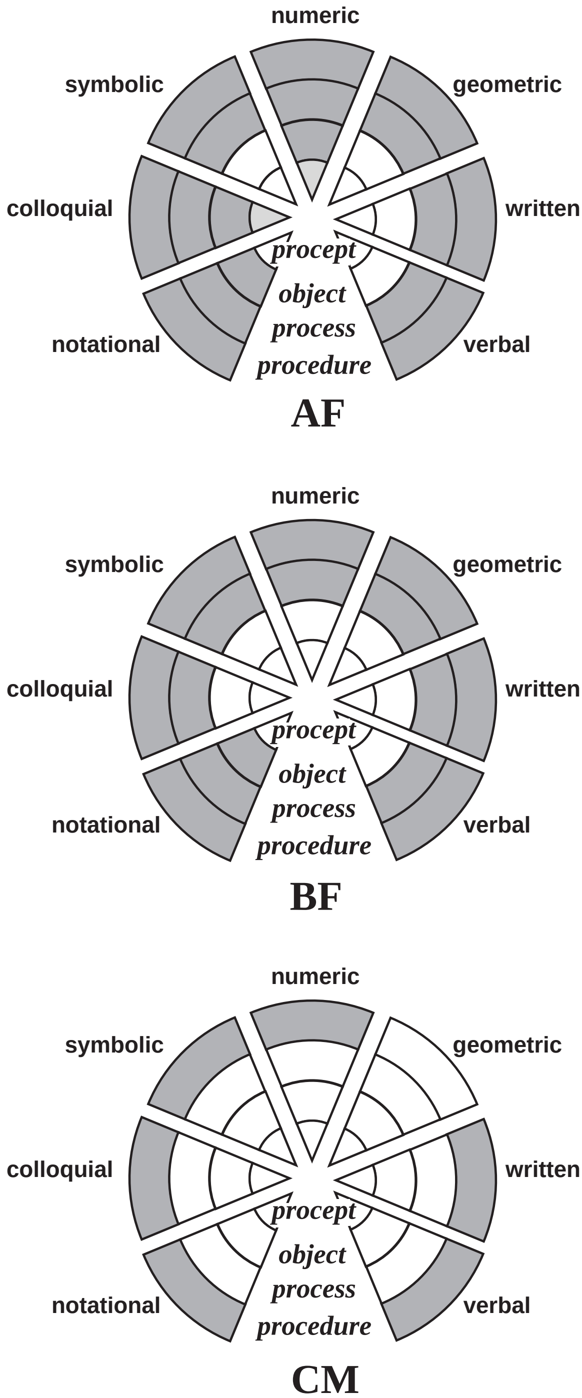

Figure 6: Student profiles

was able to use it coherently. She was not placed in the object layer for the geometric facet because she demonstrated a process-orientation looking at graphs rather than to seeing it as a function object. Her knowledge of the notation facet (for instance the meaning of $y=f(x)$) appeared strong and consistent except for an occasion when she was asked to substitute 44 for y in an equation containing $y(x)$ and said "44 of x." She quickly withdrew this statement and described 44 as replacing $y(x)$. AF was the only student interviewed able to distinguish between $3f(2)$ and $2f(3)$.

Of the three students, BF exhibited the most growth during the course. At the beginning she was judged to be at the procedure layer only on the symbolic facet. By the end, she appeared to be at or near the process layer on all facets surveyed. The numeric and colloquial facets showed some difficulties with process. She was highly procedural in creating an equation from a function machine writing down the steps of the function machine literally. This result carried over to the interview. Her choice of tables that represent functions focused on those tables in which a clear procedure or pattern was present. Her strongest facet seems to be notation which she interpreted flexibly in both post-course survey and interview although she exhibited difficulty interpreting $3f(2)$ and $2f(3)$ and substituting 44 for y in an equation involving $y(x)$. In the interview she was placed in the object layer for notation because of her ability to discuss the notation as an object. On the symbolic facet, she accepted the constant function as a function, but had trouble with piecewise-defined functions. She was the only student of the three that was able to correctly apply the vertical line test to graphs both on the post-course survey and during the interview. While consistent in her verbal and written definitions, BF was not as comfortable as AF in adopting alternate definitions. She had more difficulty crossing boundaries between facets. She did not easily move from a function machine to an equation and was procedural in using equations. This caused difficulty when given a variable input. She was unsure what to do and was not sure the output made much sense.

CM was the least successful of the three. At the beginning of the course he demonstrated procedure layer knowledge in both numeric and colloquial facets placing him slightly ahead of BF. By the end of the course, he was procedural in every facet except for some movement into the process layer of the symbolic facet. On the post-course survey he showed some ability to reverse a table and some hints of process when selecting tables as functions. The interview suggested that CM was at the procedure layer on all facets except geometric where he remained pre-procedural. In addition, his interview answers in the symbolic, geometric, numeric, and verbal facets were highly inconsistent with those on the post-course survey. He looked for specific procedures when identifying equations or tables as functions and was unable to identify any usable rule when looking at graphs. His written and verbal definitions of functions varied and he could not assimilate any alternate definitions of function into his own. At best, he indicated some use of prototypes when looking at graphs and demonstrated some knowledge of function notation relating only to procedural aspects of equations and the function machine. Neither written nor geometric facets seemed connected to any other facet at all.

Quantitative Data

The class as a whole reflected this spectrum from procedure to mental object conceptions of function. On the pre-test in the colloquial, symbolic and numeric facets, around 70% were able to cope with input-output as procedure or process but only 3% were at this level handling graphs (table 5).

	Colloquial (Function Machine)		***Symbolic (algebra)***		***Numeric (Table)***		***Geometric (Graph)***	
(N = 92)	pre	post	pre	post	pre	post	pre	post
pre-procedure	32%	10%	26%	8%	30%	9%	97%	50%
procedure	20%	18%	54%	51%	14%	11%	2%	21%
process	49%	72%	20%	41%	55%	80%	1%	29%

Table 5: Changes in levels of responses for four facets between pre-survey and post-survey

The table reveals improvement in all four facets. Other data collected during the project implies a corresponding improvement in the verbal, written and notational facets. This suggests that the function concept is accessible as procedure or process for many of these remedial students. The function machine appears to be a sufficiently primitive structure to serve as a cognitive root on which to build the function concept. However, the manner in which these students link the function machine to other facets suggests real difficulties in building sophisticated ideas upon it. All three students AF, BF and CM moved to other facets via algebraic symbolism and only AF used standard algebraic expressions. Many others in the class exhibited similar difficulties moving from the function machine to other representations. Although the function machine is a good candidate as a cognitive root for the full function concept, for many of these students the total concept is too complex to allow a full development.

For instance, student competence with the geometric facet was almost non-existent at the beginning of the course and difficulties persisted throughout even though (or perhaps in part *because*) students had regular access to graphic calculators. While there was a significant increase in response handling graphical problems, by the end of the semester less than half the students were able to use a graph to find output given input and only 19 percent were able to reverse the process. Of our cross-section of students, AF showed good depth in understanding of this facet, but BF and CM had enormous difficulty.

Function notation was also interpreted inconsistently, with many students (including AF) using it correctly in some settings yet unable to translate it to a new, similar setting.

Students are often competent at “plug and chug” mathematics and use this ability to hide weaknesses in their understanding. CM, for example, used the more abstract symbolic facet when the more primitive table failed him. He indicated little understanding of the symbolism, but demonstrated several times that he could evaluate a function. This appears to be an example of “pseudo-conceptual” understanding where he attempted to respond in a manner he sensed was desired by the teacher, yet failed to make appropriate internal connections (Vinner, 1997).

Summary and Reflection

This study underlines the complexity of the function concept. Its inherent richness allows it to be considered as an *organising principle* in mathematical courses such as algebra. The use of function machines provides a new approach in remedial algebra which does not simply reproduce the procedural errors of earlier experience, There are gains in moving students to procedural and process levels of thinking in several facets, but the graphic facet and some of the links between different facets remain problematic. The function machine provides a primitive idea that the majority of the students recognised at the beginning of the course, at least at a procedural level. Theoretically it contains the basic idea of long-term growth. It has an inner procedure that can be viewed externally as an interiorized process and potentially as a mental object that can be operated upon. In this sense the function machine can operate as a cognitive root for the function concept itself. However, for many students, the complexity of the function concept is such that the making of direct links between all the different representations is a difficult long-term task. In this course using graphing calculators, the development of graphical ideas had to start almost from nothing and only partial progress was made. For a course on algebra the function concept can theoretically be used as an organising principle, but is it a cognitive root for general long-term development of the algebra curriculum? In our judgement the jury is still out.

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