The Roles of Cognitive Units, Connections and Procedures in achieving Goals in College Algebra

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The purpose of this paper is to develop a means to illustrate and analyse the cognitive paths taken by students in solving problems. The approach is built upon the notion of cognitive unit (small enough to be consciously manipulated). Our interest is in the nature of the student's cognitive units and the connections between them. We find that a student may have an overall strategy and even formulate goals to achieve all or part of a solution. However, if conceptual structures are too diffuse, the student may concentrate on procedures that occupy most of the focus of attention. This may cause them to lose touch with the ultimate goal and be faced with sequences of activity that are longer, more detailed, and more likely to break down.

Introduction

Why is it that some students find algebra so essentially simple, yet others struggle so badly that they fail in school and need to take remedial algebra courses in college? The literature abounds in distinctions between the conceptual thinking of some students and the procedural thinking of others (e.g. Hiebert & Lefevre, 1986). But *why* does this occur? What is the nature of procedural thinking that makes it the default position for so many? Hiebert and Carpenter (1992) suggest two metaphors for cognitive structures, as *vertical hierarchies* or as *webs:*

> We believe it is useful to think about the networks in terms of two metaphors … structured like vertical hierarchies or … like webs. When networks are structured like hierarchies, some representations subsume other representations, representations fit as details underneath or within more general representations. Generalisations are examples of overarching or umbrella representations, whereas special cases are examples of details. In the second metaphor a network may be structured like a spider's web. The junctures, or nodes, can be thought of as the pieces or represented information, and the threads between them as the connections or relationships. Hiebert & Carpenter (1992, p. 67)

Such ideas have long been part of mathematics education. However, they are often used as general philosophical structures rather than explicit techniques to analyse empirical evidence. Our plan here is to extend these ideas and use the extended theory to analyse the specific solution processes for specific individuals in specific contexts. Here we focus on the activities of students working in college algebra.

Varifocal webs and cognitive units

Skemp (1979) proposed a "varifocal learning theory" in which the nodes of webs are themselves subtly connected schemas when viewed in detail. With this in mind, webs and hierarchies may occur within the same model. As an example, consider the equation

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"$y=mx+b$". As a concept it can be viewed in more detail with a network of internal ideas: that m is the slope, b the intercept; that any linear equation can be represented by substituting numbers for the parameters m and b; that the graph can be drawn if one knows two points on it, or one point and the slope, etc. Some students therefore may see "$y=mx+b$" as a *single structure* with rich connections easily brought to the focus of attention.

Barnard & Tall (1997) introduced the notion of "cognitive unit" as "a piece of cognitive structure that can be held in the focus of attention all at one time". We see cognitive units as forming the nodes of a cognitive structure linked to other units using the web metaphor of Hiebert and Carpenter, incorporating the varifocal element of Skemp. There is a great deal of flexibility as to how the units and their connections may be laid out in a diagram. The notion of whether a link is "internal" within a unit, or "external" between units is largely a matter of personal choice. The actual connections within the brain are not topologically divided into an inside and an outside.

However, there are situations in which the idea of "inside" and "outside" can be helpful as a metaphor to represent the different strengths of connections, as we now consider. For instance, any of the following:

- the equation $y = 3x+5$,
- the equation $3x–y = –5$,
- the equation $y–8 = 3(x–1)$,
- the graph of $y = 3x+5$ as a line,
- the line through (0,5) with slope 3,
- the line through the points (1,8), (0,5),

may be considered as cognitive units which can be linked together as representing the same underlying concept—the single straight line or equivalent linear relation between x and y. This may be represented diagrammatically as six separate nodes with appropriate connections between each. In this sense the connections are *external* to the six cognitive units. However, an alternative, more powerful, view is to consider all six ideas to be various aspects of the same phenomenon, the linear relation/equation or straight line which all of them represent. This allows the separate ideas to be seen as different aspects of a *single entity* that is itself a single node in a larger network.

The move from conceiving of separate ideas to a single idea with different aspects is called "conceptual compression" (Thurston, 1990, Gray & Tall, 1994). For conceptual compression to occur, the individual's cognitive structure must have matured in such a way that the separate elements have an intimate connection enabling the individual to move flexibly from one to another. It is not just that there is a cognitive link between, say, the line through (0,5) with slope 3 and the line with equation $y = 3x+5$, but that both describe exactly the same thing—they are different aspects of the same entity.

In terms of Skemp's varifocal theory, this entity is itself a concept which has internal links as a schema in its own right. What is important to be able to compress a collection of related ideas into a cognitive unit is that the whole entity can be conceived as a unit that is "small enough" to be considered consciously, all at one time. The way that the human mind usually copes with this is to give it a *name* or *symbol*. The name or symbol (assuming it is "small enough") can be held in the focus of attention and manipulated. Such a concept has rich interiority through carrying "within" it many powerful links that enable it to be manipulated and invoked to solve problems.

If the diverse elements are not connected sufficiently fluently, then it may be impossible for the individual to regard the totality as a cognitive unit. It follows that it may be impossible for the individual to make links to it, simply because there is no "it". Any links that are made by such an individual are not made to a flexible conceptual entity but to one element in a loosely connected structure. We conjecture that it is this situation that underlies the often-heard cries of the remedial student saying "don't explain it to me, just tell me how to do it." An explanation—which may be perfectly clear to the teacher with a rich personal cognitive structure—is not perceived as an "explanation" to the student hearing words which do not link to adequate cognitive units in the student's mind.

Focus of attention and working memory

The way in which the human brain works enables certain ways of thinking and constrains others. Crick (1994) views the brain as a complex, multi-processing system which can be used coherently only if much of its activity is suppressed at any given time to focus consciously on a small number of important ideas (cognitive units). These in turn are linked to others that can be brought into focus as appropriate. This idea was expressed succinctly over a century ago:

> There seems to be a presence-chamber in my mind where full consciousness holds court, and where two or three ideas are at the same time in audience, and an ante-chamber full of more or less allied ideas, which is situated just beyond the full ken of consciousness. Out of this ante-chamber the ideas most nearly allied to those in the presence chamber appear to be summoned in a mechanically logical way, and to have their turn of audience.
>
> (Galton, *Inquiries into human faculty and its development*, 1883)

The "presence-chamber" of Galton is the current focus of attention and its "ante-chamber" extends it to the *working memory* consisting of closely linked cognitive units that can be evoked for problem-solving. However, it is important not to allow the physical metaphor of a "chamber" to suggest a single fixed area of activity in the brain. The "focus of attention" may be spread over many disparate areas currently resonating together in conscious thought. It therefore remains susceptible to other activities that can interrupt and override the current thought process. Such interruptions may result from unrelated external sensations, such as hearing a school bell ring to end the mathematics class, or more intimately linked strategic activities, such as a mental process monitoring whether a longer-term goal is being achieved.

Skemp (1979) theorizes that a specific problem-solving context provides a goal to be achieved, in which sub-goals may be formulated to achieve parts of the solution process. He hypothesizes that a *comparator* activity occurs at various times which considers whether the solution process is getting suitably close to the goal or to one of the intervening sub-goals. When following a routine sequence of actions we conjecture that the focus on successive remembered steps may be so great as to temporarily fill the focus of attention and suspend the activity of any comparator. This would suggest that the inflexibility of procedural thinking can become so dominant as to cause the individual to lose sight of the goal and so fail to solve the problem. Skemp also suggests the dual idea of an “anti-goal”, something to be avoided—such as the anti-goal of avoiding failure—bringing with it a sense of anxiety that may negatively affect creative activity.

We therefore hypothesise that the difficulties encountered by remedial students relate to the nature of their ideas: that powerful concepts—which others can compress into manipulable cognitive units—remain, for them, as more cumbersome structures too diffuse to employ in a novel context. Our empirical evidence reveals that remedial students may have goals to achieve, indeed may articulate sub-goals, but the dominant procedures they use to attempt to achieve these goals seem to take up so much conscious thought as to prevent them from making necessary cognitive links to complete the exercise. While the successful mathematical thinker may have flexible cognitive units with powerful internal relationships which allow them to be used in diverse productive ways, the less successful may therefore be faced with longer procedural routes which actually make the mathematics *harder*. In other words, the *weaker* students are following longer more detailed cognitive paths that cause *greater* cognitive stress and further increase the chance of failure.

An example

As an example consider the following problem from a college algebra course:

> Find the x-intercept and y-intercept of the graph with equation $3x+4y=12$.

For students with a sense of the symmetry between the occurrences of *x and y* in this equation, it may be possible to “see” the answers in the equation itself. For instance, to obtain the y-intercept, imagine the “$3x$” part to be zero and focus on $4y=12$ to *see* the solution $12/4=3$ (Figure 1). A similar route for the other intercept gives a compressed solution of the problem as two immediate links without any need to write down intermediate steps. However, students who do not see this instant solution may resort to formulating sub-goals using lengthier procedures.

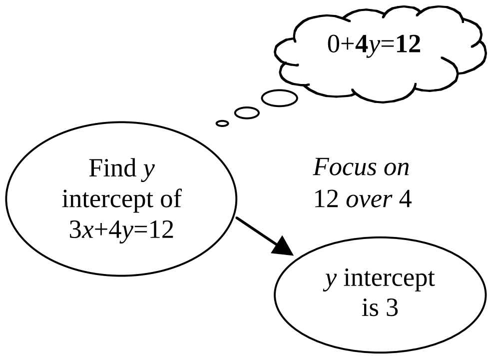

Figure 1: compressed solution

Kristi

Kristi is a community college student taking a remedial Intermediate Algebra course using a graphing calculator to produce tables and graphs. She needs to pass it before she can attempt the college mathematics courses required for her degree in psychology. She had met the concept of a straight-line equation in its various forms before the course and when interviewed afterwards she was able to discuss problems dealing with lines, their equations, slopes, graphs, etc. However, she had a strong focus on the equation in the form "$y=mx+b$", not least because she had been taught to use it to type into her graphing calculator to draw a graph. She could also read off the slope as the number before the x, and the y-intercept as the number at the end. So when asked for the slope of $y = 3x+5$ she could see this as 3, and the y-intercept as 5. For her, this standard form was the starting point for many solutions to problems, and she was frequently successful using it. She therefore began to use the sub-goal of "putting the equation into the form $y=mx+b$" before attempting the question under consideration, whether or not this was appropriate.

Her second major strategy stemmed from the first. If the standard form is known, it can be typed and the graph drawn on a graphing calculator. Kristi frequently used a graph–either a mental one, a graph on a piece of paper, or one on a calculator screen.

> If I were to just look at it, to **visualize** it in my mind—it's a line …

The interviewer said, "what's the y-intercept on the graph?" Kristi responded

> that's where the . . . it intercepts the *y*- … I know it's just a line, so I know it's going to have to cross up here somewhere.

She had a piece of paper with axes drawn on it and pointed to a spot on the y-axis of the grid on the paper, above the origin. Kristi tried to visualize it—she had a mental graph—but seemed unable to use it to solve the problem at this point. The interviewer said "Can you graph it?" and she replied:

> Yes, if I have my graphing calculator …

She has had success graphing with her graphing calculator, and was comfortable with it. Without it, however, she could still have some success . . .

> it's like . . . I need a point. … zero? [she seems to seek support, but then proceeds on her own] . . . if x is zero, then . . . okay, x is zero. Zero, five. Okay.

She plotted the point (0,5). Implicitly she had found the y-intercept she was seeking, *but she failed to recognise it*. Either her comparator is failing to operate or she does not (at this moment) link the point she has found to her ultimate goal, the y-intercept. She continued in her strategy to produce a line by evaluating a second point. She let x be 1, and wrote the point (1, 4). She plotted the points, drew the line through them, and decided that the y-intercept was 5.

The interviewer then asked her to find the y-intercept for $2y + x = –6$. Using her "general strategy", Kristi began to put it into slope-intercept form, "move the x over", "divide by 2". When asked to do it without putting it into slope-intercept form, she said

> I don't know what to do . . . I can't visualize it in my mind . . . like, if I get back the value, I don't know what to do unless I divide everything by 2. So far, that's what I know to do . . . put it into slope intercept.

Asking her to do the problem without putting it into slope-intercept form severed her links with her coping strategies. She attempted once more to graph the equation by plotting points.

Later in the session the researcher asked her to find both the *x*- and *y*-intercepts of $3y + x - 12 = 0$. When asked, "What would you do here?" she replied:

> Divide everything by 3. In my mind I'm visually moving everything, and dividing *x* by 3, its … one third *x* plus … , so the *y*-intercept is 4.

Once again she put the equation into slope-intercept form to find the *y*-intercept. Had she had the conceptual link to do so, it would have been much simpler to set *x* to zero to find the *y*-intercept. She was then asked, "What are you trying to do? What do you graph?" and she immediately plotted the point (0,4). When she was then asked how to find the *x*-intercept, she replied:

> on the calculator screen, where *x* is . . . if *y* is what, then hit intersect and try to find where the *x* is.

Her general strategy of attack is represented diagrammatically in figure 2.

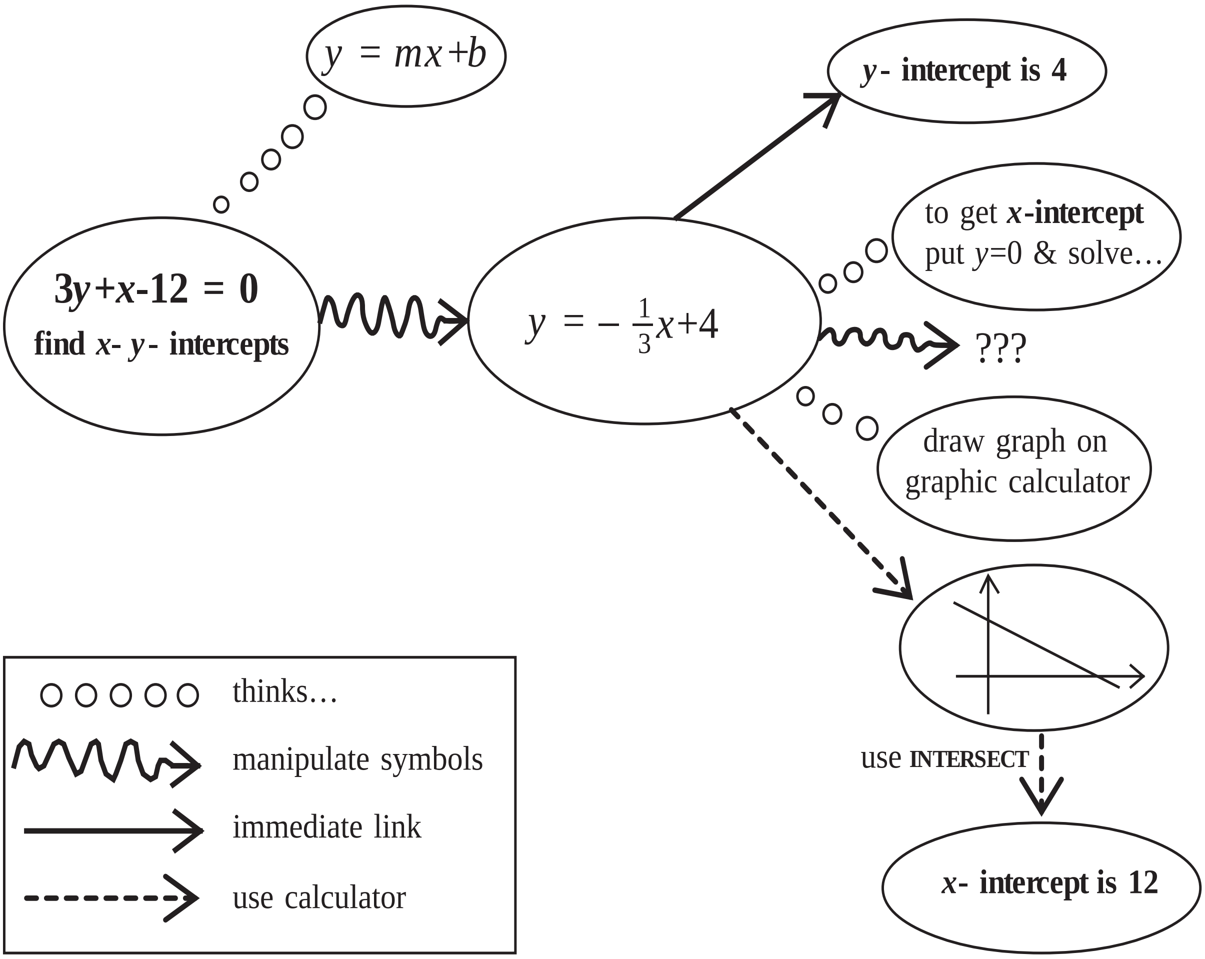

Figure 2: Kristi's strategies for finding *x*- and *y*- intercepts of $3y+x-12=0$.

This interview shows the complications that can appear when the student uses perfectly legitimate procedures to solve a problem. In this case, a compressed solution to find the *x* and *y* intercepts need involve only two very short computations in a symmetrical manner. However, the student's experience of the graph as a function provides an *asymmetric* relationship in which the roles of *x* and *y* (as input and output) are radically different and in which the methods of finding the corresponding intercepts are radically different. Kristi thinks about the sub-goal of putting the equation into her favored slope-intercept form, itself a procedure requiring effort. From this the *y*-intercept is easily read off but the *x*-intercept requires a second lengthy procedure. The structures of the compressed solution and the more lengthy procedure are represented in figure 3.

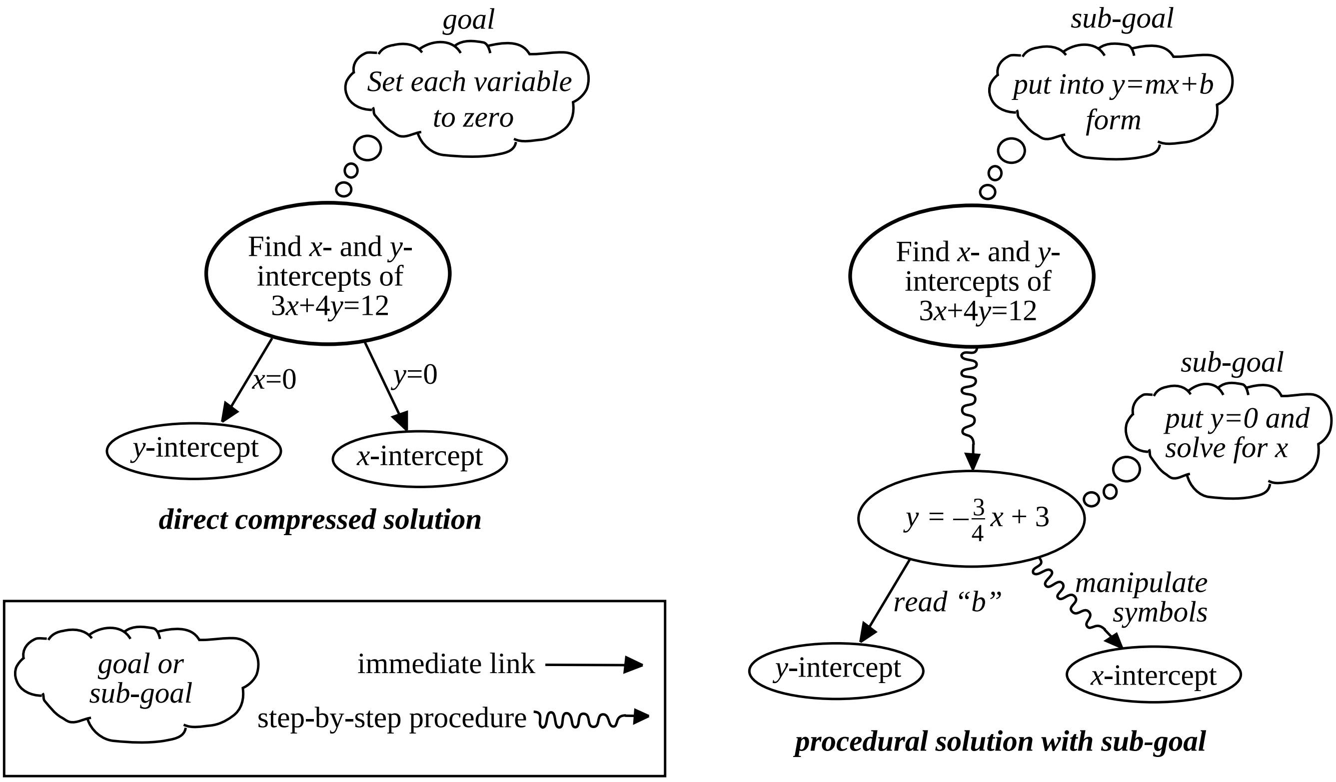

Figure 3 : compressed and procedural solutions for finding intercepts for a specific equation

This use of familiar uncompressed processes with sub-goals occurred repeatedly in Kristi's work. For example, she was asked to write the equation of the line through (1,4) and (4,–2), which she did successfully. She was then asked whether the three points (1,4), (4,–2), and (5,2) were on the same line. Rather than check (as the interviewer expected) whether the third point satisfied the equation of the line she had just found, she calculated the line through (1,4) and (5,2) and compared it with the one she had, saying:

> The way I know how to do it is to take the slope that I got, and get the line through these two points, and see if they are the same. That's the only way I know how to do it.

She used the idea of a line through two points again, repeating a familiar activity that had just been successful. However, she did not exhibit the flexibility that she needed to cope with different problems in new contexts.

The inflexible use of procedures occurred in many other students. Sometimes they were even more diffuse and error-prone than those attempted by Kristi. Kim, for instance, solved the equation $3y + x - 12 = 0$ to obtain:

$$y = -\tfrac{1}{3}x + \tfrac{12}{3}.$$

For this student the equation was doubly difficult; it involved not only fractions, but also negative numbers. We can hypothesise that the notions of fractions and negatives have not become cognitive units that can be used fluently. Kim therefore compounds (at least) two levels of difficulty. First there are the uncompressed, inflexible procedures that are onerous to handle. Within these are uncompressed conceptual structures for negatives and fractions that render the difficulties even more burdensome.

Summary and reflections

In this paper we have highlighted the difference between the use of flexible cognitive units on the one hand and more diffuse uncompressed structures on the other. We give evidence that a student who has yet to compress external relationships between concepts into tight cognitive units with strong internal links will find it more difficult to cope with problems requiring their use. The case studied here showed that a simple problem of finding intercepts of a linear equation contains subtleties easily handled by a student with a compressed cognitive unit encompassing the properties of algebra and the graph of a linear equation. The student with a more diffuse cognitive structure is at a serious disadvantage; this places a strain on the focus of attention at this stage and may prevent powerful theory building for the future. In this way there develops a spectrum of performance in which those who are struggling use even more complicated solution processes that place them in greater danger of failure.

References

Barnard, A. D.. & Tall, D. O. (1997). Cognitive Units, Connections, and Mathematical Proof. In E. Pehkonen, (Ed.), *Proceedings of the 21st Annual Conference for the Psychology of Mathematics Education*, Vol. 2 (pp. 41-48). Lahti, Finland.

Crick, F. (1994). *The Astonishing Hypothesis*. London: Simon & Schuster.

Galton, F. (1883). *Inquiries into human faculty and its development*. London.

Gray, E.M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: a "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116–140.

Hiebert, J. & Carpenter, T. P. (1992). Learning and Teaching with Understanding. In D. Grouws, (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65–97). New York: MacMillan.

Hiebert, J. & Lefevre, P. (1986). Procedural and Conceptual Knowledge. In J. Hiebert, (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Kuchemann (1981). Algebra. In K. M. Hart, (Ed.), *Children's Understanding of Mathematics*: *11-16* (pp. 102-119). London: John Murray.

Skemp, R. R. (1979). *Intelligence, Learning, and Action.* New York: John Wiley & Sons.

Thurston, W. (1990). Mathematical education. *Notices of the American Mathematical Society, 37,* 844–850.