From School to University: the Transition from Elementary to Advanced Mathematical Thinking

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This paper presents my current thinking on the problems that students face in transition from school mathematics to university mathematics. At school the accent is on computations and manipulation of symbols to "get an answer", using graphs to provide imagery to suggest properties. At university there is a bifurcation between technical mathematics that follows this style (with increasingly sophisticated techniques) and formal mathematics, which seeks to place the theory on a systematic, axiomatic basis. There is a broad spectrum of student thinking styles, partly genetic, partly influenced by social experience and teaching, which predispose students to different kinds of learning techniques. Using theory developed by Duffin & Simpson (1993), and Pinto (1997), it may be hypothesised that "natural" learners build from their experience and try to make sense of the mathematics through their current knowledge, "formal" learners are willing to take the mathematics and its rules as a game to be played, to make sense of it within itself. Both natural and formal learners face cognitive difficulties. When natural learners meets new ideas which do not fit their current ideas (and therefore seem "alien"), some cannot proceed because they lack any conceptual structure to build on, others can only proceed by reorganising their own knowledge into a form that allows them to build the new. Meanwhile, the formal learner is more able to take the new ideas in their own restricted context and attempt to make sense of them in a separate new compartment. The formal learners may later encounter problems when attempting to relate the new constructions to old knowledge. Students are rarely at one extreme or the other, often being a combination of the two; some are fortunate enough to be "flexible" learners who can utilise each to best advantage. In this paper we consider how different styles of learner have different cognitive problems in making the transition from elementary to advanced mathematical thinking.

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Introduction : Thinking about Symbolism

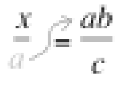
This paper was prepared for an audience of school mathematics teachers and university mathematicians. It might therefore be appropriate to talk about some topic in mathematics, but instead I wish to turn our attention from the mathematics alone to the way in which we *think* about mathematics. We all have the experience of thinking in our own personal way and may have sensed that others do not always think in exactly the same manner. In particular, we do not always think *mathematically* about mathematics. Let me explain what I mean by this.

Just before I started writing this article on my computer, I decided to tidy up some of the files on my computer desk-top, throwing some into the waste-basket and moving others to a choice of two different folders. At each stage I had to decide which of the folders or the waste-basket the file should go to and move it. After doing several of these I moved the icon for the waste-basket from down in the left-hand corner up to the right in a more convenient place. As I took each decision, it was not long before I became immersed in the process and found myself moving unwanted files not to icon for the waste-basket, but down to the bottom left corner where it was previously found. So, I had failed to make the actions more efficient because deeply ingrained in my subconscious was the physical action where to put unwanted files. Although I *consciously* chose to do the action efficiently, unconsciously there were deeply ingrained habits of old that took command of my thoughts and actions.

It is this kind of idea that I am trying to focus on at the moment. When we operate on mathematical symbols we may learn to operate with rules formulated in a mathematical sense, but in the longer run we use all kinds of deeply ingrained mental and physical processes to carry out a given mathematical operation, and these ingrained processes are not the same as natural laws of mathematics. For instance, though we might learn to solve the equation

$$\frac{x}{a} = \frac{b}{c}$$

by "multiplying both sides by *a*", we may later "see" the *a* move across from the left-hand denominator to the right-hand numerator.



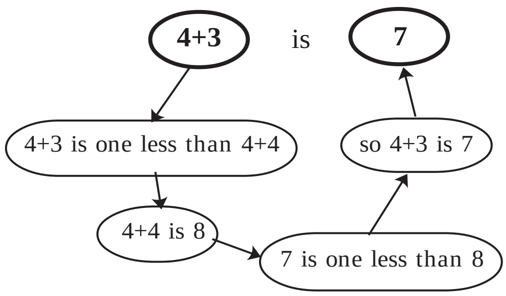
It proves efficient to utilise our human actions of "moving" symbols around, yet there is also a need to be able to reflect on the action to relate it to mathematical decision-making processes. The problem is that these often get translated into unconscious acts that may be associated with rules of thumb such as "change sides, change signs", "cross-multiply", "two minuses make a plus" etc.

The tensions that arise in such mental activities are complex and can manifest themselves in a variety of ways. Not only may old rules remain unchanged and be used inappropriately, new rules may supplant old rules and be used incorrectly when earlier work is recalled. For instance, rules of thumb in arithmetic can be mistranslated in algebra, and those in algebra can be mistranslated in arithmetic. An example of the former is when a fraction is seen to involve "dividing the top number by the bottom number" so that $\frac{12}{6}$ is correctly computed by dividing 12 by 2. Yet in the algebraic expression $\frac{a^{12}}{a^{6}}$ "dividing the top number by the bottom number" may be incorrectly given as a^2. In algebra the rule to compute $3a^2 \times 4a^3$ to give $12a^5$ by "multiplying coefficients and adding powers" may be mis-applied back in arithmetic to compute $3^2 \times 4^3$ as 12^5. Both of these errors (and a variety of others) prove to be made by a significant proportion of students.

Such errors have led to a theory of "buggy arithmetic" where children's mistakes are seen to be a result of "mal-rules" rather than arbitrary slips. But such a theory is already missing the boat. To attempt to teach someone how to get correct answers only by correcting their errors may simply replacing one rote-learnt routine by another. Such a limited strategy is likely to fail to help the individual build up a coherent overarching sense of mathematical conceptualisation.

Instead we must look more closely at the subtle relationship between the mathematics and the manner in which it is conceived by the individual. It is my observation that the routines that we learn in mathematics serve in two very distinct capacities. For most they enable individuals to carry out a specific computation, but then there is a "parting of the ways", first described eloquently to me by Eddie Gray (eg. Gray, 1991, 1993). Some remain fixed in the mechanics of the routine — able to perform it, able to build up a collection of routines to operate in different circumstances. Others develop a more flexible way of using symbols — seeing them both as processes to be computed and also as mental concepts to be manipulated. For such fortunate individuals arithmetic takes on a more generative form, where known facts, such as 4+4 makes 8 are used to generate derived facts, such as 14+4 makes 18 or 4+3 makes 7.

A consideration of the possible formal links required (as illustrated) suggest that these are more complex than counting on (4), 5, 6, ***7***. The only way that the cognitive links can give an advantage is that they must be tighter and therefore different from the formal deductions.



Symbols as process and concept

Inspired by a succession of thinkers on cognitive development—including Dubinsky (1991) and Sfard (1991)—Eddie Gray and I noted, as had others before us, that symbols in arithmetic, algebra, calculus, and a wide range of other mathematical contexts had a certain characteristic. The following symbols illustrate this:

$$5+4,\ 3\times 4,\ 3a+2b,$$

$$\lim_{x\to a}\frac{x^3-a^3}{x-a},\ \frac{d}{dx}\left(\frac{\sin x+\cos x}{x^2+3x+1}\right),\ \int_0^{2\pi} e^{2x}\cos x\,dx,\ \sum_{n=1}^{\infty}\frac{1}{n^2},$$

These all play a dual role representing both a mathematical *process* to be carried out and the *result* of that process. For instance 5+4 evokes the *process* of addition to produce the *concept* of sum 5+4, which is 9, $3a+2b$ is a both process of evaluation and a concept of algebraic expression, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is the process of evaluating an infinite sum to find the limit value (which happens to be $\pi^2/6$).

The name *procept* was introduced for the combination of symbol, process and concept which occurs when a symbol evokes a process to give the resulting concept (Gray & Tall, 1994). We were interested in the way in which individuals interpret symbols in arithmetic, algebra and calculus, causing some students to find mathematics essentially easy yet others finding it increasingly difficult.

We emphasise that the cognitive notion of procept carries with it no implication as to how that cognitive structure is built. Indeed one of our purposes was to investigate the concept-building of such symbols. However, in many practical contexts, we often found that the meaning of symbols developed through a sequence of activities:

(a) *procedure*, where a finite succession of decisions and actions is built up into a coherent sequence,

(b) *process*, where increasingly efficient ways become available to achieve the same result, now seen as a whole,

(c) *procept*, where the symbols are conceived flexibly as processes to *do* and concepts to *think about*.

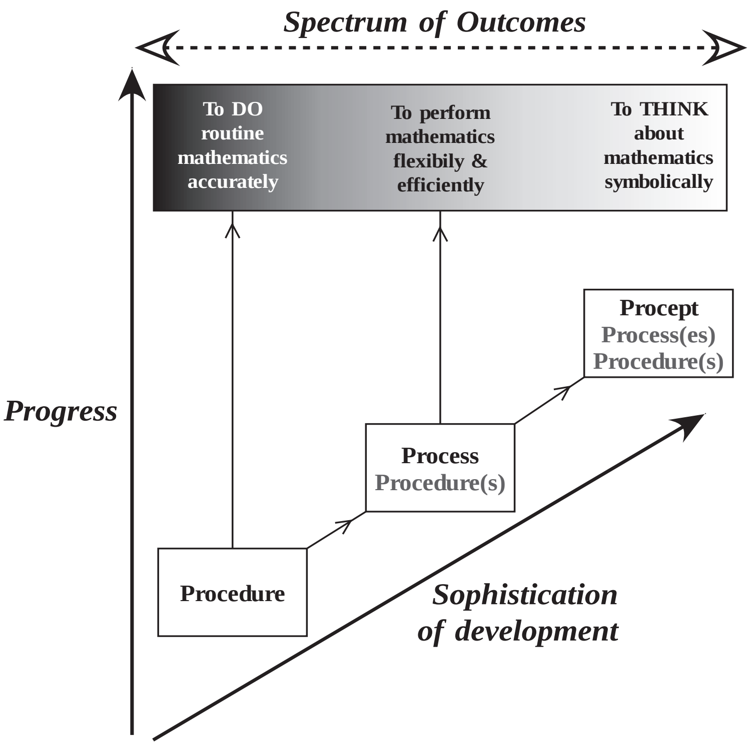
Initially the individual builds an "action schema" (in the sense of Piaget) as a coordinated sequence of actions. At the *procedural* level, the focus of attention concentrates on how to do each step and how this leads to the next. Following Davis (1984), we use the term "procedure" for a specific finite sequence of decisions and actions. In contrast the term "process" is used in a more general sense, such as "the process of addition" or the "process of solving a linear equation". A process may have several different procedures which give the same result. For instance, the symbols $2(x+3)$ and $2x+6$

involve two different sequences of computation, but represent what we consider to be the same process. In this way the function $f(x)=2(x+3)$ is the same function as $g(x)=2x+6$ because they have identical input and output.

The addition 2+7 might be performed in a variety of ways, say by counting two sets, then both together, or starting at 2 and counting on 7, or counting on 2 starting at 7, or simply knowing that 2+7 makes 9. Now the symbol 2+7 may be seen not only as a *process* (of addition), but also as a *concept* (of sum), so that 2+7 not only *makes* 9, but 2+7 *is* 9. This can lead to a rich web of relationships, so that, if "2+something" is 9, then the "something" is 7, and on to other facts involving place value, such as 3**2**+**7**=3**9** or **7**0+**2**0 =**9**0.

The child who sees addition only as a "counting-on" procedure is likely to see subtraction as a "counting-back" procedure, counting back 9–2 in two steps as "8, ***7***", or 9–7 as count-back seven steps "8, 7, 6, 5, 4, 3, ***2***" incorporating lengthy counting procedures that prove to be increasingly more difficult to carry out correctly.

Procedures allow individuals to *do* mathematics, but learning lots of separate procedures and selecting the appropriate one for a given purpose becomes increasingly burdensome. *Procepts* allow the individual not only to carry out procedures, but to regard symbols as mental objects, so they can not only *do* mathematics, they can also *think* about the concepts. For such a student with powerful mental connections, greater abstraction gives greater simplicity, whilst the less successful student is left with ever increasing complexity and the greater likelihood of failure.



A consequence of this is that those students who do *not* make enough appropriate mental connections may be able to do the current problems but have a far greater mental burden and fall back on the need to routinise mathematics to be able to “do” the procedure to get an answer. They can therefore “do” a problem in a limited context and see this as “success” but are not developing the long-term connections to be able to think about more sophisticated ideas.

I conjecture that this is a major reason why many students are “damaged” by their experiences in school, apparently learning how to “do” mathematics but unable to link together ideas which are, for them, either meaningless or too complex. The common fall-back position is to routinise specific procedures to be able to do routine examples, but without the flexibility to cope when the problem is dressed up in a different manner or in a new context. I regret that this is also a common fall-back position for many school teachers, especially in the later stages of secondary education where the going is getting rougher for students meeting symbols that they need to manipulate but do not fully understand.

This problem is far more prevalent than one might wish to acknowledge, for instance, MacGregor and Stacey (1993) asked the following question of 255 mixed ability year 9 students (aged 13 to 14):

> z is equal to the sum of 3 and y. Write this information in mathematical symbols.

Of the total questioned, 43% gave either an incorrect response (often z=3y or y=3z), or made no attempt.

Students very often learn to do what pleases the teacher, and is it no wonder, when what they are asked to do often has no real meaning for them. A classical case is the definition of a negative or fractional power. For a student who knows that a product of three fives, (5×5×5) can be written as 5^3, and a product of four threes (3×3×3×3) can be written as 3^4, the rhythm of the notation soon becomes part of their inner harmony “da times da” is da raised to the power two, “da times da times da” is da raised to the power three, and so on. After experience of meaning such as this, what can $a^{1/2}$ mean? Can it mean “half an a multiplied together”? Or can a^{-2} be “minus two as multiplied together”?

For many students, the idea that a^m means “m lots of a multiplied together” has, *as a consequence*, the property that $a^2 \times a^3$ is a^5 and, more generally that $a^m a^n$ is a^{m+n}. But this explains a *natural property of whole number powers*, which disintegrates when the powers are taken to be rational or negative numbers. The concept *has no meaning*, at least it has no meaning in the conventional sense. Hence the fact that the value of $a^{1/2}$ can be *deduced* from

$$a^{1/2} \times a^{1/2} = a^{1/2+1/2} = a^1 = a$$

also seems somewhat peculiar.

Some students who focus on the essentials of the notation are happy to accept this argument. Such students have the potential to become mathematicians. But those who see it as a piece of conjuring, aimed to give a meaning to the unmeaningful, to use rules without reason, are only naturally likely to see mathematics at this stage as rules that are to be learned without having to have a reason.

Natural and Formal Learners

When faced with new mathematical ideas, individuals behave in different ways. In arithmetic the more successful students already have flexible interlinked structures which support the use of symbolism both as process to get results and concept to think about. The less successful focus more on the security of performing the algorithms and have limited success with routine problems. As their development continues through mathematics, the differences begin to diverge even more. In facing new ideas, some have little cognitive structure to build on and are likely to fall back even further on rote learning. But even those with a growing richness of cognitive structure develop different personal approaches.

One method of categorising different approaches is to say "does the learner build on current structures to make sense of the new mathematics, or does the learner try to make sense of the mathematics as a task in itself?" In other words. does the student synthesise from their experience to build the new mathematical ideas or analyse the new mathematical ideas to build a system in itself which may perhaps later be integrated with old knowledge. Duffin & Simpson (1993) call the former "natural" learners and the latter "alien". As I struggle with these ideas myself, I prefer the names "natural" and "formal". A natural learner tries to make sense of new ideas using current knowledge, a formal learner gives the new knowledge a chance to develop its own meaning by playing with it without initially feeling the need to link it to other knowledge.

In faced with the introduction of fractional and negative powers, natural learners are likely to meet a conflict. Their experience of a^m is of "m lots of a multiplied together" is fine for a^2, a^3, ..., a^{27}, … . But the concept of $a^{1/2}$ does not fit this meaning. This can have a variety of effects for the natural learner, it may mean that the learner cannot give any meaning whatsoever to the notation and can only cling to anything that is given or said in class to help the student cope. This is the road to meaningless manipulation, learning rules to give the answer required by the teacher.

An alternative is to concede that $a^{1/2}$ means *something*, and to concede that the rule for powers could conceivably also apply to fractional powers and then to see, for consistency, that $a^{1/2}$ can be taken to be $\sqrt{a}$. A "formal" learner may say "OK, I don't know what $a^{1/2}$ means, but for a time, let me play the game presented to me and see what happens." This allows the individual to suspend belief for a time and to manipulate the symbols to gain a sense of what might happen, leading to the possibility, indeed, coherent likelihood that $a^{1/2}$ can be taken to be $\sqrt{a}$. The formal learner, with no immediate qualms about manipulating things that don't seem to have meaning, can eventually develop meaning from the activities that are carried out. Perhaps later they can also look back with hindsight and see that the extended definition of a^m to include rational (and negative) values of a, indeed, when the graph of 2^x or e^x is drawn (perhaps by a computer), the "filled-in" values of x to include fractional and negative values begin to give a global alternative meaning of its own. Thus both a natural learner, willing to restructure knowledge, and a formal learner, willing to take new ideas at their face value and give meaning through playing with them, have long-term possibilities of success. But the natural learner who is confused by the new meanings or the attempted formal learner who is unable to get the new ideas to fit together in a coherent way will face more serious long-term difficulties.

Long-term difficulties with symbols in elementary mathematics

As the mathematical curriculum develops through arithmetic, algebra and calculus, the symbols operate in subtly different ways, of which the following are examples:

(i) *arithmetic procepts*, such as 5+4, 3x4, $\frac{1}{2}+\frac{2}{3}$, or $1{\cdot}54\div2{\cdot}3$, have explicit algorithms to obtain an answer, but become increasingly difficult for the procedural learner.

(ii) *algebraic procepts*, such as $2a+3b$, do not have an "answer" (except by numerical substitution), but they can be manipulated using more general strategies, which again coerces the procedural learner into rote-learning of isolated techniques,

(iii) *implied procepts*, such as $a^{1/2}$ or a^{-2}, which are the results of operations which initially have no meaning but can be deduced from assumed properties of the symbolism,

(iv) *potentially infinite procepts*, such as

$$\lim_{x\to a}\frac{x^3-a^3}{x-a},\ \sum_{n=1}^{\infty}\frac{1}{n^2}$$

involve a potentially infinite process of "getting close" to a limit value, which may be computed by a numerical approximation and sometimes by a symbolic algorithm.

(v) *calculus procepts*, such as

$$\frac{d}{dx}\left(\frac{\sin x+\cos x}{x^2+3x+1}\right),\ \int x\cos x\,dx$$

which are potentially infinite as limit concepts, but can (sometimes) be computed by using the standard rules of differentiation and integration.

At this moment in time I do not claim that these are the *only* different types of procept, but they illustrate my point that the use of symbolism for computation and manipulation involve different problems in different contexts. Each of the above requires new ways of thinking about the symbolism, a change of conceptualisation that proves difficult for many, particularly for "natural learners" who are unable to cope with changes in meaning. A child who thinks of a sum 4+3=7 as a counting procedure in which "4 plus 3 *makes* 7" may find it difficult to cope with a symbol such as $4+3x$ which does not "make" anything, except perhaps to "do the bit 4+3 that makes sense" and get $7x$, which does *not* make sense. This leads to great confusions for many students starting algebra.

Likewise, a student who is used to "doing" mathematics in a finite number of steps may find it difficult to cope with the potential infinity of the limit process. In practice, therefore, even if students are given "explanations" of the potentially infinite limit process of calculating the derivative at a point, they are very likely to be far happier with the security of the calculus computation because it is reminiscent of the operational procepts of arithmetic. The rules for differentiation may not give a *numerical* answer, but they provide an algorithm that produces an answer in a finite number of steps.

The variety and differences between the performance of these different types of procept demonstrate the subtle difficulties underlying the mathematical curriculum. Instead of being a comfortable sequence of successive pieces of knowledge building coherently on what went before, the mathematics curriculum is actually littered with subtle hurdles that trip up the learner, but are not always apparent to the expert. At each of these points of difficulty some students succeed, but an increasing number fail. The "failure" can be disguised by learning how to do procedures and this becomes increasingly so as students get older, including many who return to teach in school, believing that repetition and rules of thumb are the (only) way for all to learn mathematics.

The Transition to Advanced Mathematical Thinking

There are various possible points in the mathematics curriculum which may qualify as signalling a change to "more advanced forms of mathematical thinking", including those in the previous section coping with symbols defined by their properties, such as $a^{1/2}$, or those involving potentially infinite processes, such as the notion of limit in the calculus, or those which have not been discussed here involving proof in geometry. However, the significant change which has been widely considered as a major change in thinking processes is that in which definitions are given as axioms to build up systematic mathematical theories (see, for instance, Tall, 1991). This change often coincides with a mathematics students move from school to university.

Definitions as *criteria* for building concepts formally

In thinking about a mathematical concept, such as *limit*, *group*, *vector space*, *continuous function*, and so on, there is a great deal more cognitive structure involved than the formal definitions and formal deductions made from those definitions. The student must have previous experience which develops intuitions that suggest what will make a good definition, and what theorems are likely to be proved. Tall & Vinner (1981) defined the ***concept image*** to consist of "the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes". It is the concept image which provides the underlying structure for thinking about the concept, both consciously and subconsciously.

In elementary mathematics we build up concepts from our experiences in which the experience often comes first and the words are used to describe it. It can also happen that words are used initially to describe a concept, but then that concept is something which the individual is about to encounter and gain experience of through interacting with it.

In advanced mathematics something new happens. When we say "a *group* consists of a set and a binary operation on that set satisfying the following properties …" we are doing more than stating some of the properties which hold for a concept about to be encountered. We are also stating *precisely what is necessary for something encountered to be called "a group"*. In other words, the definition we give state a number of criteria which must be satisfied in order for us to know we are indeed dealing with (an example of) the given concept.

A ***concept definition*** is a statement in words and symbols which identifies a particular concept. In advanced mathematics a concept definition specifies *precisely* what is required for a given notion to be considered an instance of that concept. It may (and usually does) have more attributes, but it can *never* have less. We use the attributes specified in the definition to deduce other properties, so that we then know that any system which satisfies the criteria specified in the definition has all the properties deduced from the definition. This gives a huge power of generality. Once the properties required are deduced from a definition, *any* system also satisfying the definition also has those properties.

However, this can also create a huge potential conflict for the student. A *formal concept image* must be constructed consisting *only* of those properties which can be formally deduced from the definition. Yet, in building such properties, the concept image is used to suggest possible theorems and proofs. The individual must make an almost schizophrenic separation between the intuitive appeal to the concept image that senses mathematical truth and the formal deduction processes that establishes it. In practice, this separation is rarely, if ever, accomplished. For instance, if one studies the Peano Postulates, how does one "know" that 2+2=4 or 999+2=1001? Is it through *deductions* from the postulates, or an elemental knowledge built up since childhood? In practice it is of course a mixture of the two, with a belief that those things that have not been formally deduced could be proved if this were ever desired. Experts become accustomed to this convention, but novices may be profoundly disturbed in not knowing what it is they are "allowed to know" from their experiential knowledge as opposed to "knowing" from deductive proof.

The incorporation of experiential knowledge into formal knowledge presents a subtle and deeply rooted conundrum from which even the greatest mathematician cannot escape. It causes great difficulty to students whose life experiences make them familiar and comfortable with a vast range of computations to *do* mathematics, yet wary of handling what may be (to them) needless and complicated proof.

For instance, I sat in a colleague's analysis class not long ago as he attempted to avoid too deep a discussion of the axioms for the real numbers by suggesting that they could assume any properties of the *arithmetic* of numbers, but needed to prove things about their *order*. One of the problems was to use the rules that any non-zero number *a* was either positive, or –*a* was positive (but not both), and that the sum and product of positive numbers was again positive. To prove that 1 is positive, they were asked to note that either 1 or –1 is positive, and if –1 were positive, so is the product (–1)(–1), which is 1. Since only one of 1, –1 can be positive, the only possibility is that 1 is positive and not –1. Hence they could conclude that 1 is positive.

What a *foolish* exercise for students to be asked to perform at this developmental stage! Their experience in number as young children *starts* with the ordered sequence, 1, 2, 3, … The number 2 comes "after" 1, and 3 "after" 2; 2 is "bigger" than 1 and 3 is "bigger" than 2, and 1 is certainly "bigger" than nothing! So "1 is positive" is part of their essential belief structure going back to their earliest memories. How foolish to ask them to prove something they *know* so intimately, using later developments in which they are far less secure!

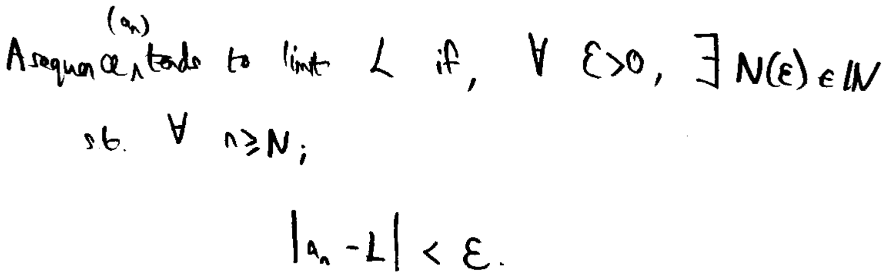
The effects of different cognitive styles in advanced mathematical thinking: the case of the limit concept

What happens to natural and formal learners when they encounter the definitions and deductions of advanced mathematics? The natural learner must take his or her current knowledge and attempt to fit the definition into place. This usually requires a considerable amount of reflection and reorganisation of knowledge which defeats many. Indeed, those "natural learners" who have yet to understand the role of definition as formalising an entirely new concept and deducing its properties, already "know" many of the properties and are confused by the whole issue. Others, however, *can* be successful and are characterised by *giving* meaning to the definition from their richness of experience.

Formal learners, on the other hand, are those who attempt to take the verbal definition at its face value and use it to extract meaning by playing with it. Once again, some are successful and some fail, as we shall see by considering specific individual experiences in confronting the definition of limit of a sequence.

(i) Students building of operable definitions

Both formal and natural learners are capable of building a coherent definition of limit. A formal learner may concentrate on the definition as given and try to make sense of it. Ross, for example, a "formal sense-deducer" first routinised the definition by repeating it over and over again. He also reflected on the definition and began to be able to use it in a meaningful way. He wrote down the definition as follows (Pinto, 1996):

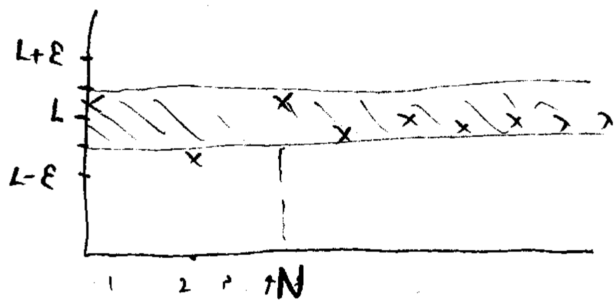


(Ross, first interview)

He went on to explain that he coped by:

> "Just memorising it, well it's mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it get imprinted and then I remember it." (Ross, first interview)

He did have some visual ideas as to what was going on, and drew the following picture:



(Ross, first interview)

However, he explained:

> "Well, before, I mean before I saw anyone draw that, it was just umm … thinking basically as *n* gets larger than N, a_n is going to get closer to L, so that the difference between them is going to come very small and basically, whatever value you try to make it smaller than, if you go far enough out then the gap between them is going to be smaller. That's what I thought before seeing the diagrams … something like that." (Ross, first interview)

Three weeks later, when asked to write down what was meant to state that a sequence was *not* convergent to a limit *L*, he first wrote down:

$$\forall\, \varepsilon>0,\ \exists\, N(\varepsilon) \in \mathbb{N} \text{ s.t. } \forall\, n \geq N. \quad |a_n - L| < \varepsilon.$$

(Ross, second interview)

then negated it by using the formal negation of quantifiers (passing a "not" over a universal or existential quantifier changes one to the other), to give:

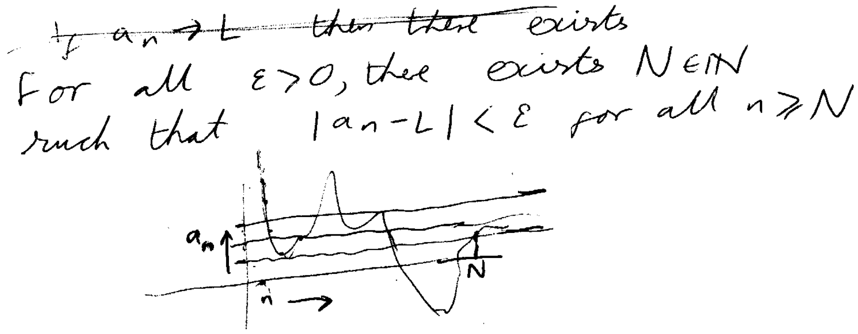
$$\forall L,\ \exists\, \varepsilon>0 \text{ s.t. } \forall\, N(\varepsilon) \in \mathbb{N}\ \exists\, n \geq N, \text{ s.t. } |a_n - L| \geq \varepsilon$$

(Ross, second interview)

Note how he introduces $\forall L$ in the second statement, corresponding to the implicit unwritten $\exists L$ in the first statement. But despite this subtle understanding of the original definition and its formal negation, there is an error in the negation. By writing "$\forall\ \varepsilon>0$, $\exists\ N(\varepsilon)$" in the original definition to indicate that *N* depends on ε, he then erroneously wrote the negation as "$\exists\ \varepsilon>0$, $\forall\ N(\varepsilon)$" but now *N* does *not* depend on ε. Had he simply written "$\forall\ \varepsilon>0$, $\exists\ N$", then the portion of the negation "$\exists\ \varepsilon>0$, $\forall\ N$" would have been satisfactory, by default. In other words, by giving *additional* meaning to the definition, he exposed an error in the routine negation that would not have been visible had he written out the definition in its basic form. Such a problem proved easy to remedy by asking him to think it through, which led to him being able to self-correct his error.

In essence, Ross's initial method of working is to begin by familiarising himself with the definition, *routinising* it by practising saying it, then, when he has the grasp to write it down he plays around with it to *extract* sense from it (Pinto, 1997).

Another student, Chris, approached the task as a "natural learner", but one who had great experience of reflecting deeply on what he was doing and reorganising his thinking to take account of new meanings. He worked from his current knowledge structure by drawing a picture and using it to *build* a meaning for the verbal definition. When asked to write down the definition, he wrote:



(Chris, first interview)

He explained:

> "I don't memorise that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down."
>
> "I think of it graphically ... you got a graph there and the function there, and I think that it's got the limit there ... and then ε once like that, and you can draw along and then all the ... points after *N* are inside of those bounds. … When I first thought of this, it was hard to understand, so I thought of it like that's the *n* going across there and that's a_n. ... Err this shouldn't really be a graph, it should be points." (Chris, first interview)

His description is suffused in physical movement, pointing at the graph, fixing the limit value, gesturing above and below to show the range from ε below to ε above, moving along to the left to show increasing *n*, then pointing to the number *N*, and gesturing once more to the region to the right where the values of the sequence lie within ε of the limit.

Notice that Chris also made an error (drawing a curve instead of points for the values of the sequence), but again, he was able to self-correct using his wider cognitive connections.

When it came to negating the definition, he first clarified the issue by asking "did you mean *does not tend to a limit L* or does not tend to *any* limit" and then thought through the whole thing meaningfully, writing:

> "A sequence (a_n) does not tend to a limit if for any L, there exists $\varepsilon>0$ such that $|a_n-L| \geq \varepsilon$ for some $n \geq N$ for all $N \in \boldsymbol{N}$." (Chris, second interview)

This is an exceptional feat of thinking not found in many students. It is more demanding than the formal method of negating quantifiers used by Ross. Of 250 students majoring in mathematics asked to describe how they remembered the definition of limit later in the course, only five mentioned the use of a picture. Even if pictures are used in the first place (which may very well have happened with far more than the five who later recalled it), it seems that the visual approach is largely supplanted by the symbolic definition when the latter becomes operable.

These two students exhibit two quite distinct approaches to success with the limit concept. One, the formal sense-deducer, focuses on the verbal definition, remembering it and reflecting upon it to extract its meaning. He works *from the mathematics* to extract meaning that is implicit in the definition. The other, the natural sense-constructor, works *from his own experience*, using pictures and physical actions to give a sense to the concept from which he can build the written definition (Pinto 1997) The formal thinker focuses on the given formalities of the mathematics and deduces properties of the concepts by using the formal rules given, the natural thinker builds from personal ideas to construct meaning for the definitions and their properties and then uses his naturally developed intuitions to suggest theorems and to use definitions to prove them.

(iii) Less successful students

A large number of students cannot cope with the definition of limit and remember it inaccurately (Pinto, 1996).

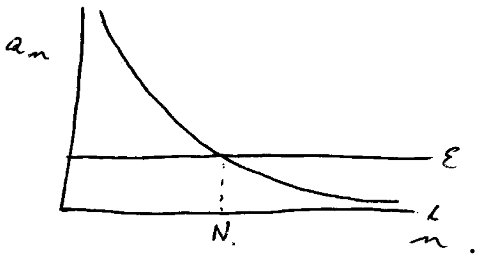
> A sequence (a_n) tends to a limit L for $\varepsilon > 0$ if there exists $N \in \mathbb{N}$ s.t.
> $|a_n - L| < \varepsilon$ provided $n \geq N$.

(Robin, first interview)

> If $a_n \to l$, then there exists $\varepsilon > 0$, such that $|a_n - l| < \varepsilon$ for all $n \geq N$, where N is a large positive integer.

(Colin, first interview)

Colin used a picture at this stage to support his thinking:



Unlike the visualisation used in the previous section by Chris, this picture involves a restricted concept image of a decreasing function or sequence. His imagery therefore is not being distilled to give the essential meaning of the limit Instead he is using a limited image to draw out limited meaning. Although Colin denoted the limit by l, he considered it as a lower bound (a common concept image, see for example, Cornu, 1981, 1991). He also wrote ε instead of $l+\varepsilon$ possibly relating the definition to the introductory work in the course which began by studying sequences with limit zero. He explained:

“.. umm, I sort of imagine the curve just coming down like this and dipping below a point which is ε ... and this would be N. So as soon as they dip below this point then ... the terms bigger than this [pointing from N to the right] tend to a certain limit, if you make this small enough [pointing to the value of ε].”

Again he uses physical gestures to demonstrate the ideas, but his written definition is flawed, with an existential quantifier for ε and the intimation that N is just “a large positive integer” without linking it explicitly to ε.

Unsuccessful Negation

Neither of these students could cope with the formal idea of a non-convergent sequence. For instance, Robin wrote:

> “A sequence a_n does not tend to the limit L if for any $\varepsilon > 0$, there exists a positive integer N s.t. $|a_n – L| > \varepsilon$, whenever $n \geq N$.”
>
> (Robin, second interview)

The original quantifiers for the definition of limit here remain unchanged, and all that is changed is the inner inequality $|a_n – L| < \varepsilon$ incorrectly negated to give $|a_n – L| > \varepsilon$. He is unable to treat the whole definition as a meaningful cognitive unit, and simply focuses on the inner statement as something which he can attempt to handle.

The other student, Colin, said:

> “Umm ... I would just say there doesn’t exist a positive integer because we can’t work it out ... no ... you cannot find an integer N ...”.

and wrote

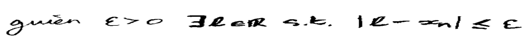
> There exists a term where $|a_n – L| \geq \varepsilon$ where $n \geq N$, where N is a positive integer. (Colin, second interview)

(ii) Further along the spectrum

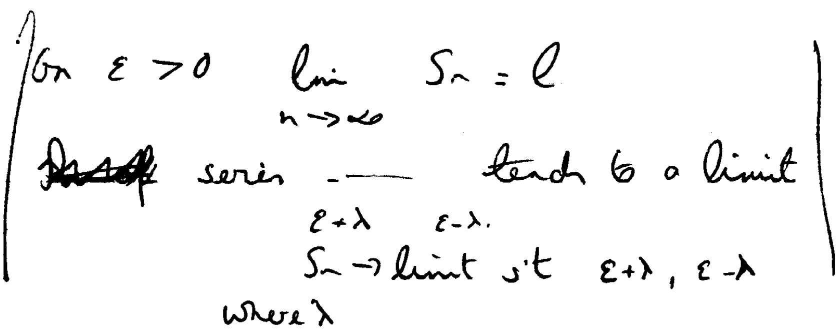
Other students’ remembrances of the definition of limit of a sequence (two weeks after encountering it) reveal a collage of isolated ideas (Tall, 1994).

(a)

(b)



(c)



(d) $\lim_{n\to\infty} s_n = l$

as $n\to\infty$ the terms approach and get closer + closer to l (or may reach it) but l is not exceeded

ie $s_n - s_{n+1} \to 0$ as $n \to \infty$

In these cases the students remember isolated facets from the definition and from their concept image. The first remembers something about the limit l, then changes the symbol to a δ, relating to the definition of limit of a real function rather than limit of a sequence. The second remembers x_n getting within ε of the limit, $|l-x_n|<\varepsilon$, but fails to remember the role of N. The third evokes the term “series” (which was more recent in their experience than sequence), remembers the range of values “something ± something” but gets it as $\varepsilon\pm\lambda$ rather than $l\pm\varepsilon$. The fourth remembers dynamic imagery, including a warning that the sequence *can* reach the limit, yet has an image of an increasing sequence not exceeding the limit, and adds a further fact that $s_{n+1}-s_n$ tends to zero, which is true but not part of the definition.

Bewilderment shines throughout these responses. In the exam the students pick up marks by using procedural methods for proving convergence of series involving computations which make them feel comfortable, but few have any conception of the role of definition. Some find the world of formal analysis at variance with their real-world experiences. Some training to be teachers daily teach young children by example and see no relevance for esoteric proof in their future profession. In interviews remarks such as the following arise:

> I can do examples with numbers and things like that but I can’t do things with definitions. I just don’t know what they are about.
>
> I mean it’s not as if these things are real. You need to swot them up to pass exams but you are never going to use them again.
>
> When I try to prove things in my teaching, I show examples and do it in particular cases. In school I’ll never have to teach stuff like this.
>
> I work at the example sheets but after a while I get so mad, all I want to do is throw the papers all over the kitchen.

None of these even attempts to be a formal learner. Their natural focus is on their experience, rooted in the real-world, where a definition involves saying enough about a particular concept to enable someone else to be able to identify it. Instead of “distilling the essence” of the definition in a minimal, essential manner, they build a growing knowledge base, which is neither well-focused nor well-connected by adding information that might perhaps be of some use to pass the examinations.

The two ends of the spectrum are illustrated by the reactions of two students on first being given the thirteen axioms for the real numbers. When asked why they thought the lecturer had introduced the axioms, Caroline, a mathematician in the making said:

> Well, when you prove things properly you need to say exactly where to start, what it is you are assuming, and that is what the axioms are for.

She already had a grasp of the formal approach and also brought with her a rich personal knowledge base.

But Martin, who later gained a good degree in economics was bemused:

> I dunno really. I've seen most of it before. I knew most of this stuff when I was about five.

He lives in the real world where his pragmatic grasp of economics will probably earn him a higher salary than a research mathematician and he has no conception of the world of formal definitions distinct from his powerful concept imagery.

Summary

In this presentation we have seen how the long-term development of mathematical thinking in individuals leads to a bifurcation of thinking styles. In arithmetic, for example, the more successful build a flexible knowledge structure with symbols both representing processes to get answers and concepts to think about problems. The less successful concentrate more on the procedures themselves, seeking the security of being able to "do" mathematics in a routine sense but with less long-term prospect of building flexible mathematical thinking structures.

As the curriculum content develops, new ideas require not only expansion of knowledge, but reconsideration and reconstruction of old knowledge. Natural learners, building from what they know to make sense of the new, can only succeed in these contexts by a process of mental reconstruction — involving cognitive conflict and confusions that need powerful fortitude to succeed. A few are successful, but many fail, stretching the spectrum of success yet again.

Those who develop techniques to work with new knowledge as a formal game to first be experienced and then understood, can also gain success, but for them the greatest success occurs when later reflection allows them to see their new knowledge fitting coherently within a wider context, again almost certainly requiring cognitive reconstruction.

In the transition to advanced mathematical thinking, the focus of attention changes to considering new theoretical worlds built upon clearly stated axioms. Here the natural learner may attempt to give meaning to the definitions from personal experience, *giving* meaning to the theory, whilst the formal learner attempts to gain meaning from the definition, *extracting* meaning from the theory. Each of these approaches (and combinations of the two) have the potential for success and failure. Natural learners, with powerful and well-connected knowledge structures, who are also willing to reflect on their ideas and struggle to reorganise it, have the potential to build a new logical structure built from both intuition and the axiomatic foundations. But many natural learners (often including, I regret to say, many intending primary mathematics teachers) find the interplay of intuition for concepts they “know” and formalisms they must “prove” to be a bridge too far. Meanwhile, formal learners have the potential to play with the new ideas and make sense of the new theory as a separate conceptual structure. However, this too may fail, as the definitions may appear too complex to be handled in their entirety.

What does this tell us as we plan for the mathematics curriculum of the future? It shows that, even given apparently the same learning contexts, some will succeed and some will fail. Those who succeed at the highest level inevitably have a more flexible mental structure whilst the learning of routine procedures can give limited success to *do* mathematics without necessarily helping to *think* about it. Different learning styles exist, for example, those described here as *natural* and *formal*, and each can lead to success or failure, depending on the individual’s ability to cope with the essence of ideas within an apparent profusion of detail. Our problem in the future is to use these insights to maximise the success of as many different individuals as possible. The evidence here shows that a single method will not work with all students. There is still a role for the sensitive teacher, aware of the needs of the student.

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