A Versatile Theory of Visualisation and Symbolisation in Mathematics

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*This presentation will consider the roles of visualisation and symbolisation in the cognitive growth of mathematics. Visualisation plays a fundamental role throughout, both in the global overview afforded by visual diagrams and the processing of symbolism through the ability to scan written symbols and shift attention to different aspects at will. Reference will be made to the author's development of a visual computer approach to mathematics (the theory of generic organisers) and to the subtle role of symbols standing for both **pro**cess and **concept** (the notion of **procept**). This will be placed in an extension of the theory of Bruner in which each mode of mental representation has its distinct form of objects, actions and proof.*

Modes of mental representation

Almost thirty years ago, when current graphic computer environments were not even a dream, Bruner (1966) distinguished three different modes of mental representation – the *sensori-motor*, the *iconic* and the *symbolic*. In his essay "Patterns of Growth" he wrote:

> What does it mean to translate experience into a model of the world. Let me suggest there are probably three ways in which human beings accomplish this feat. The first is through action. […] There is a second system of representation that depends upon visual or other sensory organization and upon the use of summarizing images. […] We have come to talk about the first form of representation as **enactive**, the second is **iconic**. […] Finally, there is a representation in words or language. Its hallmark is that it is **symbolic** in nature.   Bruner, 1966, pp. 10–11.

Bruner considered that these representations grew in sequence in the individual, first enactive, then iconic and finally a symbolic representation, where the latter may have a power of its own which then depended less on the first two. Furthermore he hypothesised that "any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in recognizable form" (ibid. p. 44).

Although this claim is far-reaching, it has proved possible to *appear* to explain certain sophisticated logical ideas in, say, a simple visual form. However, different representations use different forms of knowledge which has advantages and hidden difficulties.

In figure 1 the bold lines show Bruner's proposed sequence of growth, together with other connections between the systems. For instance, the symbolic system passes written communication to the enactive system for writing and typing, the iconic system passes its drawing
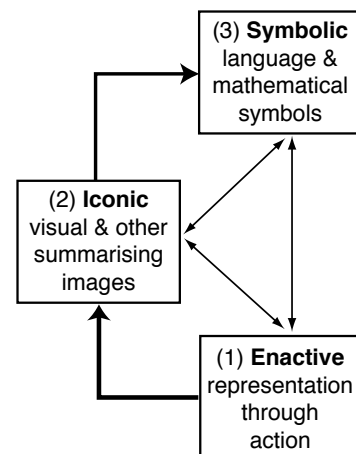

Figure 1: Bruner's three modes of representation

actions to the enactive system and there are many links between iconic and symbolic in mathematics, for example between symbolic functions and graphs.

Modern psychologists and neurophysiologists regard the brain as a system of modules with groups of neurons working together both within groups and between groups to receive sensory perceptions and process them to pass messages to effectors to take appropriate actions (see, for example, Gazzaniga, 1985; Edelman, 1987, 1992; Paivio, 1986). However, I hope to show that it proves useful to focus on Bruner's sub-systems to uncover subtle aspects of mathematical growth.

In the early 80s I had begun working on a *visual* approach to the calculus using the computer corresponding to Bruner's *iconic* mode to underpin the symbolic mathematical ideas (e.g. Tall, 1985a). This might suggest a division between *visual* and *symbolic* modes. However, Michael Thomas and I (Tall & Thomas, 1991) found a significant visual element in algebra too. Bruner's *symbolic* mode, must in essence be a *visuo-symbolic* mode in which the symbols are written or drawn and *seen*. Once written, the page may be scanned in any order and any part may be made a focus of attention to reflect on its meaning and to consider its relationship, at leisure, with other parts. Written symbolism allows far greater flexibility in use than the spoken word.

We proposed the idea of *versatile learning* combining visual global gestalts for overall ideas and sequential manipulation of symbols. At that time we began to be aware of algebraic symbols being used in two very different ways – as generalised arithmetic where an expression such as $2+3x$ stood for a *process* of evaluation (once x is given a value) or as a manipulable mathematical *object*. The former causes great difficulty (sometimes termed "lack of closure") for children unable to accept a symbol that could not be evaluated for unknown x .

Mathematical symbols which represent either process or concept – the notion of procept

In subsequent work with Eddie Gray on children's arithmetic exactly the same phenomena occurred in arithmetic that had occurred in algebra (Gray & Tall, 1991, 1993, 1994). Children use number symbolism both as a notation representing a *process* (counting) and a *concept* (number). Addition also begins as a process, usually with some form of counting so that "3 + 2 *makes* 5", before it is encapsulated as the known fact "3 + 2 *is* 5". This encapsulation of process into object uses the symbolism as a pivot to represent either **pro**cess or con**cept**. We coined the term **procept** to represent this amalgam of process and concept with a symbol operating dually for either. The symbol can be spoken, heard, seen and read and the combination of these sensory perceptions and actions gives it a cognitive existence as a mathematical object. But it is more powerful than that – the process can be used to *do* mathematics and the object can be used to *think* about it.

A few examples of procepts throughout arithmetic, algebra, calculus and analysis are as follows:

$$3+2 \qquad \text{(process of addition, concept of sum)},$$
$$+2 \qquad \text{(process of "add two", concept of "+2" (sometimes noted as } {}^+2)),$$
$$3/4 \qquad \text{(process of division, concept of fraction)},$$
$$3+2x \qquad \text{(process of evaluation, concept of expression)},$$
$$\lim_{x \to 0} \frac{\sin x}{x} \qquad \text{(process of tending to limit, concept of limit)},$$

$$dy/dx \qquad \text{(process of differentiation, concept of derivative)},$$
$$\lim_{n\to\infty}\sum_{k=1}^{n} a_k \qquad \text{(process of } n \text{ tending to infinity, concept of limit)}.$$

The notion of 'procept' enabled us to focus on how some children see symbolism mainly as process and others use it flexibly as either process or concept, giving them a huge advantage.

We found at least three different kinds of procepts which helped our analysis at different levels:

- *operational procepts* such as 3+2 which have a built-in process of computation to produce the result,
- *template procepts* such as 2+3*x* whose built-in process cannot be performed until certain place-holders are given values, but the symbol can itself be manipulated,
- *structural procepts* such as $\sum_{n=1}^{\infty} 1/n^3$ which have an associated process (tending to a limit) but do not have a built-in method of calculating the value of limit itself.

The notion of process being encapsulated as object has a wide coverage in the literature (e.g. Greeno, 1983;. Dubinsky, 1991; Sfard, 1991; Harel & Kaput, 1991; etc.). As individuals mature they often see symbolism first as a process to be performed and only later (if at all) encapsulate symbols as manipulable objects – for instance, in *arithmetic*, (Gray & Tall, 1991, 1993, 1994), in *algebra*, (Tall & Thomas, 1991; Crowley et al, 1994); in *limits* (Schwarzenberger & Tall, 1977, and other studies summarised by Cornu, 1991), programming limits in a structured computer language, (Tall & Li, 1993), investigating limits with a symbol manipulator, (Monaghan, Sun & Tall, 1994).

Refining Bruner's symbolic category

Not all symbols in mathematics are procepts. For instance, classical Euclidean geometry uses symbols such as *ABC* for a triangle acting as names for geometrical objects without being manipulable in the same way as numbers or algebraic expressions. These symbols are essentially mathematical extensions of everyday *verbal* language. Moving on to advanced mathematical thinking, there are yet new methods of construction to conquer – the logical system of *defining* mathematical objects and making formal deductions from the definitions.

I therefore suggest a refinement of Bruner's symbolic mode to take account of (at least) three different types of symbol system which Bruner (1966, pp. 18, 19) had alluded to in passing as "language in its natural form" and the two "artificial languages of number and logic." These sub-systems I call *verbal*, *proceptual*, and *logical*. Each treats mathematical objects in a different way, has different ways of acting on mathematical objects and has its own form of proof (Figure 2).

The enactive *sensori-motor* system is at the root of our activity, beginning with perceiving objects in the external world and acting upon them. The notions of *object* and *action upon object* provide two very different motive forces in mathematics; emphasis on (geometric) objects leads to Euclidean geometry and emphasis on actions (encapsulating process as object) to arithmetic and algebra. The enactive system has a simple method of proof – verification by physical experiment.

The iconic *visuo-spatial* system uses iconic representations to enable us to sense spatial relationships and consider "thought experiments" where we imagine certain conditions holding in a
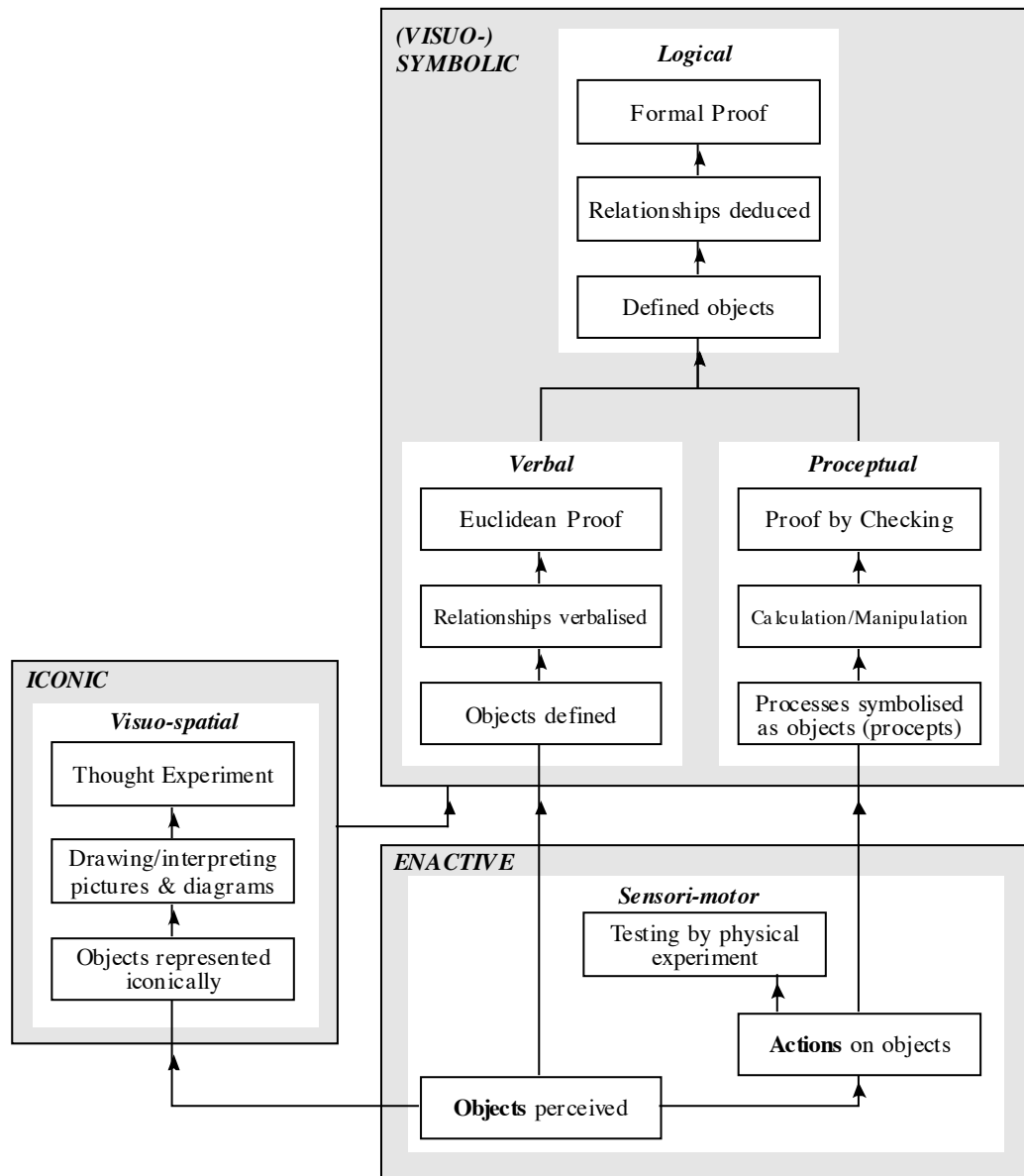
Figure 2 : Objects, actions and proof in different representational modes

static or dynamic gestalt to "see" if a conclusion holds. Such proofs are holistic, without necessarily having any logical deduction.

In the *verbal* system everyday language is used to define idealised versions of geometric objects (such as a point having "position but no size") and to describe constructions and relationships. The mental objects rather than the definitions are dominant in the thinking processes and for this reason I call them "objects defined" to emphasise that the object comes before the definition.

The *proceptual* system concerns the symbols of arithmetic, algebra, calculus and so on, which are used to represent the result of a desired calculation or manipulation such as:

$$3+5, \, 2(x+3), f(x), \, g(f(x)), \lim_{n \to \infty} \sum_{k=1}^{\infty} 1/k^3 \, , \, \frac{d}{dx}(\sin x) \, , \, \int_1^t \frac{1}{x} \, dx \, .$$

Because these symbols (with the exception of certain structural procepts) have built-in algorithms to carry out the processes, it is this area that is most easily converted to operate on the computer.

The *logical* mode uses formal definitions of concepts as the basis of systematised proof. To emphasise the fact that the definition is now the fundamental notion (and *any* object satisfying the definition will have the deduced properties) I use the term "*defined object*" to distinguish it from an elementary "object defined".

Focus of attention and the need for mental compression of information

All the various mental subsystems work with the strengths and weaknesses of the human brain, in particular the combination of vast storage capacity with a narrow focus of attention. Data needs to be *compressed* into a form which

Figure 3 : spatial chunking to 'see' a number

can be used effectively in the focus of attention and a mechanism is required to shift appropriate data in and out from the memory store. A primitive, but useful, compression method is to *chunk* several pieces of information together to think of them as a unit. For instance, the array on the right in figure 3 is more quickly seen to contain 20 items because they are grouped (chunked) into four sets of five.
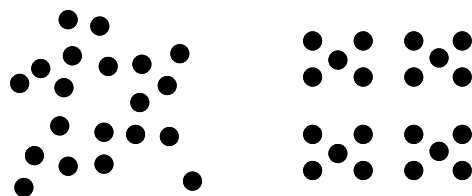
A far more powerful method of compression is *language* where a single word can stand for a whole complex of interlinked ideas. But procepts are even more powerful – they flexibly cue either processes to *do* mathematics or mental objects to *think* about mathematics.

The role of the computer

The computer can carry out procedures almost instantaneously, producing visual output on the screen. It is therefore valuable for giving *visuo-spatial* and *proceptual* support. (Figure 4). Geometric software such as *Cabri Géomètre* or *Geometer's Sketchpad* allows the user to specify properties of geometric figures and then pull them around enactively with the mouse to perform "what-if" *thought experiments*. However, the individual must still reflect on these experiences to build the relations necessary for *Euclidean proof*.

Proceptual software accepts symbolic input from the keyboard (or mouse-driven actions), performing computations internally and providing
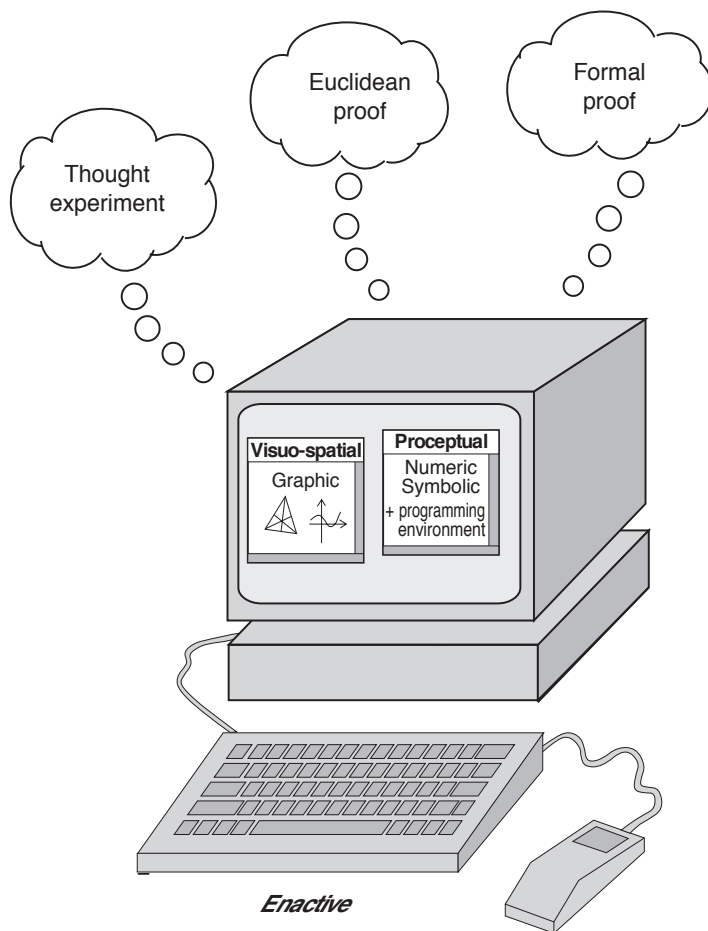

Figure 4 : Using the computer to provide visuo-spatial and proceptual support for various levels of mathematical thinking

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the user with the results of those computations to consider.

Calculators and symbol manipulators currently work with symbols in "closed form", (given by arithmetic, trigon-ometric and exponential functions and their inverses). They thus tend to focus on conceptual models *that work*, rather than *non-examples* of concepts *that fail* and consequently give limited experience appropriate for formal proof.

Programming in appropriate procedural languages can be proceptual in nature. Early versions of BASIC were essentially procedural, allowing the programmer to specify the algorithms required. More sophisticated functional and object-oriented languages raise the level to where the procedures may be conceived as objects to be input into other procedures. The language ISETL (Interactive SET Language) is particularly valuable in that it mirrors the language of set theory operating on finite sets, allowing the user to specify a wide variety of constructs appropriate for logic, calculus, linear algebra, group theory, etc. By reflection on such computer activities the student may develop cognitive support for higher level formal proof (although the mental objects may be related more to programming experiences than formal "defined objects").

The foregoing discussion emphasises the difference between deduction and proof available in the different modes of thinking and the essential role played by individual learners in constructing appropriate meaning from software by reflecting on their interaction with the computer constructs.

The computer also allows new curriculum sequences to be developed, using appropriate representations for the stage of development of the learner. Before the computer was invented, calculations in arithmetic, algebra and calculus needed to be mastered, practised and routinised and only when the symbolism had a stable meaning could it be studied in any sophisticated way. The computer makes other routes possible. For instance, using software to carry out a process internally so that the learner may study the resulting objects *before* mastering the mechanics of the process.

Visuo-spatial overview, selective focus and generic examples

Visualisation allows a large amount of data to be conveyed holistically enabling the individual to change focus from one part of the picture to another so that relationships may be observed. I illustrate this by briefly alluding to some of my earlier work.

In *Graphic Calculus* (Tall 1985b, Tall, Blokland & Kok, 1990) I used visualisations to encourage students to give graphical meaning to concepts such as *rate of change*. The latter used the notion of zooming in on a graph.
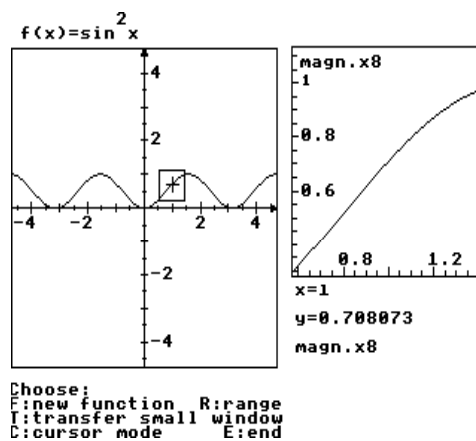
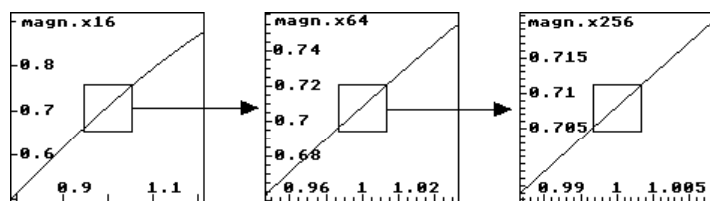figure 5 : Magnifying the graph of $y=\sin^2 x$ by a factor of 8

Figure 6 : Further magnifications by an extra factor of four each time

Although the graph may look curved at a normal scale, at a higher magnification it may look less curved (figure 5) and as the magnification grows, the curvature may lessen until the graph may appear to look straight (figure 6). Successive magnifications then do not change in gradient as homing in yields essentially the same picture as the magnification increases.

A graph with this property is said to be *locally straight*. The gradient of the graph at a point can now be *seen* as the gradient of the magnified (almost) straight image. With this mental image a student can learn to look along the graph to see the changing gradient. This gives a new function, the *gradient function* which can be calculated numerically for a given function and its graph drawn, enabling the student to explore the properties of the gradient function of a graph and to construct intuitive meaning for what will be computed symbolically as the derivative.

Figure 7: Magnifying a *Van Der Waerden* fractal function (with wrinkles)
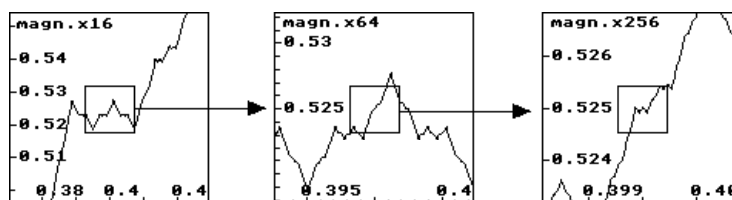
Figure 8: Successive magnifications show wrinkles at each level

The notion of "locally straight" not only embodies an intuitive notion of differentiability, it also intimates that a *non*-differentiable function has a graph which will *not* magnify to look straight. (Figures 7, 8). In this way the learner may develop intuitive notions of *nowhere* differentiable functions and other subtle ideas such as different left and right derivatives (where a graph magnifies to having a "corner" with

Figure 9 : Enactively building a solution of a differential equation

different left and right gradients) *before* the mechanism of symbolic differentiation is introduced.

Such insight enables the student to "see" a first order differential equation, say *dy/dx=1/y*, as a curve whose gradient at (*x,y*) is given by the equation, in this case 1/*y*. It is easy to program software to move a small line segment around the screen (figure 9) either by mouse or cursor movement and to leave a trace of appropriately positioned line segments to build up a solution curve. Here enactive mouse movement leads to a visual representation before any symbolic solution processes need be studied, in true Brunerian sequence (Tall, 1991a).
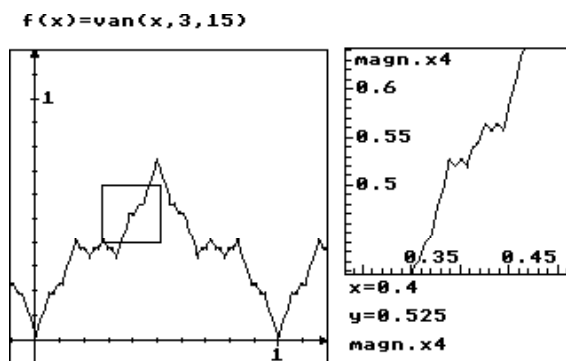
A software environment enabling the user to explore examples (and preferably non-examples) of a concept in preparation for a general concept embodied by the examples is called a *generic organiser* (Tall, 1989). Such environments do not picture the *general* concept; a specific quartic or quintic may represent a typical polynomial, but it is not possible *in a single instance*, to draw a *general* polynomial which may be symbolised as $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$.

For such reasons visualisation alone is eventually inadequate in mathematics (Dreyfus, 1991). Verbal or symbolic means become essential to represent general concepts and to formulate general arguments. At the research level, mathematicians often use visual imagery to inspire mathematical thinking that eventually leads to formal proof. But, as Hadamard (1945, p. 88) says succinctly, it is vital to be "guided by images without being enslaved by them".

Symbols, computation and generality

Processing symbols is quite different from processing mental imagery. For instance, the problem:

> Amy's older brother Ben is now three times as old as her. In four years he will only be twice as old. How old is Amy now?

can be modelled algebraically by letting Amy be x years old so that Ben is $3x$ years old. In four years Amy will be $x+4$ and Ben $3x+4$. As Ben is now twice as old as Amy, we have

$3x+4 = 2(x+4)$.

This equation drastically compresses the information into a compact form which can be solved algebraically by a sequence of manipulations:

	$3x+4 = 2(x+4)$
expand brackets:	$3x+4 = 2x+2{\times}4$
simplify:	$3x+4 = 2x+8$
subtract 4 from both sides:	$3x+4-4 = 2x+8-4$
simplify:	$3x = 2x+4$
subtract x from both sides:	$3x-x = 2x+4-2x$
simplify:	$x = 4$.

Experience leads to compression of several steps into one, such as:

	$3x+4 = 2(x+4)$
expand brackets:	$3x+4 = 2x+8$
subtract 4 from both sides:	$3x = 2x+4$
subtract x from both sides:	$x = 4$.

The solution process is a visually moderated sequence (VMS) of symbol manipulations (Davis, 1984), where each manipulation is performed and written down to be used as a cue for the next one. This process has a strategy in which both sides are simplified by multiplying out brackets and collecting together like terms before getting all the numbers on one side and all the xs on the other, then divide by the coefficient of x (if it is not 1). A VMS uses mental resources to good advantage, focusing successively on the salient features to make the best use of our short-term attention.

This solution of this equation can be generalised to include solutions of all equations of the same type, say $ax+b=cx+d$, has solution

$x=(d-b)/(a-c)$, (provided that $a{\neq}c$).

By programming algorithms to solve this and a range of other equations, a **solve** command can be implemented so that **solve**($ax+b=cx+d,x$) gives the solution for x The **solve** command compresses the solution sequence into a *procept* whose built-in process is computed by the software.
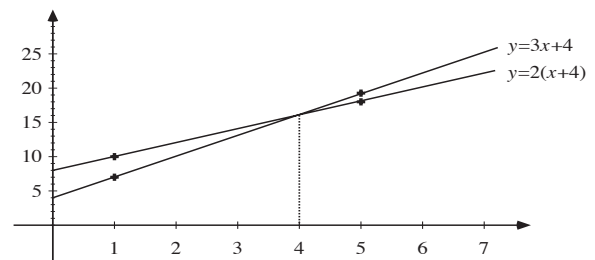

Figure 10 : Solving an equation graphically

Visual and symbolic computer solutions of problems

Visual and symbolic software allow the learner to concentrate on different aspects of the solution process while the software does other parts of the process internally. For instance, to solve the equation of the previous section by practical drawing requires two points on each of the lines $y=3x+4$ and $y=2(x+4)$ to be calculated to draw the lines through them (figure 10).

A graph plotter eliminates the computation and the act of drawing and focuses on finding the value of x for which the two sides are equal. However, this admirable short-cut can have unforeseen circumstances; skills not used may be lost. For instance, a class of 17 students were asked the following question before and after a course using *Derive* on hand-held computers:

"What can you say about u if $u=v+3$, and $v=1$?"

The students had used the software to draw graphs but had no practice at substituting values into expressions. No student improved from pre-test to post-test and six who were successful on the post-test failed to respond satisfactorily on the post-test (Hunter, Monaghan & Roper, 1993).

The use of symbol manipulators also causes students to go through very different processes for computing a result. To calculate the derivative no longer requires working from first principles or using the formulae for the derivative of a product, composite etc. Instead the process becomes a sequence of keystrokes, such as that in *Derive*:

- select **Author** and type in the expression,
- select **Calculus**, then **Derivative**,
- specify the variable (e.g. **x**),
- **Simplify** the result.

This fits the human system of routinising a sequence of operations to perform them with little conscious effort, but it may not result in conceptual understanding. A group of students with personal symbol manipulators were able to compute derivatives, but when asked to explain, most referred to specific formulae and how to carry out the sequence of key-presses to get the answer (Monaghan, Sun & Tall, 1994).

The transition to formal mathematics

The differences in the nature of mathematical objects and of proof cause serious difficulties for many students starting university mathematics. Their previous experience is usually of "objects

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defined" and they have built rich concept images which suggest (by thought experiments) that many theorems of mathematical analysis are trivial. (One only has to imagine a continuous function negative somewhere and positive somewhere else to *see* that it must be zero in between!) Regrettably these concept images often conflict seriously with formal definitions, including function (e.g. Bakar & Tall, 1992; Dubinsky & Harel, 1993 etc.) and *continuous* function (Tall & Vinner, 1981; Tall, 1992). A limit may be seen as an unencapsulated limiting *process* of getting closer rather than a limit *object*, or as a variable *infinitesimal* quantity (Cornu, 1991).
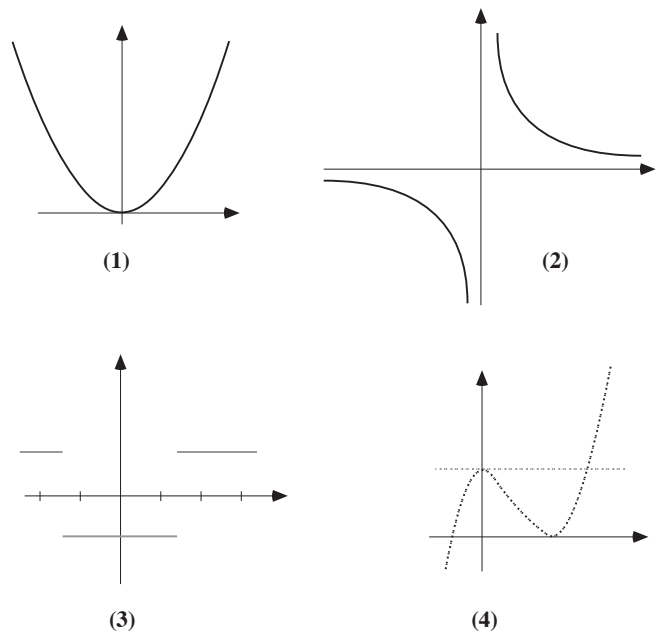

Figure 11: Which of these are graphs of continuous functions?

Given the human propensity to interlink a wide range of mental images, it is natural to link visual, verbal, symbolic and other representations of a concept to give a wide array of possibly conflicting concept images. For instance, which graphs in figure 11 represent continuous functions?

For many learners (1) is the graph of a continuous function, but often because it seems to have a formula $y=x^2$ which "varies continuously" (Tall & Vinner, 1981). Graph (2) "jumps" discontinuously at the origin and seems *discontinuous* (even though $y=1/x$ $(x{\neq}0)$ is continuous in the formal sense everywhere in its domain). Graph (3) also seems discontinuous because it jumps. But I cheated because I didn't say it is only defined for *rationals*:

$$y = \begin{cases} +1 \text{ if } x \text{ is rational and } x^2 > 2 \\ -1 \text{ if } x \text{ is rational and } x^2 < 2 \end{cases}$$

and this function is *continuous* (everywhere in its domain). Graph (4) looks crazy, but the formula

$$y = \begin{cases} 1 & \text{if } x \text{ is rational} \\ (2x+1)(x-1)^2 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous everywhere except $x=1\frac{1}{2}$ where it is continuous and $x=0$ where it is differentiable!

Formal continuity (even these extreme kinds) can be given a visuo-spatial basis by stretching the graph horizontally (figure 12). If a graph eventually pulls out into a horizontal line, one may imagine this line to have $f(x_0)$ in the middle and to be a strip of thickness $\pm e$ (for error). The horizontal interval $x_0{-}d$, $x_0{+}d$ can then be selected suitably small so that, when x is in it, the point $(x, f(x))$ lies in the strip with $f(x)$ between $f(x_0){\pm}e$. This gives the formal definition of continuity:

Given $e > 0$, $d > 0$ can be found so that $x_0{-}d < x < x_0{+}d$ implies $f(x_0){-}d < f(x_0) < f(x_0){+}d$.

The visual meaning easily yields a thought experiment to prove a differentiable function is continuous (zoom in with equal scales to see a locally straight graph, then stretch the picture horizontally). Such visualisations also support concepts in integration (Tall, 1993), including the
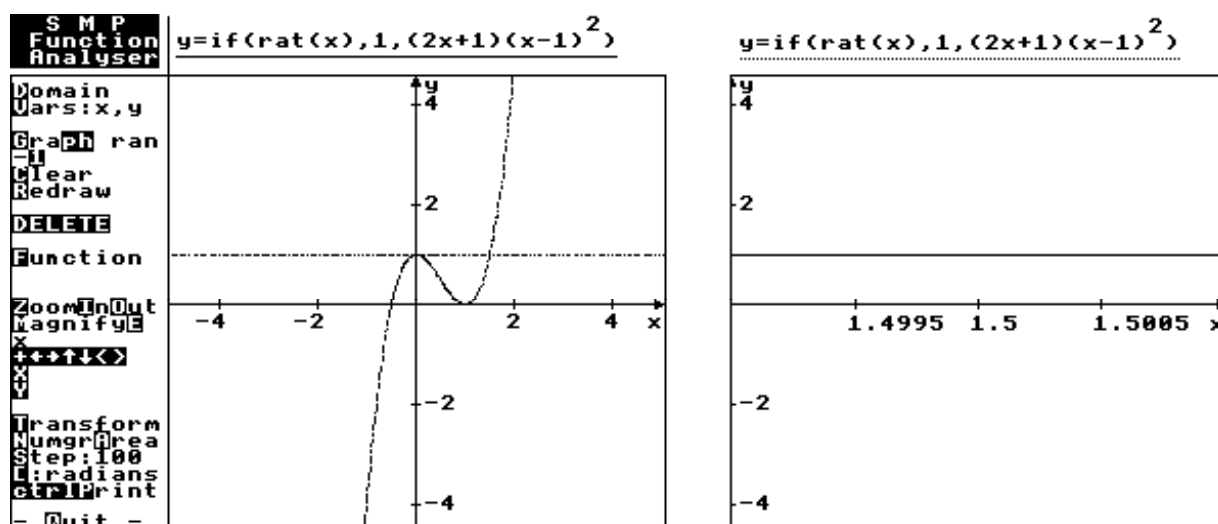
Figure 13 : The graph of y=if(rat(x),1,(2x+1)(x–1)^2, drawn to equal scales, then stretched horizontally to pull flat at x=1.5

proof of the fundamental theorem for a continuous function. When I drew a computer picture for the area under a continuous, non-differentiable function in class, a student remarked without prompting that "the area function must be differentiable everywhere once but nowhere twice"!

A visuo-spatial approach like this can give students powerful intuitions but, once again, it is not by itself sufficient. A definition derived from a picture is more likely to be conceived as an *object defined* (with powerful concept image) rather than a *defined object* (with formal definition to be used in formal proof). It requires considerable further effort (which often ends in failure) for the student to construct the meaning and implications of the definition with its subtle use of quantifiers.

A versatile combination of mathematical activity

We have outlined the manner in which mathematical ideas grow in various modes of representation – enactive, iconic, verbal, proceptual and logical and the ways in which each supports different aspects of mathematical thinking. A *versatile approach* uses whichever method is most appropriate to support the required thinking at a given time. But as we have seen, this is easier said than done.

Visual approaches can give valuable cognitive underpinning to support thought experiments and may be valuable to those not pursuing formal mathematics for it gives insight into more sophisticated ideas and problems which may occur. But those who need to study formal mathematics will still need to make the difficult transition from thought experiments to formal proof.

Proceptual approaches calculating with numbers and manipulating symbols are powerful for modelling real world activities, for *doing* mathematics and for providing a vehicle for *thinking* about mathematics. Symbol manipulators may help by avoiding long sequences of algebraic manipulation, especially at a time when less effort in practising algorithms means that students have less algebraic fluency. But they are not a universal panacea, indeed, they may become new procedural devices in which meaningless sequences of key-presses replace rote-learned sequences of algebraic operations. They still need to be placed in a context which gives a rounded meaning to

both the procedural and conceptual, and considerable cognitive re-construction remains necessary if the individual is to proceed to more sophisticated formalism.

Mathematical thinking is not a coherent structure in which successive ideas build continuously on previous ones. Working in one context invariably has specific connotations which require reconstruction in more sophisticated contexts. By attempting to understand how different cognitive sub-systems of the brain work and interact, we may yet make some small progress towards developing a versatile approach to mathematics using human resources to their best advantage.

References

Bruner ,J. S.: 1966. *Towards a Theory of Instruction*, New York: Norton.

Cornu, B.: 1991. Limits. In D. O. Tall (Ed.), 1991b, pp. 153–166.

Crowley, L., Tall, D. O. & Thomas, M. O. J.: 1994. Algebra, Symbols, and Translation of Meaning, *Proceedings of PME 18*, Lisbon, Portugal, II 240–247.

Davis, R. B.: 1984. *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*, Norwood NJ: Ablex.

Dreyfus, T.: 1991. On the status of visual reasoning in mathematics and mathematics education, *Proceedings of PME 15*, Assisi, I, 32–48.

Dubinsky, E.:1991. Reflective Abstraction. In D. O, Tall (Ed.), 1991b, pp. 95–123.

Dubinsky, E. & Harel, G. , (Eds.): 1992. *The Concept of Function: Aspects of Epistemology and Pedagogy, MAA* Notes, Vol. 25.

Edelman, G.: 1987. *Neural Darwinism*, New York: Basic Books.

Edelman, G.: 1992. *Bright Air, Brilliant Fire*, New York: Basic Books, reprinted Penguin, 1994.

Gazzaniga, M. S.: 1985. *The Social Brain*, New York: Basic Books.

Gray, E. M. & Tall, D. O.: 1991. Duality, ambiguity & flexibility in successful mathematical thinking, *Proceedings of PME XV*, Assisi, Italy, II 72–79

Gray, E. M. & Tall, D. O.: 1993. Success and Failure in Mathematics: The Flexible Meaning of Symbols as Process and Concept, *Mathematics Teaching*, **142**, 6–10.

Gray, E. M. & Tall, D. O.: 1994. Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal of Research in Mathematics Education,* **26,** 2, 115–141.

Greeno, G. J.: 1983. Conceptual entities. In Genter & Stevens (Eds.). *Mental Models*, 227–252.

Hadamard, J.: 1945. *The Psychology of Invention in the Mathematical Field*, reprinted Dover.

Harel, G. & Kaput, J. J.: 1991. Conceptual Entities. In D. O. Tall (ed.), 1991b, pp. 82–94.

Hunter, M., Monaghan, J. & Roper, T.: 1993. The Effect of Computer Algebra Use on Students' Algebraic Thinking, *Working Papers for ESRC Algebra Seminar*, Inst. of Ed., London.

Li, L. & Tall, D. O.: 1993. Constructing Different Concept Images of Sequences and Limits by Programming. *Proceedings of PME 17*, Tsukuba, Japan, II, 41–48.

Monaghan, J. D., Sun, S. & Tall, D. O.: 1994. Construction of the Limit Concept with a Computer Algebra System, *Proceedings of PME 18*, Lisbon, Portugal, III. 279–286.

Paivio, A.: 1986. *Mental representations: A dual-coding approach*. Oxford: OUP.

Schwarzenberger, R. L. E. & Tall, D. O.: 1978. Conflicts in the learning of real numbers and limits, *Mathematics Teaching*, **82**, 44-49.

Sfard, A.: 1991. On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, **22** 1, 1–36.

Tall, D. O.: 1985a. Understanding the calculus, *Mathematics Teaching*, 110, 49–53.

Tall, D. O.: 1985b. *Graphic Calculus*, [Software for BBC compatible computers], ABCo: London.

Tall, D. O.: 1989. Concept Images, Generic Organizers, Computers & Curriculum Change, *For the Learning of Mathematics*, **9** 3, 37–42.

Tall, D. O.: 1991a. *Real Functions and Graphs*, [software for the Archimedes Computer]. Cambridge University Press.

Tall, D. O. (Ed.): 1991b. *Advanced Mathematical Thinking*. Dordrecht: Kluwer.

Tall, D. O.: 1992. The Transition to Advanced Mathematical Thinking In Grouws D. A. (ed.) *Handbook of Research on Mathematics Teaching & Learning*. NY: Macmillan, 495–511.

Tall, D. O.: 1993. Real mathematics, rational computers and complex people, *Proc. Fifth Int. Conference on Technology in College Mathematics Teaching*, Addison-Wesley, 243–258.

Tall, D. O., Blokland, P. & Kok, D. 1990: *A Graphic Approach to Calculus*, [for IBM compatibles] Pleasantville, NY: Sunburst. (also French, Spanish, Dutch, German, Italian from elsewhere.)

Tall, D. O. & Thomas, M. O. J.: 1991. Encouraging Versatile Thinking in Algebra using the Computer, *Educational Studies in Mathematics*, **22** 2, 125–147.

Tall, D. O. & Vinner, S.: 1981. Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, **12**, 151–169.