Success and Failure in Mathematics: Procept and Procedure

2. Secondary Mathematics

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Introduction

In our last article (Gray & Tall 1992), in response to the question

> "Why is it that so many fail in a subject that a small minority regard as being trivially simple?",

we showed that

> *the more able are doing qualitatively different mathematics from the less able.*

A major source of the generative power of mathematics is in the use of symbols which are used ambiguously to evoke both a process of calculation and the product of that calculation. The amalgam of process and concept, represented by the same symbolism we called a *procept*. The more able treat such symbolism flexibly, as process or concept, whichever is more appropriate in a given context, the less able tend to conceive of mathematics more as separate procedures to carry out computations. The manipulation of concepts is cognitively easier than the coordination of procedures, and so the more able are doing an easier form of mathematics than the less able. In a nutshell, the more able are using flexible procepts, the less able, procedures. This difference in mathematical thinking we termed the *proceptual divide*. We believe that the proceptual divide is a major cause of the difference between success and failure.

In this article we move on into concepts that occur in secondary mathematics and through into advanced mathematics at the sixth-form and university. We find that procepts abound in secondary mathematics, and this leads to further divisions between success and failure at subsequent stages. But the proceptual divide is now occurring higher up the ability range, so that now more children are failing. Furthermore the procepts that occur in higher levels of mathematics begin to take on new features. In the earlier stages of mathematics, the associated processes are given by explicit procedures, such as counting, use of multiplication tables, algorithms for multidigit arithmetic, and so on. But procepts in the sixth form will be found to include processes such as

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"tending to a limit" or symbolic integration where there may not be a simple procedure to carry out the process. This leads to further confusion. Those who need the security of a procedure to compute an answer become unsettled by situations where concepts cannot be computed directly by a single procedure. Thus the proceptual divide widens into a chasm between the majority who have failed at some time in their development and the few who continue to find mathematics (proceptually) easy.

Procepts in secondary mathematics

Recall that we defined a *procept* to be a combined mental object consisting of both process and concept in which the same symbolization is used to denote both the process and the object which is produced by the process (Tall & Gray, 1992). Procepts in secondary mathematics involve processes that act on symbols which are themselves procepts. They include:

- The notion of a fraction, say $\frac{22}{7}$ which represents both the process of dividing 22 by 7 and the result of that process,
- The algebraic symbol $3x+2$ stands both for the process "add three times x and two" and for the product of that process, the expression "$3x+2$",
- The trigonometric ratio sine = $\frac{\text{opposite}}{\text{hypotenuse}}$ represents both the process for calculating the sine of an angle and its value,
- The function notation $\mathrm{f}(x)=x^2-3$ simultaneously tells both how to calculate the value of the function for a *particular* value of x and encapsulates the complete concept of the function for a *general* value of x,
- The derivative $\mathrm{f}'(x)$ of a function such as $\mathrm{f}(x)=\sin x$, evokes both the process of finding the derivative and the value of the derivative,
- The integral $\int \mathrm{f}(x)\,\mathrm{d}x$ evokes both the process of calculating the integral and the symbolic function produced by this process,

- The notation $\lim_{x\to a} f(x)$ represents both the process of *tending to a limit* and the concept of the *value of the limit*,
- So does $\lim_{n\to\infty} s_n$,
- and $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$,
- and $\lim_{\delta x\to 0} \sum_{a}^{b} f(x)\ \delta x$.

These procepts begin to take on more complex forms, for instance, the derivative is given by an algorithm (if you know the derivative, write it down, otherwise, use the formulae for derivatives of a sum, product, composite etc to build up the derivative). The integral has several associated algorithms (integration by substitution, integration of rational functions, integration by parts, etc) but these only give the solution for a limited number of special cases and do not guarantee an integration formula in every case. (What is $\int \frac{1}{(\sin x + \ln x)\ dx}$, for instance?)

Worse is yet to come. We may know that $\Sigma 1/n^3$ tends to a limit l, and be able to write

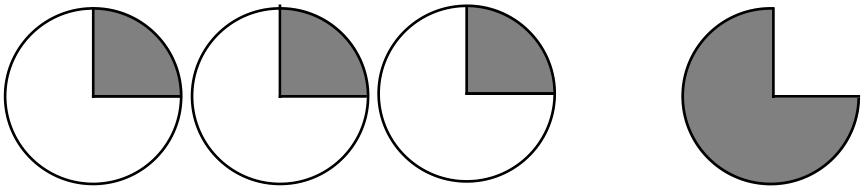
$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = l,$$

using a notation that represents both the *process* of tending to a limit and the *value* of that limit. Yet there is no simple procedure to calculate the value of l.

The examples which follow look briefly at a number of different procepts and consider the problems faced by pupils who see them procedurally rather than proceptually. We shall see the proceptual divide between those who succeed and those who fail, occupying a moving position in the ability spectrum, so that, as the concepts become more advanced, it is not only the less able who find them difficult, but the less able amongst the more able!

Example 1: Fractions

A fraction is a classical example of a procept, for the notation $\frac{3}{4}$ means both the process – divide 3 into four equal pieces – and the product of such a subdivision. The *meaning* of a fraction poses considerable problems. There are different procedures which produce the same product: "divide three cakes between four people (and take one share)", "divide one cake into four and take three pieces".



There are also *equivalent* processes that produce the same product: $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, …

At the very simplest level, children see a difference between the procedure and product, sometimes by using a different notation. For instance, only 44% of a sample of 12 and 13 year old children following a standard secondary mathematics course thought that 6÷7 meant the same thing as $^6/_7$, because "$^6/_7$ is a fraction, 6÷7 is a sum" (Thomas 1988).

For anyone who sees a fraction as a flexible procept, the arithmetic of fractions is relatively trivial. For those who see it only as a routine symbolic process to be learnt, it is likely to be meaningless. The proceptual structure of the arithmetic of fractions is subtly different from that of whole numbers. For whole numbers, multiplication is repeated addition. For fractions this relationship is more deeply hidden. Indeed, the algorithm for multiplication is actually *simpler* to carry out than that for addition in a purely procedural manner. But if the child has failed to master the flexible procepts of sum and product of whole numbers, then the arithmetic of fractions is almost certainly impossible to understand. The large number of children who fail to understand fractions is consistent with a proceptual divide occurring at this stage.

Example 2: Signed number

Signed number is a good example of procept. The signed number "+3" is both the process of "add on three" or "move three units right on the number line" and also the position +3 on the number line. The signed number "–2" is both the process of "subtract 2" or "move two units left on the number line" and the position –2 on the number line.

"Modern" mathematics attempted to distinguish between the concept of the signed number +2 and the procedure "add two". Signed numbers were introduced by some schemes as transformations on the number line (as a process of shifting the whole line to the right or left). New notations such as black numbers (positive) and red numbers (negative) were introduced so that "take away a red number" is conceptually different from "add a black number". As *procedures* these are different, but as *products*, they are the same. The more successful child realises that that –(–2) and +(+2) are the same, and it is this flexible use of symbolism which gives them great power. The carefully worded distinction between them, which may well be an important cognitive stage to pass through in the learning process, involves considerable difficulties which may freeze out the less able child, leading once more to a proceptual divide.

Example 3: Algebraic notation

Traditionally algebra is often regarded as "generalized arithmetic". Thus $2x+2y$ means "when I know x and y, I can work out the sum of twice x plus twice y". To many children the symbolism $2x+2y$ means an instruction to carry out a procedure, just as 2+3 is an instruction to add 2 and 3. But in the case of arithmetic they can produce an answer: 5. In algebra the instruction $2x+2y$ is asking them to carry out an operation that they cannot do. For children who regard $2x+2y$ only as a procedure, it is a process that they are unable to carry out. It is the first time the children meet a procept whose implicit process cannot readily be carried out as a simple procedure to give an answer. It causes an enormous conceptual obstacle, which Tall & Thomas (1991) call the *expected answer* obstacle. These and other difficulties are catalogued in greater detail in Küchemann (1981).

There are added problems with the algebraic notation as a process, for instance, the learner may be confused with the order in which the process is to be carried out. Instead of being read left to right, as with normal speech, $2+3x$ must be organised with multiplication taking precedence over addition as "first multiply 3 by x then add the result to 2". Seeing it simplistically as a procedure reading from left to right may lead to "2+3 is 5", giving the erroneous answer "$5x$".

These difficulties make algebra enormously problematic for less able pupils who simply do not understand the proceptual nature of the notation as both process and concept. By this stage in the curriculum those who fail are a far greater proportion of the total population: they not only include the leass able who were the focus of the first article but also the less able of the more able. Such pupils see a procedure they cannot carry out and are unable to encapsulate the meaning into a concept. Tall & Thomas used various techniques to give meaning to algebraic notation, involving the pupils in action games (to carry out the procedure), programming (to specify the procedure which was then carried out by the computer) and appropriate software (which accepted standard algebraic notation). Because the computer carries out the procedure, the pupils are more likely to see that $(2+3)x$ always gives the same result as $2x+3x$, but is usually different from $2+3x$. So the computer activity focuses their attention on the notation as *product* rather than as process. By this method, more children were helped to see algebra as *procept*, rather than as a collection of procedures to carry out tasks ("multiply out brackets", "collect together like terms", "change sides, change signs", etc).

Again there is a proceptual divide between those who successfully see the algebraic notation as flexible procept and those who see it only as process and fall into routines using instrumental procedures to carry out computations. Those using the computer approach were more likely to fall into the former category.

Example 4: Equations

The idea of an equation, such as

$3x–1=5$,

is one which will cause great difficulties to children who see an algebraic expression as a process. It is much more likely to be meaningful to those who see it as procept. Those who see the solution of an equation purely as a collection of procedures which enable them to carry out the process – "add the same thing to both sides", "change sides, change sign", etc – are likely to be less versatile in solving equations. Those who see each side as a procept, to be flexibly decomposed and reorganized are more likely to solve the equation in a meaningful and versatile way. The former might add one to both sides to get $3x=6$ and divide both sides by 3 to get $x=2$. The latter might see that "one less than $3x$ is 5, so $3x$ is 6 and so x is 2". A versatile thinker seeing the equation as a procept would be able to recognize

$3s–1=5$

as the same equation, with solution $s=2$. The student solving by procedure might even recognize it as essentially the same, but might then need to go through the formal procedure to confirm the result. Faced with the equation

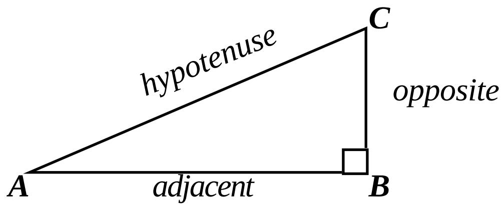
$3(s+1)–1=5$,

the proceptual divide opens up clearly. The procedural student multiplies out the bracket, collects together like terms to get $3s+2=5$, takes 2 from each side, and divides by three to get $s=1$. The proceptual thinker, able to chunk together the expression s+1 as a single entity, sees once more the same equation, with s+1=2, so s=1.

In Tall & Thomas (1991), empirical evidence is given to show that students using a computer approach to give meaning to algebraic notation as product of a process are more likely to respond in the more versatile way which we now hypothesize is due to thinking with procepts rather than just procedures.

Example 5: Trigonometry

Trigonometric formulae are all procepts. For instance, saying that



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

involves both process (to divide the length of the opposite side BC by the length of the hypotenuse AC) and product, the number which is the ratio of these two lengths. But it involves much more than this. It involves the flexible ability to see that $BC=AC\sin A$, and that $AC=BC/\sin A$, and to see these relationships in other triangles with the same angles but having different sizes and different orientations. It requires the flexibility to perceive that as the angle increases from 0˚ to 90˚ then the sine increases from 0 to 1 and to give meaning to the singular cases when the angle is zero or 90˚ and the triangle ceases to exist as a proper triangle. Later it requires the flexibility to extend to the case when the angle increases beyond 90˚ or becomes negative.

Thus it is that children may learn the rule for the sine ratio, they may even be able to carry out the *process* of calculating the ratio, but unless they see the *procept* of sine as a plastic amalgam of related processes and concepts, they will fail to understand trigonometry in a meaningful way.

Once again, Blackett (1990) showed that pupils using a computer program designed to enable them to link numerical data dynamically with accurate geometric representations, are more likely to respond in a versatile manner characteristic of procept rather than just procedure.

Example 6: The Function Concept

The function concept is another archetypal procept. As a *process* it is an input–output machine, transforming an element x in the domain to an element f(x) in the range. The composite of two functions is found by coordinating the processes, one after the other, first transforming x to f(x), then f(x) to g(f(x)). The difficult process of thinking of a function *as a concept* is enshrined in the use of the symbol f to denote the function and the composite gf of two functions f and g. Prior to the advent of modern mathematics and set theory, the use of the symbol x to (ambiguously) represent either a specific or a “variable” element allowed the notation $f(x)$ to represent the function both as process and as concept. If x is a variable, then it embodies the whole process; for instance if $f(x)=x^2$, then $f(x)$ embodies the act of taking *any* number x and transforming it to x^2. On the other hand, if x is a specific number, say $x=2$, then $f(x)$ is the corresponding *value* of the function, $f(2)=4$.

The function procept becomes more plastic as the learner gains experience. It exists in several representations: procedural, numerical, symbolic, graphical, tabular, ... The powerful way to use the function procept is to be able to use whichever representation is appropriate, moving from one to the other without any sense of transition, because there is no transition – they are all the same procept.

Examples from more advanced mathematics

In advanced mathematics, procepts proliferate. The limit concept in its various guises is regarded as both process and concept. For instance, calculating the derivative of $f(x)=x^2$ using the notation:

$$\text{as } h\to 0, \frac{f(x+h)-f(x)}{h} \to 2x$$

has a definite dynamical sense of h getting close to zero, whilst the notation

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is both process (h approaching zero) and product (the limiting value itself). *The process here has a less explicit procedure of calculation than examples earlier in the curriculum*. This limit process has many representations: as numerical approximations, as a graphical image of a secant approaching a tangential position, from “first principles” as a symbolic manipulation to obtain an expression in which h can be allowed to tend to zero, as a symbolic algorithm using theorems about sum, difference, product, quotient or composite of functions whose derivatives have been calculated from first principles, and so on. There is the alternative notation:

$$\frac{dy}{dx} = f'(x)$$

in which some insist $\frac{dy}{dx}$ is just a single symbol, but others allow it to be a fraction representing the quotient of the y-component of the tangent vector by the x-component.

All the following manifestations of limit are simultaneously process and product:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$\lim_{n\to\infty} \frac{1-x^n}{1-x}$$

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{\delta x\to 0} \sum_{x=a}^{b} f(x)\ \delta x\ .$$

In each case they represent both the *process* of calculating the limit and the *product* of this process – the limiting value itself. But the process is not always associated with an easy procedure for computation of the limit and again behaves very differently from procepts in elementary mathematics. Whilst a notation such as 3+2 conjures up in the mind a specific procedure to calculate the answer (count on), the limit notation

$$\sum_{n=1}^{\infty} a_n$$

does not evoke any way to calculate the value of the limit. The limits are calculated by deriving facts in a completely new way by using an ε–δ definition to develop a few special cases and a few general theorems from which the results are deduced. Here there are “known facts” of two distinct types : *particular facts* (such as $\sum_{n=1}^{\infty} 1/n^2$ converges) and *general theorems* (such as the comparison theorem which asserts that if $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms and $\sum_{n=1}^{\infty} b_n$ is a series of positive terms with $b_n < a_n$ for all n, then $\sum_{n=1}^{\infty} b_n$ converges). New “known facts” are derived from old ones by circuitous routes which only serve to cause enormous confusion in the student meeting the ideas for the first time.

Research shows most beginning university students conceive of a limit as a dynamic *process* rather than the static *limit* concept and end up with all sorts of confusions (Schwarzenberger & Tall, 1977). They fail to understand the deeply embedded cultural way that mathematicians use ambiguity of notation to bridge the difference between process and concept. It seems, too, that many mathematicians are unaware of this explicit ambiguity in their thinking processes.

The computer and the principle of selective construction

Let us close on a positive note. We have seen that there is a qualitative difference between the thinking processes of those who succeed and those who fail in which the flexible use of symbolism plays an important role. Might it be possible to improve the situation?

Initially it seems that we are in a catch-22 situation. The less able remember fewer facts, so they have less in their mind to manipulate and therefore rely on procedures which exist in time and impose greater cognitive strain to coordinate. So it seems that the traditional way of breaking up the difficulties into smaller steps will only make the situation worse. It may give the less able an even greater number of parts to coordinate. Perhaps the innate ability of the successful gives them an advantage that cannot be bridged by those less fortunate.

All is not lost. A possible way ahead might involve reducing the cognitive strain by allowing the learner to concentrate on the concept without having to carry out the procedure to obtain it. The calculator and computer can be of assistance here. We have intentionally used the word "procedure" to describe the sequence of actions the child carries out in the mathematics. Such a procedure is usually an *algorithm* that can be programmed on a calculator or computer. This includes the procedures of arithmetic, algebra, calculus, drawing graphs, solving equations, etc.

We suggest that a learning strategy to reduce cognitive strain is to separate out the doing of the procedure from the manipulation of the concept and to do them at different times. Historically the child needed to carry out the procedure to obtain the concept to manipulate. But if the computer can carry out the procedure, it may be possible to allow the child to concentrate on the concept without first doing the procedure.

Consider the case of a pie-chart. The meaning of such a chart, that bigger slices mean a bigger contribution, is apparent to a very young child. Yet the process of *drawing* a pie-chart. does not occur in the National Curriculum until level 6 when the child is in the early teens, for it requires handling angles up to 360˚, calculating fractions of 360 and coordinating the measurement of angles in the picture. If a database with graphical facilities is used to convert appropriate data into pie-chart form, the complicated arithmetic procedures can be circumvented and the child can focus on interpreting the pie-chart rather than on the intermediate process of making the calculation to draw the picture.

At another time it is possible to focus on the process of drawing a pie-chart, thus separating the two events: the procedure of drawing and the interpretation of the result. In this way one may select the portions of knowledge that the child is asked to understand at a given time, selecting some to be constructed by the child whilst others are constructed (internally) by the computer.

The traditional sequence of learning first required that procedures were practised until they were routinized and capable of being reduced to subconscious action before the products of the processes could be successfully manipulated and understood. By using the computer to carry out the procedures, the learner can be focused on the *products* and thus the higher level activities can be encouraged earlier and separately from the processes. This reduces the cognitive strain and offers the possibility of the less able breaking out of the proceptual divide wherein they cannot master the procedure because it is too complex and they therefore cannot encapsulate the procedure as a mental object because the procedure causes too much cognitive strain. Thus the computer can be used, by a process of selective construction, to encourage the formation of flexible procepts in a wider range of ability.

This philosophy is already being used with success in the new 16–19 A-level. Students meeting the derivative for the first time do so by magnifying the graph and if a small portion looks almost straight, then the gradient of this (locally) straight segment is taken as the gradient of the graph. Thus the students can conceive intuitively of the changing gradient of the curved graph before they begin to look at the algorithms for symbolic differentiation. Indeed, before they do symbolic differentiation, they carry out numerical procedures to give arithmetic methods of calculating the gradient of a graph.

Likewise, in solving a first order differential equation, they may use a piece of software (The Solution Sketcher) which will draw a small line segment of gradient given by the differential equation and use the computer to stick small segments together to build up the *concept* of a solution before studying the (numeric or symbolic) procedures to calculate a solution.

In studying the Newton-Raphson method of finding a solution of $f(x)=0$ by using a linear approximation to the graph near a root, again students can use software to see the method in action, to build up the concept *before* studying the symbolism and numerical calculations of the method.

Several research projects in various areas of mathematics show that this is a promising avenue of development, in algebra (Tall & Thomas 1991), in trigonometry (Blackett 1990), in calculus (Tall 1986), and the CAN project (Shuard et al, 1991) shows children improving their use and understanding of arithmetic and number through free access to calculators. These experiments do not show success for *all* children, but the position of the proceptual divide between success and failure is shifted in a positive direction. The philosophy of using the computer and calculator to carry out certain processes whilst the child concentrates on the resulting concepts is therefore a powerful strategy to develop.

Conclusion

It has long been known that mathematics is a hierarchical subject and if one does not understand one stage then the next stage becomes difficult and the one after that is impossible. What we have attempted to do in our two articles is to make this universal observation more precise. What we see is procedures becoming procepts. The *proceptual divide* occurs between those who complete this transition and those who fail.

If the flexible meaning of procept as both procedure and concept (through the use of the same symbolism for both) fails to occur to a satisfactory extent, then it may happen that procedural success can occur at the current level, but may not give a solid foundation for future development. Such procedural thinking is less likely to lead to flexible proceptual thinking with its feed-back loop of derived facts effortlessly generating new knowledge. Short term success (which the children crave and the teacher feels duty bound to give) may thus lead inexorably to long term failure. Allowing children to perform arithmetic in their own way at their own speed without any teacher enquiry or intervention may significantly contribute to the worsening of the gap.

Our analysis points to the weakness of the less able child turning to the security of procedures rather than the successful use of procepts. Therefore the additional practice at such procedures may only make the differences greater, not close the gap. Additional practice has been the traditional response, it can hardly claim to have been successful.

Resolution of the challenge created by those who are constrained to perform harder (procedural) mathematics rather than the more powerful and easier proceptual mathematics has been faced in two ways: a need for greater insight into the ways in which today’s success is achieved and a means of providing support learners to make the process/product links.

We believe that a helpful way ahead is to support the less able child (and the more able for that matter) with the computer as a tool, to give added power in those areas that the learner is weak, just as someone with weak eyesight might use spectacles, or someone wishing to travel from London to Glasgow might use motorised transport instead of foot-power. By using the computer to carry out the procedures, it may be possible to concentrate on the products of those procedures to build up a better conceptual structure. By focusing at one time on the product of procedures (using a computer), and at another time on the procedures themselves, the cognitive strain may be reduced and the position of the proceptual divide in the spectrum of mathematical performance may be moved to give advantage to a larger number of children.

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