

Modelling Irrational Numbers in Analysis using Elementary Programming

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The Quest

In recent years we have been teaching mathematical analysis using the computer to give experiences to help students understand the concepts (Mills & Tall 1988). It has proved to be a valuable aid in many ways, but the method depends on giving a suitably rich environment enabling students to experience the full range of possibilities. A great weakness of the use of regular computer languages is that they hold numbers in memory as a finite expansion in base two, so, effectively, all numbers in such a system are rational. This meant that we were unable to demonstrate any ideas that depend on the difference between rationals and irrationals; for instance, we could not plot graphs of the kind

$$f(x) = \begin{cases} 0 & (x \text{ rational}), \\ 1 & (x \text{ irrational}). \end{cases}$$

It meant that we could not give computer experiences appropriate for the more exotic functions that occur in mathematical analysis, and so students following such an approach were likely to have limited experiences from which to generalize. As a consequence they may not have natural intuitions which give them insights into more esoteric theorems and, in short, they would not be much different from the great majority of mathematics students who find the theory of analysis abstract and impenetrable.

Our main aim was to define a computer function $\text{rat}(x)$ whose value would be TRUE if x were rational and FALSE if x were irrational. Clearly we were on a losing wicket because we were intending to program in BBC BASIC and here every computer number x is rational! So we lowered our sights somewhat and sought out a function that would have $\text{rat}(x)=\text{TRUE}$ for most familiar rationals like, 1, $-3/2$, $2/3$, $22/7$, yet would have $\text{rat}(x)=\text{FALSE}$ for familiar irrationals, like π , $\sqrt{2}$, $\sqrt{3+1}$, and so on.

Our plan was to approximate a number x by a rational number and use the nature of the approximation to divide numbers into two kinds, *pseudo-rationals* including 1, -1.5, $2/3=0.6666666667$, $22/7=3.1428571429$ and *pseudo-irrationals*, like $\pi=3.141592654$, $\sqrt{2}=1.41421356$, $\sqrt{3+1}=1.73205081$, etc.

The Plan

We used the method of the Ancient Greeks to compute continued fractions, which gives a sequence of successive rational approximations to a real number. In the case of a rational, this eventually produces the rational expression in lowest terms. In the case of an irrational, it goes on forever, computing a rational expression with ever larger numerator and denominator. In theory, therefore, the irrationals are those whose denominator grows ever larger, whilst the rationals eventually stop. In practice we calculated the continued fraction approximation until it was within a specified error (say $e=1/10^8$) and, defined the number to be *pseudo-irrational* if the denominator exceeded a chosen large number (say $N=10000$).

The actual mechanism for computing fractional approximations to a real number x , starts by writing x as

$$x = n_1 + d_1$$

where n_1 is the integer part of x and d_1 is the decimal part. If d_1 is non-zero, we can get a better rational approximation to this by the simple expedient of noting that $1/d_1$ is greater than 1 and finding the integer part n_2 of $1/d_1$, giving

$$1/d_1 = n_2 + d_2,$$

where n_2 is an integer and d_2 lies between 0 and 1.

This gives

$$x = n_1 + 1/(n_2 + d_2)$$

which in turn contains a better rational approximation for x in the form

$$x \approx n_1 + 1/n_2.$$

This may be turned into a repeating algorithm by writing

$$r_1 = x,$$

then, when r_k is found, let

$$n_k = \text{integer part of } r_k, d_k = r_k - n_k.$$

If $d_k = 0$, the process stops, otherwise, let $r_{k+1} = 1/d_k$ and repeat the process.

This gives a sequence of fractional approximations to x in the form

$$n_1, n_1 + \frac{1}{n_2}, n_1 + \frac{1}{n_2 + 1/n_3}, n_1 + \frac{1}{n_2 + \frac{1}{n_3 + 1/n_4}}, \dots$$

For instance, for the rational number $x=22/7$, we get

$$n_1 = 3, d_1=1/7, r_2=7, \text{ so } n_2=7, d_2=0,$$

and the process stops, with the usual rational expression:

$$22/7 = 3 + 1/7.$$

For the irrational number $x=\pi$, we get

$$n_1=3, d_1=0.141592654, r_2=1/d_1=7.062513306,$$

$$n_2=7, d_2=0.062513306, r_3=1/d_2=15.99659441,$$

$$n_3=15, d_3=0.99659441, r_4=1/d_3=1.003417229,$$

$$n_4=1, d_4=0.003417229, \text{ etc,}$$

giving the sequence of rational approximations

$$3,$$

$$3 + \frac{1}{7} = \frac{22}{7},$$

$$3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106},$$

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113}, \dots$$

with ever increasing denominators, getting ever closer, but never equal to, π .

Using this algorithm, we wrote a BBC BASIC function which repeated the calculation for a given number x , until the fractional approximation was within a specified error e , then looked at the denominator of the fraction to see how large it had become, returning the value TRUE if the denominator did not exceed a specified value N and FALSE otherwise. This idea was implemented using an algorithm derived from one given in the *Mathematical Association 132 Short Programs*, but reformulated in BBC BASIC as a function:

```

DEF FNfrac(x,e,N)
LOCAL r,s,t,b,t0,b0,t1,b1,n,d
r=x : t0=0 : b0=1 : t1=1 : b1=0
REPEAT
  n=INT r : d=r-n : t=n*t1+t0 : b=n*b1+b0
  IF d<>0 THEN r=1/d : t0=t1 : b0=b1 : t1=t : b1=b0
UNTIL ABS(t/b-x)<e
= (b<=N)

```

By experimenting with the values of e (as small as practicable) and N (suitably large, but not too large), we ended up defining

```
DEF FNrat(x)=FNfrac(x,1E-9,10000)
```

It was this function that we used to distinguish between “pseudo-rationals” (for which $\text{FNrat}(x)$ is TRUE) and “pseudo-irrationals” (when $\text{FNrat}(x)$ is FALSE).

Skating on Thin Ice

We knew that as we did this we were operating a dangerous technique. *Theoretically* if x is rational, the continued fraction method terminates to give an exact answer. *In practice*, because of rounding errors, where the decimal part d_k should be zero, it may be incredibly small, but non-zero. So the routine would continue with $r_{k+1}=1/d_k$, which is now huge and leads to incredibly awful errors. By terminating the routine for FNfrac when the error gets small we were side-stepping this possibility, but we were aware that we should not try to press our technique too hard otherwise it could lead to unmitigated disaster! We found peculiar things happened in practice but the exercise proved useful in flexing our muscles for more difficult tasks that lay ahead.

The Consequences

What we found was quite remarkable. The function $\text{rat}(x)$ gives $\text{rat}(1)$, $\text{rat}(-1.5)$, $\text{rat}(2/3)$, $\text{rat}(22/7)$ all to be TRUE, whilst $\text{rat}(\text{PI})$, $\text{rat}(\text{SQR}2)$, $\text{rat}(\text{SQR}3+1)$ are all FALSE.

Because it is only an approximation we knew that the function would fail in many ways. For instance, rationals with large denominators, such as $12/12345$ would fail the test, so that $\text{rat}(12/12345)$ turns out to be FALSE and $12/12345$ would be considered a pseudo-irrational. Likewise, if we took a number we wanted to be a pseudo-irrational, such as π , then a large multiple, such as 1000π would be defined to fewer decimal places and might give $\text{rat}(1000\pi)=\text{TRUE}$, so that 1000π is considered pseudo-rational! In fact, even 10π was large enough to give a TRUE response, and one might wish to change the definition of $\text{rat}(x)$ to say $\text{FNfrac}(x,1\text{E}-9,1000)$ to lessen this type of flaw.

The definition therefore has some peculiar properties. However, we should not give up on an idea just because it seems to fail – it was bound to fail anyway. A bonus of the definition is that it provides some fascinating parallels with true rationals and irrationals. First a *randomly* generated number has a high probability of being pseudo-irrational, which is analogous to the case that most real numbers (in a cardinal sense) are irrationals.

A short program of the form

```
c=0 : n=0
REPEAT : n=n+1 : IF FNrat(RND(1))=TRUE THEN c=c+1
PRINT c/n
UNTIL FALSE
```

will generate random numbers RND(1) between 0 and 1, and increase the counter c every time a pseudo-rational is given; this shows that less than 10% of pseudo-rationals are generated. Changing the value of N in FNfrac(x,e,N), say to $N=1000$ instead of $N=10000$ will radically reduce the proportion of numbers considered to be pseudo-rational.

In the real case a random number has a probability 1 of being irrational and 0 of being rational, which is reasonable, for if we have a truly randomly generated number in which each digit is selected independently, the probability of getting a repeated block throughout the expansion is rather small.

Plotting Rational and Irrational Parts of the Domain

We used two methods of plotting graphs over an interval $[a,b]$ which were implemented in the SMP *Function Analyser* on the Archimedes computer (Tall 1991). One method makes many passes along the interval. At the first pass it plots the value at the mid-point $(a+b)/2$, then made a second pass filling in the other quarter points, $(a+b)/4$, $3(a+b)/4$, then the intermediate eighths, the intermediate sixteenths, thirty-seconds, sixty-fourths, and so on. If the original points a,b were rational, then this plotting method uses all rational numbers and, even allowing for rounding errors in arithmetic, and the peculiar definition of pseudo-rational, it turns out that this initially plots only pseudo-rationals. Figure 1 shows the eleventh pass drawing the function

$$y=\text{if}(\text{rat}(x),1,-1),$$

where $\text{rat}(x)$ is the function we have just introduced and $\text{if}(\text{expr},\text{val1},\text{val2})$ tests the truth of expr and, if it is true, returns val1 else it returns the value val2 . In this case the function returns 1 if x is

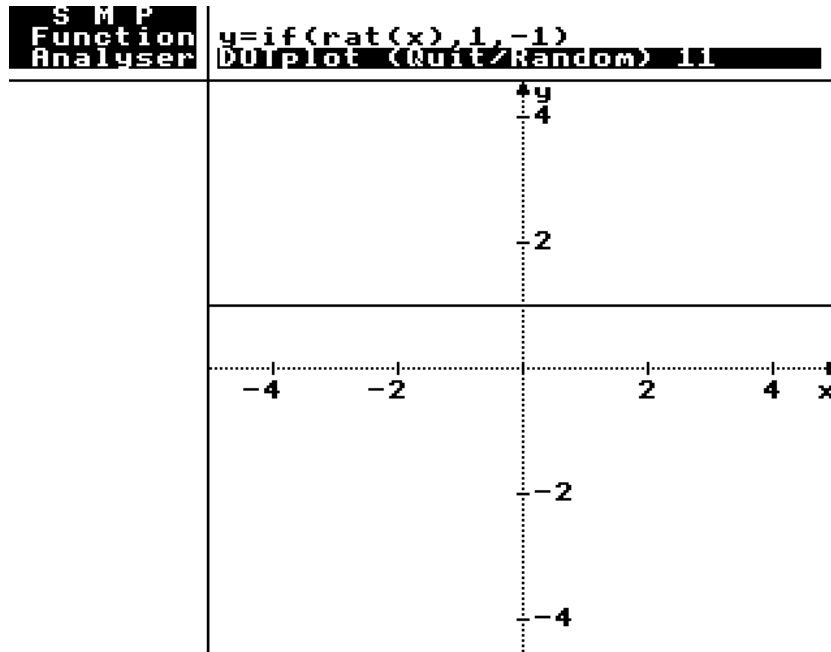


Figure 1 : Plotting rational points (all "pseudo-rational", so far)

pseudo-rational and -1 if x is pseudo-irrational. In figure 1 *all* the points plotted so far are pseudo-irrational, (the interval is 10 units wide and the 11th pass divides this into 2^{11} parts each of width $10 \cdot 2^{-11} = 10/2048$, all qualifying as pseudo-rationals). After a few more passes the points develop too large a denominator, together with rounding errors, which start to produce numbers considered to be pseudo-irrational. We should also note that if the x -interval were changed from $[-5,5]$ to an irrational interval, say $[0,\pi]$, then we would be likely to pick up pseudo-irrationals from the very first pass.

The second plotting method generates random numbers r in the interval $[a,b]$ and plots the points $(r,f(r))$. This soon fills out sufficient points to see the shape of the graph. Truly randomly generated points would have a high probability of being irrational. These computer-generated (pseudo-)random numbers have a fairly high probability of being pseudo-irrational, so this method is far more likely to plot pseudo-irrationals than rationals. In figure 2 we see that the line of pixels for $y=-1$ is virtually filled out, whilst $y=+1$ is more sparsely visited, revealing the far larger proportion of pseudo-irrationals generated.

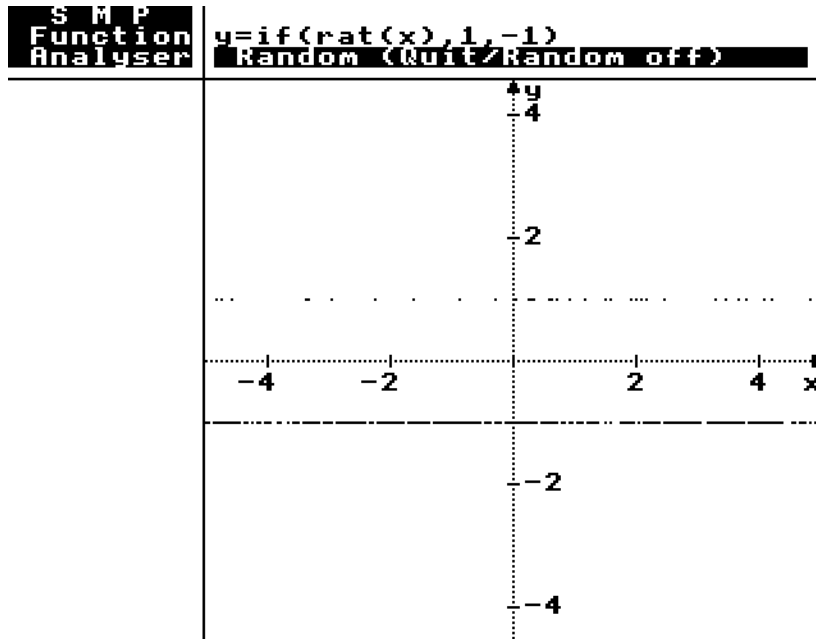


Figure 2 : Plotting random points (nearly all “pseudo-irrational”)

Hence we have two methods of plotting graphs, one giving mainly pseudo-rationals values of x , the other mainly pseudo-irrational!

Integrating highly discontinuous functions

Now we ask what happens if we try to calculate the area under a highly discontinuous graph, say

$$f(x) = \begin{cases} 0 & (x \text{ rational}), \\ 1 & (x \text{ irrational}). \end{cases}$$

Figure 3 shows the area calculated under $y=f(x)$ using strips of random

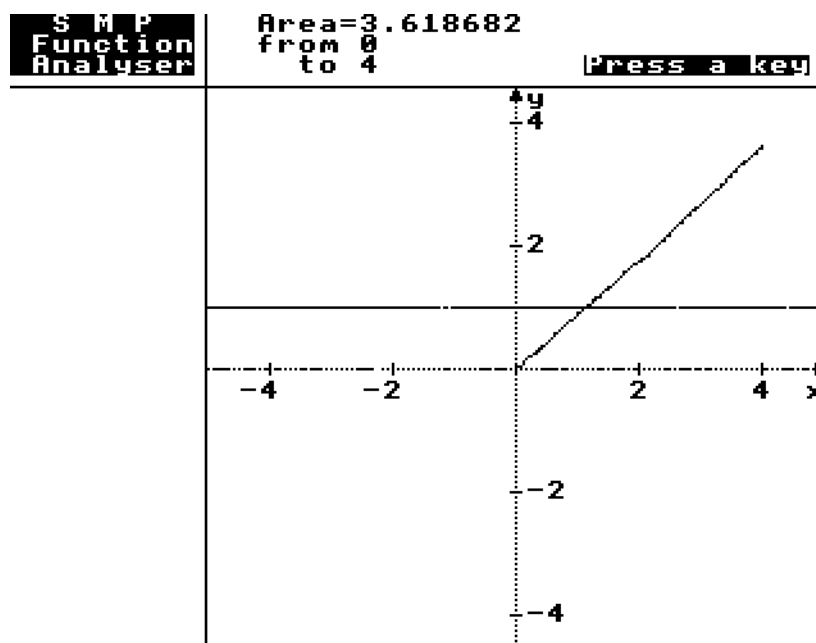


Figure 3 : Calculating the area under $y=f(x)$ using a random partition

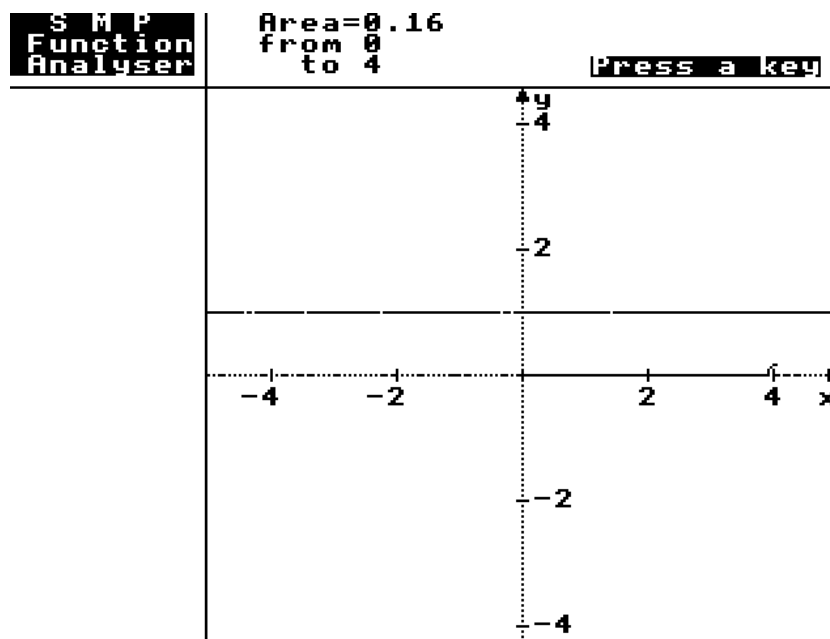


Figure 4 : Calculating the area using a (fairly) rational partition

width whose height is $f(t)$ for a random point t in the strip. The area function is drawn and is approximately $y=x$, so that the area from 0 to 4 is given as 3.618628, representing the fact that only a small proportion of random points have a pseudo-rational value (about $0.38/4=10\%$, agreeing with the figure calculated earlier). Lowering the value of N in $FNfrac(x,e,N)$ reduces the proportion further by defining fewer pseudo-rationals.

Figure 4 uses a fixed (rational) step-width and the mid-point of each strip to calculate the area by the mid-ordinate rule. Here virtually all the points picked up are (pseudo)-rational, so the area function from $x=0$ to $x=4$ is approximately $y=0$. Only at the end (just below $x=4$) is there a slight hiccup, caused possibly by rounding errors in calculation where the area limps up to 0.16.

The visions given by these pictures are intriguing. In the theoretical case every step has both rational and irrational points, so the lower sum takes the ordinate to be zero and the upper sum takes the ordinate to be 1. So the “lower area” is 0 and the “upper area” (from 0 to x) is x . For non-zero x these are not the same, so the Riemann integral is deemed *not to exist*. However, the model suggests that a random partition is far more likely to pick up irrational points than rational ones, so the area is likely to be much closer to the Riemann upper area than the lower one. In fact, assuming the probability of picking a rational to be zero, then theoretically a random area calculation will have a probability 1 of equalling the Riemann upper area.

This model is therefore likely to give intuitions that there are far more irrationals than rationals, and that there may be a more sophisticated theory of integration that takes this into account. It therefore leads more naturally to the *Lebesgue integral* which arises later in a university course (if at all) and is to do with the *measure* of the sets involved (see Weir 1974). The Lebesgue integral in this case equals the Riemann upper area.

When this approach was used in class with university students who were not mathematics majors, it led naturally to a discussion of *why* there might be more irrationals than rationals. The “randomly generated number” argument was a very powerful motivator. Although the students had previously shown that “between every pair of rationals there is an irrational” and “between every pair of irrationals there is a rational”, they now had a new way to appreciate that there were far more irrationals in any interval than rationals. This led to an animated discussion about a concept that had previously seemed to many of them to be no more than an esoteric fiction.

Although the pseudo-rational definition given in this article is at best a gross approximation, it does offer vitally important insights into higher level mathematical theory, including cardinality concepts and Lebesgue integration. In this case the authors found themselves seeing old ideas in a meaningful way for the first time. The students were intrigued by the ideas sufficiently to wish to talk about them further. In this way computer simulations, even flawed ones, can help students gain intuitions about subtle higher level theory.

References

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