Concept Images, Computers, and Curriculum Change

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Introduction

It seems self-evident that the way to teach mathematics is to start from simple concepts familiar to the learner and to build more complex ideas through a sequence of activities growing steadily in sophistication. It is a salutary experience to learn that a curriculum carefully built in this way can cause serious difficulties in learning. The problem arises because the human mind does not operate in a logical manner. Instead evolution seems to have given us a powerful patterning mechanism that recognizes implicit regularities in a given context and leads to each of us forming our own personal concept image of a mathematical concept. By presenting mathematics to a learner in a simplified context, we inadvertently present simplified regularities which become part of the individual concept image. Later these deeply ingrained cognitive structures can cause serious cognitive conflict and act as obstacles to learning.

The computer offers new possibilities. Instead of building from simple to complex, it is possible to construct appropriate software environments for the learner to explore more complex ideas from the outset. This form of learning involves a negotiation of the meaning of the mathematical concepts modelled by the computer in which the organization of the curriculum and the role of the teacher is crucial.

The presentation will consider theoretical perspectives and relate them to the results of empirical studies in algebra and the calculus designed to promote versatile learning of higher order concepts using the computer.

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Mathematical Beliefs and Cultural Values

As we face the many changes that the new technology offers we should be prepared to stand back and see the way in which shared experiences in a society cause us to react in various ways to new cultural elements. In particular it is beholden to mathematicians and mathematics educators to honestly re-evaluate their concepts in a way which will be most valuable to society:

> Mathematicians themselves seem prone to ignore or to forget the cultural nature of their work and to become imbued with the feeling that the concepts with which they deal possess a "reality" outside the cultural milieu - in a sort of Platonic world of ideals. Indeed, some mathematicians seem to be completely lacking in the insight that the modern physicist has attained - the recognition that even his observations, as well as his concepts, are coloured by the observer. How much more this must be the case in mathematics, where the conceptual has gradually gained primacy over the observable?
>
> (Wilder 1968, preface, page viii.)

Anthropologists study various cultural forces that operate when new ideas are introduced. Cultural elements move from one culture to another by a process of *diffusion*. There is often a *cultural lag*, in which new elements take time to become part of the culture and sometimes a positive *cultural resistance* where new elements fail to replace old, successful ones. The result is a complex mixture of old and new, as in Britain where the metric system has been formally adopted but the mile has been retained for geographical distances and the pint as a measure for milk and beer.

The introduction of the computer will see such cultural forces in play. Some new elements, such as desk-top publishing, are already so successful that they are quickly becoming part of the new culture. Other elements (such as the proven value of the use of the computer for improving learning) will, rightly, be subject to cultural lag and resistance, for these essential processes stabilize the system and prevent us from tripping butterfly-like from one new idea to another.

In the new technological paradigm we must all reflect deeply on the nature of the mathematics required, both in terms of subject matter and the way it is organized in the curriculum. This requires a careful re-evaluation of our fundamental premises and beliefs, for our previous experience in a pre-computer paradigm may not always provide us with appropriate intuitions to make the right judgements in the new technological era. Hence the need for both reflective thinking and carefully designed research.

Concept Images

The cultural forces operating are essentially the global product of the interactions between individuals and the way that we come to see the world. At any stage we can only make sense of our observations using the cognitive structure that we have. Physically this is produced through the connections formed in our brains due to external impulses and internal processing; we make sense of new information by making new connections and reorganizing our cognitive structure. It is no wonder therefore that this structure may not be totally coherent:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. ... As the concept image develops it need not be coherent at all times. ... We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion. (Tall & Vinner 1981, p.152)

We now know that many natural learning processes introduce conflicts. For instance the experiences in the western world of reading left to right produce difficulties in algebra when pupils try to make sense of an algebraic expression such as

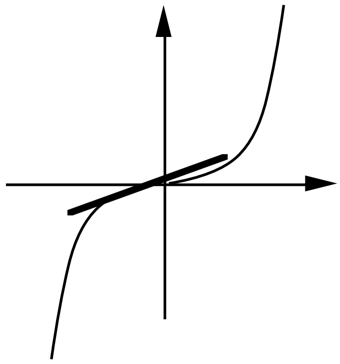
$2 + 3\,a$.

Processed from left to right, the child first computes 2+3 as 5 to give the result as 5a. When this left-right processing is further violated by the introduction of brackets, more difficulties occur. Children are told to ”calculate the expression in brackets first“, so

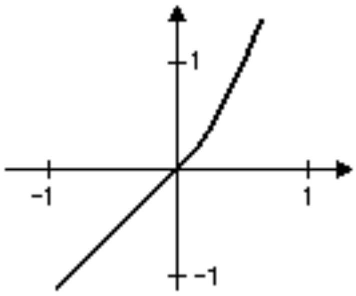
$2(a+b)$

must be calculated first by adding a to b before multiplying by 2. As a *process*, this is quite different from 2a+2b, although the results are the same. Until the child is able to conceive of an algebraic expression as an *object* rather than a process, algebraic manipulation can therefore cause cognitive conflict.

More generally, when ideas are presented in a restricted context, the concept image may include characteristics that are true in this context but not in general. For example, the tangent to a circle touches the circle at one point only and does not cross the circle. Vinner (1983) observed that many students believe that a tangent to a more general curve touches it, but may not cross it.When students were asked to draw the tangent to the curve $y=x^3$ at the origin, many drew a line a little to one side which did not pass through the curve.



In Tall (1986) a computer graph plotter was used to provide a richer context to discuss the tangent concept. The software drew graphs and was capable of drawing a line through two very close points on the graph to give a practical approximation to the tangent. This allowed investigations of graphs with corners and the tangent to graphs at an inflection point. Three experimental groups who used the graph plotter and four control groups taking a standard calculus course were asked to draw a tangent to the following graph at the origin:



$$y = \begin{cases} x & (x \leq 0) \\ x+x^2 & (x \geq 0) \end{cases}$$

Only 22 out of 65 control students (34%) drew the correct tangent $y=x$. Thirty students (46%) drew the tangent moved round a little so that it looked as if it touched the curve at only one point. The remainder were not able to cope with the task at all, asserting that the tangent could not exist because:

> The graph is two separate functions, and there is not a tangent at x=0

or

> ... because the tangent should touch the line at one specific point but this tangent would touch it constantly.

In the experimental groups 31 out of 41 (76%) responded with the correctly drawn tangent whilst only 8 (20%) moved it a little to touch at only one point (significant at the 0.01 % level).

Using the Computer to provide a Predictable Environment for Learning

Skemp (1979, page 163) makes a valuable distinction between different modes of building and testing conceptual structures in the following table:

REALITY CONSTRUCTION

REALITY BUILDING	REALITY TESTING
Mode 1	Mode 1
from our own encounters with actuality *experience*	against expectation of events in actuality *experiment*
Mode 2	Mode 2
from the realities of others *communication*	comparison with the realities of others *discussion*
Mode 3	Mode 3
from within, by formation of higher-order concepts by extrapolation, imagination, intuition: *creativity*	comparison with one's own existing knowledge and beliefs: *internal consistency*

When I wrote my first programs for exploration of the ideas of the calculus (Tall 1986b), it was my intention to provide a mode 1 environment for older students to explore mathematical concepts as a foundation for the more usual theoretical modes 2 and 3. However, Heine Bauersfeld convinced me that the tactile and enactive activities of mode 1 are absent from computer interaction, (although the Macintosh interface now re-introduces the enactive element to some extent).

Changes in cognitive structure are due to new connections in the brain as a result of either external stimuli or internal processing. From a constructivist perspective the individual may be seen as acting on stimuli either from the outside, or from within. The action within the brain is *reflective* thinking (Skemp's mode 3). I now find it helpful to distinguish at least three different sources of external stimuli:

inanimate, *cybernetic* and *interpersonal*:

Inanimate stimuli come from objects in actuality which the individual may also be able to manipulate.

Cybernetic stimuli come from systems which are set up to act according to pre-ordained rules.

Interpersonal stimuli come from other people.

The last of these corresponds to Skemp's mode 2, whilst the first two are a modification of his mode 1.

The inanimate sources are *passive* - they require the individual to manipulate them *actively* for building and testing of concepts. Typical examples are Dienes' blocks or Cuisenaire rods. Cybernetic sources are *reactive* - again the individual may act on them, but the environment now provides feedback according to the inbuilt rules.

Generic Organizers

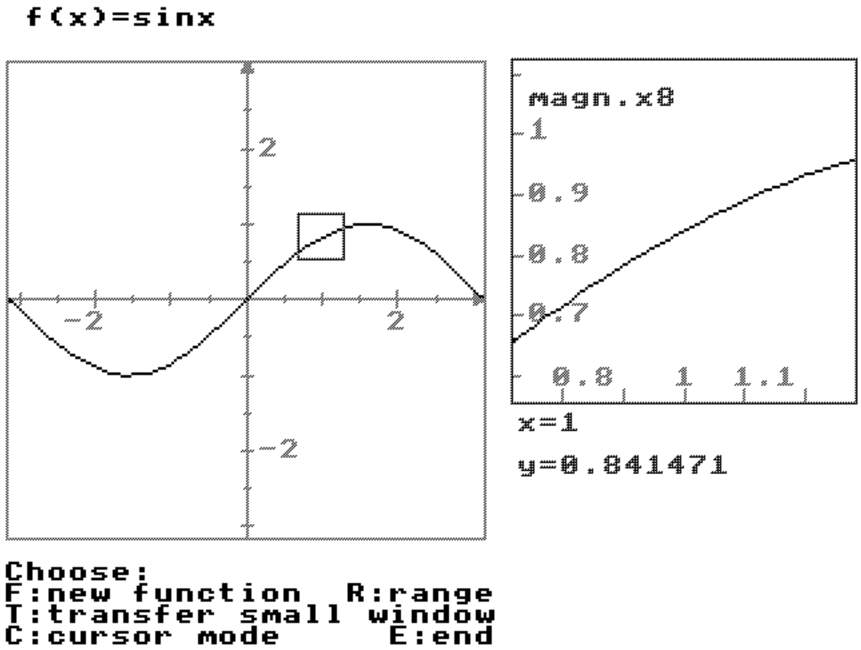
Ausubel et al (1968) defined an *advance organizer*, as

> "Introductory material presented in advance of, and at a higher level of generality, inclusiveness, and abstraction than the learning task itself, and explicitly related both to existing ideas in cognitive structure and to the learning task itself ... i.e. bridging the gap between what the learner already knows and what he need to know to learn the material more expeditiously."

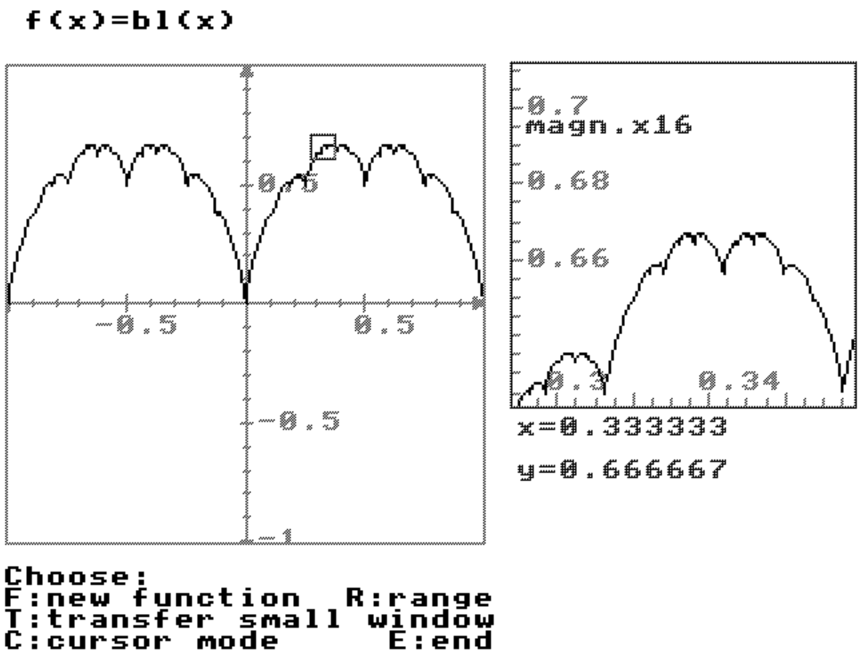
Such a principle requires that the learner *already has the appropriate higher level cognitive structure* available to him or her. In situations where this may be missing, in particular when moving on to more abstract ideas in a topic for the first time, a different kind of organizing principle will be necessary. To complement the notion of an advanced organizer, in Tall 1986a I defined a ***generic organizer*** to be an environment (or microworld) which enables the learner to manipulate *examples* and (if possible) *non-examples* of a specific mathematical concept or a related system of concepts. The intention is to help the learner gain experiences which will provide a cognitive structure on which the learner may reflect to build the more abstract concepts. I believe the availability of non-examples to be of great importance, particularly with higher order concepts such as convergence, continuity or differentiability, where the concept definition is so intricate that students often have difficulty dealing with when it fails to hold.

Many of the pieces of concrete apparatus used in mathematics teaching, such as Cuisenaire rods or Dienes blocks, function as inanimate generic organisers on which children may operate to build anchoring concepts for mathematical abstractions. However, these often focus on what a concept *is*, rather than what it is *not*. (Dienes blocks embody the notion of number base and the process of handling place value when performing arithmetic operations. Non-examples of these concepts seem to be of little relevance.)

A simple instance of a generic organizer embodying both examples and non-examples is the "Magnify" program from Graphic Calculus (1986) designed to allow the user to magnify any part of the graph of any input function.



Tiny parts of certain graphs under high magnification eventually look virtually straight and this provides an anchoring concept for the notion of differentiability. Non-examples in the program are furnished by graphs which have corners, or are very wrinkled that they never look straight, providing anchoring concepts for non-differentiability.



Generic difficulties

Given the human capacity for patterning, and the fact that the computer model of a mathematical concept is bound to differ from the concept in some respects, we should be on the lookout for abstraction of inappropriate parts of the model. Visual illusions in interpreting graphs have been documented by Goldenberg 1988 and by Linn & Nachmias, 1987. In the latter case, one third of the pupils observing a cooling curve of a liquid on a computer VDU interpreted the pixellated image of the graph as truly representing what happened to the liquid - constant for a time, then suddenly dropping a little (to the next pixel level down).

Even when there appears to be a large measure of understanding of what is happening on the computer screen this involves some mental construction as to what the computer is actually doing. In Tall & Winkelmann (1988) we described three different kinds of insight:

External, analogue, specific

External insight occurs when the user has no idea what is going on inside the computer, but has knowledge which allows him or her to check that the results are sensible.For instance, the user may have no idea of the algorithm being used by the computer, but may have other knowledge that allows him or her to check the result, or the user might explore the software extensively to note any regularities and propose a model of how it is working.

Analogue insight occurs when the user has an idea of type of algorithm in use, for instance, knowing that a root of an equation is being computed by the Newton Raphson rule, but is unaware of precisely how this implemented.

Specific insight is when the user is fully aware of how the software is programmed (though this, in practice, remains only partial for, even if the user knows how a high-level language works, the implementation within the hardware is likely to include features that are not understood).

Specific insight into computer software is rarely possible or even desirable for the majority of computer users, but it is helpful for the student to have at least external insight or, preferably, analogue insight. Here an external agent - a teacher - is desirable. The concept image of a cybernetic system constructed in the mind of the user is likely to be idiosyncratic and the teacher has a fundamental role to play through guidance and discussion. I have elsewhere described the combination of a human teacher as guide and mentor using a computer environment for teaching, pupil exploration, and discussion as the *Enhanced Socratic Mode* of teaching and learning (Tall 1986a).

Starting points for new curriculum sequences

Generic organizers on a computer allow us to develop new sequences in curriculum development starting, not from *mathematical foundations,* but from *cognitive roots*. A cognitive root is an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built. For example, students playing with a graph magnifying program invariably formulate the notion of local straightness (looking straight under high magnification) - in fact they often propose an idea which is more subtle: the higher the magnification, the lower the curvature. Local straightness proves to be a cognitive root for the calculus. A student with this idea can look along the graph of a locally straight function and see the changing gradient, particularly if a generic organizer is available which scans along the graph and builds up the gradient graph at the same time.

Without guidance the student would be unlikely to discover all the subtle and powerful ideas embodied in this organizer. First there is an implicit assumption that the drawing of the original graph faithfully represents the gradient, in the sense that there are no small oscillations on the graph, too small to represent. It usually needs a teacher, as mentor, to suggest that students might reflect on hidden oversimplifications in their ideas by trying to think of non locally-straight graphs, such as those with “corners” (such as $y=|x^2-1|$) or with tiny oscillations (such as $y=\cos x+\sin(100x)/100$) or fractal graphs which don’t magnify to look straight at all. These non-examples allow students to understand not only the concept of a differentiable function, but to obtain some experience of what may happen to cause a function to be non-differentiable.

Those experienced in the current curriculum initially tell me that they consider such ideas very deep and too subtle for pupils to grasp, but their experience gained over many years in a pre-computer technology is misleading. Given the appropriate facilities, the reverse is true - students of quite modest abilities can attack the ideas with a vigour that surprises the classroom teacher.

The difference between a mathematical and a cognitive approach

In designing a curriculum the mathematician tries to “simplify” things, to help the student, by reducing the number of variables involved. For example, the mathematical concept of a derivative requires the limit of the expression

$$\frac{f(x+h)-f(x)}{h}$$

as h tends to zero, which can be made mathematically "simpler" by fixing x and only allowing h to vary. The mathematician's sequence of activities for the beginning of the calculus consists of:

(a) an "intuitive" approach to limits

(b) fix x to calculate the limit of $\frac{f(x+h)-f(x)}{h}$ as h gets small and call the limit $f'(x)$

(c) vary x in $f'(x)$ to get the derivative as a function.

Mathematics education research shows cognitive obstacles at each stage. In the first place, the geometric idea of using a secant approaching a tangent is not cognitively intuitive in the sense that it does not occur spontaneously. It also produces a number of cognitive obstacles, for example, that many students encapsulate the *process* of getting smaller as an *object* that *is* arbitrarily small - a cognitive infinitesmal (Cornu 1981).

An alternative sequence is:

(a) explore the notion of local straightness

(b) visualize the changing gradient of the graph as another graph

(c) relate the visual picture of the gradient to the numerical algorithm to provide analogue insight into the underlying numerical computer process

(d) relate these experience to other representations, including the numerical and algebraic limiting processes.

The new School Mathematics Project 16-19 curriculum - designed by teachers for their own use in schools - is based on this approach. However, I found that cultural forces in the teachers operated in an unexpected way. So disenchanted were they with student difficulties with symbolic differentiation from first principles that the final link (d) has been postponed to the second year of the syllabus. It concerns me to see that this final link is not cemented early - in the first year the students do not even differentiate x^2 from first principles.

The cognitive root of local straightness is recalled as an anchoring concept whenever new concepts are introduced in the S.M.P. curriculum. For instance, first order differential equations - where dy/dx is known as a function of x and y - are approached using a generic organizer called the *Solution Sketcher* - a piece of software that will draw a short line segment through a point with the gradient calculated from the expression for dy/dx. The student moves the line segment around with the cursor keys (or mouse if the computer has one) and can leave the imprint of the line segment at any stage by pressing the space bar (or clicking the mouse). In this way the user can enact the building of a solution of a differential equation by fitting together short line segments end to end to give an approximate solution curve. Once the cognitive root is established, the process is speeded up by automating the numerical process. Again the formalities of solving the equations symbolically have been postponed to a later time.

Versatile learning

Generic organizers may be used to give a more overall holistic grasp of concepts, linking them together in a global, often visual way, as distinct from the accent on learning sequential processes of mathematics in the traditional curriculum. The plan is to use global/holistic insight to provide a context for relational understanding of logical/sequential processing. The combination of the complementary modes of global/holistic and logical/sequential learning is termed *versatile* (Brumby 1982).

Thomas (1988) developed a versatile approach to variables in algebra combining several different modes of operation: a *cybernetic* environment involving both pupils programming and using software to evaluate algebraic expressions numerically, an *inanimate* organizer using boxes with letters for labels and numbers inside for values, and *interpersonal* cooperation with other pupils and the teacher. The purpose of the physical manipulation of boxes is to give analogue insight into the way the computer handles variables. The software to evaluate expressions accepts either computer notation or mathematical notation and provides a cybernetic environment to evaluate single expressions or to compare the results of two expressions, such as 3+4a and 7a or 2(a+b) and 2a+2b or $(x^2-1)/(x-1)$ and x+1.

The improvement of experimental students over matched controls has been documented elsewhere (Tall & Thomas, 1988), but an analysis of student comments in interviews is even more revealing (Thomas & Tall 1989). Control pupils learning algebra through traditional manipulation of symbols were much more likely to read expressions sequentially from left to right and less able to see expressions or subexpressions as objects in their own right. Thus they were more likely to misread 2+3c as 5c or consider $\frac{6}{7}$ as different from 6÷7 because the first is a fraction and the second is a sum.

Having solved 2p-1=5 to find p=3, faced with the equation 2p-1=5, the experimental students were more likely to see the equations were essentially the same and be able to say why, as can be seen from these comments taken from Thomas & Tall 1989:

Experimental (computer) group :

> Pupil 1 : I can say that p and s have the same value...it’s the same sum.
>
> Pupil 2 : Well they are both the same...Yes, because they are both the same but different letters.

Pupil 3 : They are both...p and s both equal 3.

Pupil 4 : It's just a different letter but it would have to be 2 times 3 minus 1 equal to 5.

Pupil 5 : The same. Just using a different letter.

Pupil 8 : It is 3 the p and s...because they are basically the same sum, but are different letters.

Pupil 9 : They are both the same. It's the same apart from the letters,exactly the same except the letters.

Control group :

Those unsure of the relationship :

Pupil 10 : s could be 3 as well.

Pupil 12 : So s could be 3 as well.

Pupil 13 : They could both equal 4.

Those who needed to solve both equations:

Pupil 11 : Well what I have put is 2p equals 6 and 2s equals 6.

Pupil 14 : 2s...add the 1 and 5, 6 er 2 and 2, 6, 3 times, so s is 3 as well.

When they were then faced with the equation

$$2(p+1)-1=5,$$

the experimental students were more likely to see the link with the previous example, noting $p+1=3$, so $p=2$, whilst the control students were more likely to feel the need to start from scratch by multiplying out the brackets.

In addition to the evidence from the calculus and the early learning of algebra, we have further evidence of the enhancement of versatile learning in the linking of the algebraic symbols for linear forms and the shape of the straight line graphs using a graph plotter (Blackett 1988), and in the use of software to give pupils a better insight into trigonometric ratios (Blackett, in preparation). In the latter case, pupils were asked what happens to the angles of a triangle if the side-lengths are doubled. Those using software which drew triangles when given appropriate data all realized that the angles were unaltered. Of a less able group taking a more standard paper and pencil approach (including drawing pictures) approximately three quarters asserted that the angles would be doubled in size.

In the early seventies, long before computers became a practical reality in the classroom, Skemp commented that it is an accident of evolution that *homo sapiens* has modes of verbal output and input through speech and hearing, a mode of visual input (sight) but no mode of visual output:

> "We have built-in loudspeakers, but not built-in projectors." (Skemp,1971)

He analyzed the different characteristics of algebraic and visual symbolism: the former he saw as analytic, detailed, logical and sequential, the latter as integrative, holistic, and capable of simultaneous representation of ideas. It is my belief that traditional mathematics emphasizes the symbolic and sequential at the expense of the integrative and holistic and that, whilst the *proving* of mathematical ideas requires the former, the *building* of such ideas also requires the latter. Suitably programmed software can provide a tool which compensates for the human deficiency in visual communciation.

The computer gives us an unrivalled opportunity of building new curricula that redress the balance towards a more versatile form of thinking appropriate for the new technological age.

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