The Nature of Advanced Mathematical Thinking

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The Working Group on Advanced Mathematical Thinking was formed at the Conference of P.M.E. in 1985, yet discussion since that time has revealed difficulties in specifying the distinction between Advanced Mathematical Thinking (AMT) and Non-Advanced Mathematical Thinking, which will here be called "Elementary Mathematical Thinking" (EMT). The purpose of this paper is to initiate further discussion on the topic to encourage a more refined and specific meaning for the subject.

An initial problem that occurs is that the name can be interpreted in at least two distinct ways, as "advanced forms of mathematical thinking", or as "thinking related to advanced mathematics". Earlier discussions in the Advanced Mathematical Thinking Working Group tended to prefer ambiguity on this point, accepting either, or both. From a pragmatic viewpoint, the work of the group is concerned with extending the theory of the psychology of mathematical education to later age groups, which we have arbitrarily fixed at the age of 16+, to include the later years of secondary education, the transition to university mathematics, undergraduate mathematics as a major study and as a service subject, through to the kind of thinking employed in mathematical research. Our intended aim is to produce a coherent view of the cognitive development of mathematical thinking from its earliest beginnings in childhood to the most abstract thought of the research mathematician.

During this development, mathematical thinking becomes progressively more sophisticated, and by their early teens children are already using quite abstract ideas in algebra, so the distinction between elementary and advanced mathematics is not a simple matter of abstraction, although we shall see that there is a difference in manner in which the abstraction is used.

Generalization and abstraction

Generalization is the process of forming general conclusions from particular instances. The term also applies to the concept produced by the process, for instance "$a+b=b+a$" is considered an algebraic generalization of the arithmetic statement "3+2=2+4" and $\boldsymbol{R}^n$ is a generalization of $\boldsymbol{R}^2$. Generalization (and the

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complementary process of specialization) is common to both elementary and advanced mathematical thinking.

Abstraction is the isolation of specific attributes of a concept so that they can be considered separately from the other attributes. Abstraction is often coupled with generalization. But the two are by no means synonymous. For instance, the solution of linear equations in two variables may be seen as a generalization to the process of solving linear equations in three variables. Although one may argue that there is an implicit abstraction of the solution process, the more general process is, in this case, certainly no more abstract.

On the other hand, it sometimes happens that when an abstraction occurs, the properties abstracted are such that they uniquely determing the original concept. One example is the abstraction of the notion of a complete ordered field from the real numbers. Another is the abstraction of the group concept from groups of transformations – Cayley's theorem shows that every abstract group is isomorphic to a group of transformations – so abstract groups are no more general than transformation groups.

However these latter examples are singularities in the process of abstraction. In general (!) abstraction serves two purposes:

> (a) Any arguments which apply to the abstracted properties apply to other instances where the abstracted properties hold, so (provided that there *are* other instances) the arguments are *more general*.
>
> (b) By concentrating on the abstracted properties and ignoring all others, the abstraction *should* involve less cognitive strain.

In the latter case, however, although *mathematically* there is concentration only on the salient properties, *cognitively* there are obstacles to overcome.

There is a clear cognitive difference between generalizations and abstractions. A generalization involves the *expansion* of a cognitive schema: the generalization sets the particular cases in a broader context which enhances their properties without violating them in any way. If there are difficulties, the difficulties lie in the comprehension of the generalization. The mental process of abstraction involves a *reconstruction* of the cognitive schema: any properties of the abstraction (which may also be properties of the original concept) must be deduced from the abstracted properties alone and seen not to depend on any implicit assumptions concerning other properties of the original concept. This is almost always likely to be accompanied by a period of confusion as the cognitive structure is reorganized.

An example is seen in linear algebra. The generalization is to $\boldsymbol{R}^n$, so that all the processes are involved with n-tuples of "real numbers" where the latter are just familiar decimals, without any abstraction of the field properties of $\boldsymbol{R}$. The abstraction is to a vector space $\boldsymbol{V}$ over the field $\boldsymbol{F}$ of real numbers. Both superficially have the same ideas : solving linear equations, linear independence, spanning sets, dimension, subspaces, linear maps, kernel, image, isomorphism, determinants etc. The generalization (hopefully) involves familiar ideas in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, complicated by having more coordinates, thus expanding the cognitive structure, and only causing problems in the increased detail that must be grasped. The abstraction involves deduction from concept definitions, giving power and generality, abstract simplicity yet creating cognitive problems of conceptual reconstruction.

It is here that the terms *concept image* and *concept definition* are valuable (Tall & Vinner 1981):

> The term *concept image* [is] the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.
>
> ... the *concept definition* [is the] form of words used to specify that concept.

The concept definition has been abstracted by previous generations of mathematicians from their mathematical experiences. The student is asked to accept this concept definition as a list of abstracted properties that characterize the concept, involving reconstruction of the concept image to generate the logical deductions and to isolate the subset of properties deduced from the concept definition.

Justification and Proof

The 16+ School mathematics curriculum in England claims to introduce students to the idea of proof, though at this early stage the concept is usually developed in an informal way. However, I would contend that the type of proof envisaged by many students at this stage is more in the nature of a *justification* than a *logical necessity*. For example, in using Graphic Calculus to show that the gradient of x^n is nx^{n-1}, I have many times worked with students as they establish that the numerical gradient of the graph of x^2 approximates to $2x$ and the numerical gradient of x^3 is approximately $3x^2$. Students usually generalize to state that the gradient function of x^n is nx^{n-1}. However my experience is that when asked “don't we need to *prove* this for general n”, they often respond with the comment “Tell me n, and we’ll check with the program” – for instance, if n=1/2 then, checking with the program will confirm that the gradient is again nx^{n-1} (for n=1/2)). The form of proof here is analogous to scientific proof: to show that water boils at 100˚ C, just boil it and confirm that it does. It is also dangerously close to the format of the proof of continuity - give me an ε and I'll find a δ - only the student says “give me a specific value of n for $f(x)=x^n$, and I'll find a small value of h so that the graph of $\frac{f(x+h)-f(x)}{h}$ looks like nx^{n-1} ...”

The difference between justification and proof is also related to the mathematical objects on which the argument is based. In elementary mathematics the objects are first concrete objects in the environment whose existence is confirmed by a combination of the five senses, and then mathematical abstractions (such as number) which can be related directly to reality. Advanced mathematics is more often based on mathematical objects which are defined in terms of their *properties*. The latter may have been abstracted from reality, and the initial definitions may implicitly depend on the students’ previous experiences. For instance a function may be defined in the following terms:

> A function f from a set A to a set B is a rule or process that associates with each element of A a unique corresponding element of B

which actually required a sophisticated concept image in order that the terms used are given appropriate meanings. Having so “defined” a function, further definitions may be given in terms of properties, for instance:

> A function f from A to B is *injective* if $f(x)=f(x')$ implies $x=x'$.

To be able to cope with this definition, the student has to realize that x and x' denote elements of the set A which have different names (x and x') but the elements may be the same, indeed, in the end *they are the same*! Once one understands the game, it is a trivial matter to prove that:

> If f and g are injective functions, then the composite $g \circ f$ is injective.
>
> Proof: If $g \circ f(x)=g \circ f(x')$
>
> then, by definition, $g(f(x))=g(f(x'))$,
>
> so $f(x)=f(x')$ (because g is injective)
>
> Hence $x=x'$ (because f is injective.
>
> Thus $g \circ f$ is injective.

However, for many beginning university students, this is a very difficult proof because they do not understand the way advanced mathematicians *define* concepts by properties and then prove theorems by manipulating the properties of the concept definition.

Deduction and implication

It is also important cognitively to distinguish between two kinds of proof that are *mathematically* equivalent:

(1) The *deduction* of properties of mathematical concepts in a given context,

(2) The *implication* that *if* certain properties hold, *then* others will follow.

Typical examples of (1) might be

There are an infinite number of primes,

The derivative of *cosx* is -*sinx*,

In a group, $(a*b)^{-1}=b^{-1}*a^{-1}$,

Typical examples of (2) are:

If a function is differentiable, then it is continuous,

If f and g are injective, then so is the composite $g\circ f$.

In (1) there is an implicit conceptual field in which certain properties are assumed to hold and other properties then *exist* within this field. In (2) specific assumptions are made explicit and the existence of other properties is contingent on the explicit properties being satisfied. Given the conceptual field, the properties proved by deduction then have a *cognitive existence* and can be used to build up a powerful and stable edifice in the mind. The conditional implication has a lack of closure that means the mind must always be alert to the existence of the necessary pre-conditions. Particularly difficult from a cognitive viewpoint are statements of the kind $P \Leftrightarrow Q$ where it is necessary first to assume P is true to prove Q, then suspend one's belief in P and assume Q is true to prove P.

Thinking in advanced mathematics

Advanced mathematics in the twentieth century is built on the scaffolding of axiomatic theories, where the axioms are no longer the "self-evident" truths of the Greeks, but concept definitions which are set-theoretically formulated abstractions. The knowledge structure of advanced mathematics is based on such abstract scaffolding. Thinking in advanced mathematics is more than just the finished structure of the mathematical theory. It concerns mathematical thinking that *creates* such a knowledge structure, *communicates* it to others who then *re-create* such a knowledge structure in their own minds and *use* the forms of mathematical thinking to solve problems which are new to the individual (though perhaps not to the mathematical culture as a whole). Thinking in advanced mathematics is not always a logical process, for the creation of mathematical ideas involve associative resonances between previously disconnected ideas.

Using the notions that have been distilled by mathematicians such as Poincaré, Hadamard, Polya, and more recent mathematical educators (for instance Mason, Burton & Stacey 1982), we should recognize a number of distinct phases in mathematical thinking. For instance, Mason *et al* identify three stages: *entry*, *attack* and *review*. In the entry phase the context is clarified, identifying what is known and what is wanted, before the main phase of attack occurs. Such an attack can involve both deductive and associative processes, and once a problem has been well and truly entered, a period of mulling or incubation may well be necessary before an idea for a solution may occur. Finally the review phase checks what has been done and reflects on it before looking forward to extend the ideas in new avenues.

Advanced mathematics includes all these phases, enriching the review phase to include the building of the knowledge structure into a formal sequence of deductions from concept definitions, a stage of thinking which Hadamard terms "precising".

Advanced thinking in mathematics

It is significant to note that the book "Mathematical Thinking" (Mason et al 1982), which includes problems that can be attacked by a wide range of abilities and ages, does not explicitly refer to *proof* in its overall strategy, placing the emphasis instead on three levels of *justification*:

> Convince yourself
> Convince a friend
> Convince an enemy.

The third of these might be seen as a kind of proof, but only if the enemy is a mathematician who demands a deductive proof from concept definitions. The existence of the final "precising phase" of proof is the significant difference between EMT and AMT. This stage is what I would term "advanced thinking in mathematics", as opposed to the creative stage that occurs in attack, which involves creative "thinking about advanced mathematical concepts". There is creative thinking in both EMT and AMT. Both often involve generalization, but AMT usually involves abstraction, particularly in the use of concept definitions for associative and logical deduction.

The logical proof structure developed in the "precising" stage is one of the crowning glories of advanced mathematics. Regrettably, too often it is only the *results* of this stage that are passed on to mathematics students; giving them (in Skemp's terminology) the *product* of mathematical thought rather than the *process* of mathematical thinking.

This advanced stage of mathematical thinking requires the abstraction of generative properties from mathematical concepts to produce concept definitions which may be manipulated abstractly to develop the logical relationships between them. Whereas the keyword of EMT is *coherence* between related concepts that fit together, in AMT it is *consequence*, involving logical deduction from concept definitions.

AMT

I consider AMT to be any part of the complete process of mathematical problem-solving, from the creative processes involving deductive and associative resonances between previously unrelated, or even undefined, concepts, through to the final "precising" process of mathematical proof.

This description (for it is hardly a definition) is *global*, rather than local. A specific process may be designated as advanced mathematical thinking because it is part (or even potentially part) of the complete cycle of mathematical problem-solving. Notice therefore that the *use* of ideas developed through advanced mathematical ideas, such as an algorithm for solving a problem, is therefore designated as AMT, even though it may not involve the whole cycle of mathematical thinking. Because of the abstraction involved in the process (through concept definition and theoretical deduction), AMT occurs in a mathematical conceptual field where there are appropriate abstract mathematical structures available to build up a network of deduced relationships. In significant research, AMT includes the development of new conceptual fields.

Although an essential criterion is *proof*, or the later possibility of proof, advanced mathematical thinking can, and does, occur at times when logical deduction is absent. In particular the creative act in mathematical research through the resonance of previously unrelated mathematical ideas is classified as (an extremely important part of) AMT, even though it may involve little, or no logic, at any given time.

AMT cannot exist in a vacuum. The constructions of appropriate concept definitions rely on previous experience from which the generative properties can usefully be abstracted. There are preliminary activities laying the groundwork for AMT which introduce concepts not immediately abstracted from reality, such as the mathematical notion of an infinite process, the notion of a limit, or of cardinal infinity. These are involved in the transition from EMT to AMT and much material taught in the calculus in the late secondary school comes under this heading. It is a moot point whether this is classified as AMT or not, yet it is an important foundation for AMT. I would propose that this comes under the heading of AMT, though in a transition phase, for it is laying the foundations for ideas which will eventually be abstracted in the advanced mathematical thinking of mathematical analysis.

Distinguishing features of Advanced Mathematical Thinking

Because advanced mathematical thinking has been defined here in terms of the overall conceptual structure in which it takes place, it is not possible (at least, I have found it difficult!) to specify a list of criterial attributes that distinguish a particular instance of mathematical thinking as being advanced rather than elementary. However, for the purpose of discussion, it may be valuable to end up by listing some of the characteristics which have arisen during the course of this paper:

(1) The *abstraction* of properties to provide *concept definitions* for mathematical concepts,

(2) The use of abstract mathematical concept definitions to ease cognitive strain in thinking,

(3) The insistence on logical proof rather than coherent justification, which involves:

(4) The *deduction* of properties of mathematical concepts (from given concept definitions),

(5) The *implication* that *if* certain mathematical properties hold, *then* others follow.

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