THE MUTUAL RELATIONSHIP BETWEEN HIGHER MATHEMATICS AND A COMPLETE COGNITIVE THEORY FOR MATHEMATICAL EDUCATION

[In English, with a French Abstract]

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Il existe une relation entre les mathématiques superieures et la théorie cognitive qui devrait leur être d'un mutuel profit. La théorie cognitive sera enrichie si elle tient compte des exemples divers de la pensée mathematique, et inversement la théorie cognitive qui peut adopter ces modes de pensée peut contribuer à la compréhension des mathématiques.

Ainsi, la recherche dans ces domaines de la pensée peut être profitable de nombreuses façons:

1. *Pour l'apprentissage des mathématiques au niveau des annees terminales et universitaires.*
2. *Pour le développement d'une théorie cognitive plus complète de l'enseignement des mathématiques.*
3. *Pour la compréhension de [‘aspect cognitif des mathématiques et de l'histoire des mathématiques, du processus créateur de la recherche et de l'attitude des mathématicians de métier envers leur sujet.*

On considérera la recherche récente qui a étudié la différence entre les définitions mathématiques formelles et les significations personelles données aux concepts par les individus. Cette recherche a révélé des différences frappantes entre la théorie formelle et la perception qu'en ont les etudiants et mathématiciens professionals. Même si l'on enseigne des définitions formelles, l'imagerie conceptuelle de l'étudiant dependra des expériences de l'individu et pourra être très différente de la théorie formelle.

La recherche met en valeur la question centrale de la “signification” en mathematiques et suggère qu'une theorie générale et adaptée à l'enseignement des mathématiques devra se fonder sur une acquisition “significative” de la connaissance, c'est a dire reliant la croissance des structures cognitives chez l'individu aux mathématiques à étudier et aux processus de pensée à developper.

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INTRODUCTION

The psychology of mathematical education to date has been mainly concerned with the learning processes of children and the methods necessary to educate them in current mathematical theories. There have been far fewer studies of cognitive development at higher levels, with implications that cut two ways. On the one hand, the lack of knowledge of cognitive processes at more advanced stages of education can lead to weaknesses in the teaching of mathematics at college and university. On the other hand, the lack of study at this level has severely hampered the development of a complete cognitive theory of mathematical education by excluding the rich and varied examples of more sophisticated mathematical thinking. This lack of understanding of higher mathematical thinking has another serious implication: because the thinking processes of professional mathematicians are not well understood, this impairs our understanding of the nature of mathematics itself.

THE LEARNING OF MATHEMATICS AT COLLEGE AND UNIVERSITY

The most immediate application of cognitive studies at higher level is to provide a framework for the reassessment of teaching and learning of mathematics at college and university. My own work has concentrated mainly on the study of calculus and analysis: infinite processes, the concept of infinity, limits, continuity, differentiation, integration, the nature of number systems, use of infinitesimals, understanding of proofs, and so on.

A key idea that has helped in these studies is the distinction between a concept definition, which is the form of words used to describe a concept, and the concept image, which is the cognitive structure in the mind of an individual that is related to the concept (Vinner & Hershkowitz 1980, Tall & Vinner 1981). The concept image is more than a mental picture; for instance it is partially generated by the related processes experienced by the individual.

Suitably worded questionnaires have revealed the diverse nature of students' concept images in mathematics (see, for example, Schwarzenberger & Tall 1978, Cornu 1980, Tall & Vinner 1981). Mathematical terms like "function", "limit", "tends to", "continuous", and so on, all evoke a variety of concept images and the images evoked in a single individual can vary with the context.

The notion of concept image is useful for describing the development of understanding of axiomatic theories. For example, an answer to the question "what is a mathematical group?" might be to list the group axioms. But this is just the concept definition. To each individual the notion of a group is more than that: he has his own concept image (possibly empty) of the group concept developed through experience of manipulating the theory. This experience leads to a "feeling" for the concept generated by sensory input reacting with the concept image in his cognitive structure. In particular, each individual's intuition for a concept is a direct result of his own concept image.

The development of concept images may be usefully encouraged in the first place by presenting the individual with generic processes and generic examples: these are specific cases from which the individual can abstract the general theory. The technique is common in education at all levels, be it the interpretation of the specific statement 2+3=3+2 as a generic example of the commutative law or the generic method of solving any given set of linear equations through a few well-known examples. Formally such examples play a redundant role in higher mathematics: an individual case never proves a general theorem. But in cognitive terms their use may be crucial because abstraction from generic examples seems to be an essential way in which human beings form concepts.

Investigations into conceptual imagery can lead to new strategies for teaching, by providing students with experiences that help in the creation of a concept image that is consistent with, and supportive of, the formal structure of mathematics. These experiences may themselves be formally unnecessary.

In analysis, for example, there is a school of thought which excludes the use of pictures because they are thought to give false intuitions. On the assumption mentioned above, that intuition is a direct result of concept image, it follows that true intuitions are more likely to come from a suitably developed concept image. By suitably formulating concept definitions, pictorial ideas may be used with great profit. As an illustration, one may define a function $f{:}D \to \mathbb{R}$ (from a subset D of the real numbers $\mathbb{R}$) to be *pictorially continuous* if over any closed interval $[a,b]$ in D,

given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x,y \in [a,b]$,

$|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$.

It is easy to show that, given a pencil that draws a line of given thickness, the graph of a pictorially continuous function can be drawn to any specified scale over a closed interval $[a,b]$ in its domain without the pencil leaving the paper. What actually happens is that the graph lies inside the pencil line.

It is also easy to show that if f is differentiable at some point x_0, then given a piece of paper of any specified width and a pencil which draws a line of specified thickness, there is a small interval containing x_0 such that the graph over this interval scaled up to the width of the paper can be drawn inside a straight pencil line. This process can be exemplified using high-resolution graphics on a computer, giving valuable cognitive support.

Based on these ideas it is easy to give students a recursive method of drawing an everywhere continuous, nowhere differentiable function. By physically drawing the successive approximations they may gain a psychomotor feeling for the properties of the function and by using the concept definitions and properties just mentioned these intuitions may be translated directly into a formal proof (see Tall, 1981).

By reformulating mathematics, taking into account student's concept imagery, the theory may be enriched and made meaningful to a wider range of students.

A MEANINGFUL THEORY OF MATHEMATICAL THINKING

The study of mathematical thinking at higher levels demands an appropriate cognitive framework. In my own investigations, behaviourist theories which refuse to speculate on the nature of the thinking process have proved to be of little practical value. An extension of Piaget's theory of stages to higher levels also seems inappropriate. It is my belief that the best kind of overall theory of cognitive development is one which relates the developing cognitive structure of the individual to the conceptual framework that he either creates or is expected to master. Two useful existing theories which satisfy these criteria are those of Ausubel *et al*., 1978 and Skemp, 1979; they both apply to *all* individuals at *all* ages.

In a meaningful learning theory, the individual's concept image of the mathematics he is expected to master is of paramount importance. The cognitive development is likely to pass through transition phases where new information causes a restructuring of the concept image; this may involve a period of conflict before the resolution leads to a new stage of thinking, as observed by Piaget. But the theory would suggest not a small number of Piagetian stages, but many transitions in many conceptual areas throughout life. It is the study of such transitions and how they may be effected which I believe to be a matter of central importance in a cognitive theory of mathematical education.

THE NATURE OF MATHEMATICS

Given an adequate cognitive theory, the study of the processes of mathematical research may reveal insights into the nature of mathematics itself. A recent personal investigation (Tall, 1980) confirmed the classical accounts (e.g. Hadamard, 1945) that the activity is anything but logical, with the individual doing the research painfully putting together conceptual images from his cognitive structure, groping intuitively for new patterns (often inaccurately) long before they could be logically verified.

There is a subtle blend of choice and consequence in research: the mathematician chooses (or invents) his starting points, implicitly or explicitly (these may include his concept definitions, his axioms and, to a certain extent, his rules of procedure) but from then on there are logical consequences implicitly built into the system which he must *discover*.

Educationists would do well to note this balance of choice and consequence, invention and discovery, in mathematical theories. Many decisions in mathematical education have been based on arbitrary starting points, chosen by mathematicians for mathematical reasons, and such starting points may be inappropriate for cognitive development. For instance, Piaget's notion of conservation of number is implicitly built on Cantor and Frege's choice of one-one correspondences between sets for the starting point for the theory of cardinal number. The mathematical theory was never intended to take into account the cognitive development of the child, where repetitive processes of counting fit naturally into the human development of action schemata. A reappraisal of the theory of cardinal numbers, as in Freudenthal 1973 or Stewart & Tall 1979, shows that the emphasis on one-one correspondence at the expense of counting is unwarranted.

The mathematics educationist therefore needs a flexible view of mathematics, one which attempts to see it through the eyes of the learner and reformulates the structure in a potentially meaningful way. In doing so, one cannot escape the need to know something of the higher realms of mathematics, so that it can be made the servant of the educational process rather than the master.

Thus the circle closes: a theory of cognitive development enhanced by studies in higher mathematics may be applied to understand and modify the higher mathematics itself.

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