The Calculus of Leibniz — An Alternative Modern Approach

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Two closely linked items appeared in Volume 1, Number 3 of *The Mathematical Intelligencer:* the paper of Lakatos [2], which refers to the interpretation of Leibniz's continuum in non-standard terms, and a review [1] of Keisler's text for teaching analysis by non-standard methods. New light can be thrown on both of these articles by viewing the calculus of Leibniz through an alternative modern approach which is outlined here.

A central idea in the calculus of Leibniz, which is missing in non-standard analysis, is the notion of the *order* of an infinitesimal. (If a hyperreal infinitesimal were of first order, then its square root would be an infinitesimal of order one half.) The reason for the loss of the notion of the order of infinitesimals is that the hyperreal system is "too large" because it has to handle the calculus of such a wide class of functions. If we restrict ourselves to the repertoire of analytic functions, then we can develop a simple algebraic version of infinitesimal calculus without recourse to first order logic or the broad theory of hyperreals. We only require a number line, where all infinitesimals have specified orders, which we call the *superreal* numbers (to indicate that they lie between the real numbers and the hyperreals).

The superreal numbers $\mathscr{R}$ are already well-known as the field $\mathbb{R}((\epsilon))$ of power series in ϵ of the form

$$a_{-m}\epsilon^{-m} + \ldots + a_{-1}\epsilon^{-1} + \sum_{r=0}^{\infty} a_r\epsilon^r$$

where the coefficients a_i may be any real numbers whatsoever. These may be ordered by defining an element $\sum_{r=k}^{\infty} a_r\epsilon^r$ to be "positive" if the first non-vanishing coeffient a_k is positive. $\mathscr{R}$ is then an ordered field in which ϵ and $a - \epsilon$ are positive for any real number $a > 0$. If we write $\alpha < \beta$ instead of "$\beta - \alpha$ is positive", then

$$0 < \epsilon < a \quad \text{for every positive } a \in \mathbb{R},$$

in other words ϵ is an infinitesimal.

We can likewise define the *order* $o(\alpha)$ of any non-zero superreal α to be

$$o\left(\sum_{r=k}^{\infty} a_r\epsilon^r\right) = k \quad (\text{for } a_k \neq 0),$$

and then infinitesimals are precisely the elements of positive order. Elements of negative order $n = -m$, whose expansions $\sum_{r=-m}^{\infty} a_r\epsilon^r$ include a finite number of terms in $1/\epsilon$, are either larger or smaller than every real number, depending on the sign of a_{-m}. These elements are *infinite,* while elements of non-negative order are just power series in ϵ, $\sum_{r=0}^{\infty} a_r\epsilon^r$ and these are *finite.*

In a picture to finite scale we can only represent finite points $\alpha = \sum_{r=0}^{\infty} a_r\epsilon^r$, and then such an element is indistinguishable from its so-called *standard part* st $\alpha = a_0$, because the remainder of the power series $\sum_{r=1}^{\infty} a_r\epsilon^r$ is infinitesimal. Nevertheless, we can use the map $\mu(x) = (x - \alpha)/\sigma$, which translates α to the origin and then applies a scale factor $1/\sigma$ to the rest of the superreal line; by allowing α or σ (or both) to be infinite or infinitesimal we may see details not found in a normal-scale picture. Of course, we shall only see the standard part of $(x - \alpha)/\sigma$, and then only when $(x - \alpha)/\sigma$ is finite (when $o(x - \alpha) \leqslant o(\sigma)$); with all this in mind we define the *optical σ-lens* aimed at $\alpha \in \mathscr{R}$ to be $\nu_\sigma : I(\alpha, \sigma) \to \mathbb{R}$ where

$$I(\alpha, \sigma) = \{a + \theta \in \mathscr{R} \,|\, o(\theta) \leqslant o(\sigma)\}$$

and

$$\nu_\sigma(x) = \text{st}((x - \alpha)/\sigma).$$

The same idea works in the plane, defining

$$\nu_\sigma(x, y) = (\text{st}((x - \alpha)/\sigma), \text{st}((y - \beta)/\sigma))$$

where the orders of $x - \alpha$ and $y - \beta$ do not exceed that of σ. If σ is infinitesimal, then ν_σ is called an *infinitesimal microscope.*

An analytic function $f : D \to \mathbb{R}$ may be extended to a superreal function $f : D^\# \to \mathscr{R}$ where

$$D^\# = \{x + \delta \in \mathscr{R} \,|\, x \in D, \delta \text{ infinitesimal}\}$$

using the power series expansion

$$f(x + h) = \sum_{r=0}^{\infty} a_r h^r \quad |h| < K$$

and putting

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$$f(x+\delta) = \sum_{r=0}^{\infty} a_r \delta^r.$$

The derivative of an analytic function is then given as

$$f'(x) = \mathrm{st}\left(\frac{f(x+\delta)-f(x)}{\delta}\right)$$

for a non-zero infinitesimal δ. When we observe this phenomenon through an optical infinitesimal microscope, we find that an infinitesimal portion of the graph looks like a straight line of slope $f'(x)$:

Theorem. *If $f: D \to \mathbb{R}$ is an analytic function, $x \in D$, $\delta \neq 0$ is an infinitesimal and $o(\theta) \leqslant o(\delta)$,* then aiming ν_δ *at $(x, f(x))$ we have*

$$\nu_\delta(x+\theta, f(x+\theta)) = (\lambda, \lambda f'(x))$$

where λ is the real number $\mathrm{st}(\theta/\delta)$.

Proof. $\nu_\delta(x+\theta, f(x+\theta)) = (\mathrm{st}(\theta/\delta), \mathrm{st}((f(x+\theta)-f(x))/\delta)$

$$= \left(\lambda, \mathrm{st}\left(\frac{(f(x+\theta)-f(x))}{\theta}\,\frac{\theta}{\delta}\right)\right)$$

$$= (\lambda, f'(x)\lambda).$$

A picture of this is given in Figure 1.

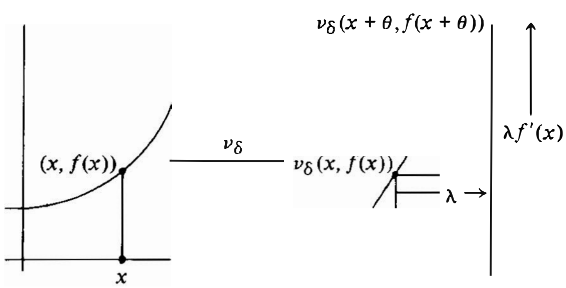

Figure 1

Integration is also easily handled by letting the area under the graph from some fixed point a to a variable point x be $F(x)$, then the area $F(x+\delta) - F(x)$ over the infinitesimal interval from x to $x+\delta$ is the sum of a rectangle area $f(x)\delta$, plus terms of higher order (beginning with a triangle of area $\frac{1}{2}f'(x)\delta^2$). Hence

$$\{F(x+\delta) - F(x)\}/\delta = f(x) + \text{infinitesimal terms}$$

and, taking the standard part, we get the fundamental theorem of calculus

$$F'(x) = f(x).$$

(Full details are given in [3].)

It is well worth noting that the part of Keisler's book least well received by students according to the review [1] was the portion on Riemann integration. In a theory restricted to analytic functions, as here, Riemann integration plays no part, nor was it part of Leibniz's original theory. The chief function of Riemann integration is to demonstrate the *existence* of an antiderivative which may be computed directly in the analytic case.

Since analytic functions serve for most practical cases, this simple model provides a useful introduction to infinitesimal techniques in the calculus, either as an end in itself, or as a concrete computational system prior to the broader theory of non-standard analysis. From a historical viewpoint it provides an alternative modern framework for the calculus of Leibniz.

References

1. Davis, M., Hausner, M.: The joy of infinitesimals, *The Mathematical Intelligencer* 1, 168–170 (1978)
2. Lakatos, I.: Cauchy and the continuum, *The Mathematical Intelligencer* 1, 151–161 (1978)
3. Tall, D. O.: Looking at graphs through infinitesimal microscopes, windows and telescopes, *Mathematical Gazette* (in press)

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