The Dynamics of Understanding Mathematics

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Recent issues of *Mathematics Teaching* have begun what looks like being a very fruitful discussion on the nature of mathematical understanding. There must be nearly as many views of what constitutes ‘understanding’ as there are mathematical educators, but I would like to inject into the discussion the suggestion that overall we must consider the *dynamics* of mathematical understanding. By this I mean the constantly changing mental patterns that give rise to mathematical understanding and which characterise mathematical thinking. The brain is a hive of activity; it takes in information, processes it, distinguishes between things, sees similarities, makes deductions, forgets things, remembers them later, and has mental blocks and leaps of insight. So many of these activities are bewilderingly quick or subconscious that we cannot say what is happening in our own minds. A verbal commentary at the time is inadequate, and subsequent explanation is often only a rationalisation of what we think (or even what we think we think) has happened.

As an example, consider the case of a four-year old child who could count up into the hundreds and knew a few number bonds. He was asked, “What’s seven and seven ?” and replied after a brief pause, “*Fourteen.*” When asked “Why ?” he replied, “*Because it’s two off sixteen.*” “oh, I see—why is that?” “*Because eight and eight are sixteen and it’s two less.*”

We can postulate that the child remembered 8+8=16 (though not in that notation) and had a partial relational understanding of number to use this to deduce that 7+7=14.

A week later he was asked, “What’s seven and seven?” and he could not remember; then after a long pause he said, “*Fourteen.*” “How do you know ?” “*It just is.*” “OK, what’s eight and eight ?” “*Eighteen.*” His elder brother (aged seven) tried to correct him. He was given his own sum to think about, “eighty plus eighty” (a real ‘toughie’), whilst his younger brother reconsidered his own problem. The elder brother replied, “A hundred and sixty,” and younger brother immediately responded, “*It’s sixteen.*” “How do you know?” “*Well, he said a hundred and sixty; then I just knowed.*”

How did he know? Did the ‘six’ in ‘a hundred and sixty’ jog his memory, or did he visualise the more complex problem as ten times his own, in some sense? He had had some experience with ‘tens’ sums, but not as big as ‘eighty plus eighty’. But at this stage the game finished because he did not want to play any more.

Any description of mathematical understanding must allow for this kind of dynamic process—understanding and deducing one week, forgetting and remembering the next.

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This example is one of success; far more difficult to describe is the process of *not* understanding. We have all experienced the blank look of a mental blockage, and here the person concerned is often completely unable to explain the cause of the difficulty. This points to an inherent weakness in the 'tetrahedron of understanding' used by Byers and Herscovics [l]. Briefly they suggest that it is helpful to consider their four kinds of understanding as a vertices of a tetrahedron, then any blend of the four may be represented as a point inside the tetrahedron. But such a representation only exhibits the *ratio* between the four kinds, as may be seen by considering the centroid, where all kinds are present in equal proportions. How does one distinguish in this representation between all four being fully present (if indeed that were possible) and all four being totally absent? (The mathematical representation of four quantities x_1, x_2, x_3, x_4 lying between 0 and 1 requires a point (x_1, x_2, x_3, x_4) in 4-space, not a point in a tetrahedron in three.)

Using a classification of understanding into four (or more) types, we must be prepared to acknowledge the kind(s) of understanding being used as a function of time. In the example given, is not the memory of 8 + 8 = 16 an instrumental act, then the realisation that 7 + 7 is "two off sixteen" a relational one and, for all we know, the final deduction ("He said a hundred and sixty then I just knowed") an intuitive one?

This minor quibble apart, the Byers and Herscovics paper gives much valuable food for thought, building on Richard Skemp's ideas [2], though it does start us on the long slippery slope of introducing more and more refined categories which may not have universal acceptance or universal interpretation. For instance their categories include 'intuitive' and 'formal'; is there an intermediate stage corresponding to Piaget's 'concrete'? Rather than tread the path towards an increase in the number of categories, I would like to suggest a dynamic interpretation which sees the various kinds of understanding as different facets of a single development.

The dynamic development of understanding

By a schema I shall mean a coherent mental activity in the mind of an individual. This exists in time and changes with time. As a child gains experience of life in general, and mathematics in particular, the constraints on his mental activity change. They can develop and become more versatile; they can decay; they can change by conscious and unconscious reformulations of ideas as the child attempts to make a coherent pattern out of the universe he lives in. Understanding which comes about through this search for coherence I would term 'relational understanding'. The example "seven and seven is fourteen . . . because it's two off sixteen" exhibits this kind of understanding, though it requires the remembering of '8+8=16' to produce the final result. 'Instrumental understanding' on the other hand can simply be an exercise of memory, no more. It may be characterised by a compartmentalisation of ideas, not wishing to make an overall pattern and preferring the comfort of a limited closed system. This closure manifests itself as “rules without reason” in Skemp’s description. I personally feel that the distinction between these two types of understanding is often one of attitude, the pieces of information being the factor which distinguishes the one from the other. Note that relational understanding, according to this viewpoint, can occur in very rudimentary situations, as with the child’s addition example, where he not only had very few number bonds available to him (the child concerned could not even write the numerals concerned) yet he understood certain relationships between numbers.

What of ‘intuitive understanding’? I consider that this occurs with a developing schema that is insufficient for the purpose at hand, yet there are facets of the problem to be understood which seem to link with the current schema available. In fact apparent linkages with the current schema may exercise such a pull that the jump in mental state to the final state may come as a blinding revelation, strongly imprinting it on the memory. The important distinction in intuitive understanding is that the person concerned has not reflected on his schema and has not rationalised the way he thinks about it.

John Backhouse [3] does not find the concept of intuitive understanding necessary, though he is more sympathetic to the notion of ‘intuitive thinking.’ However, earlier in his article he mentions the experience of ideas fitting into place—“Ah, now I get it”, “It’s clicked”, and so on. Such a feeling can eminently occur when a problem is solved ‘without prior analysis’ which characterises Byers’ and Herscovics’ definition of intuitive understanding. In fact the ‘aha’ experience of such a solution can have such a strong imprint on the individual that it seems even more true than a deduction made with cool unblemished logic.

In considering the formal category, Backhouse suggests that certain examples of ‘formal understanding’ given by Byers and Herscovics are no more than ‘understanding of form’. Looking at the matter in terms of the schematic development of the individual it may be helpful to distinguish two clearly distinct interpretations. on the one hand, formal understanding may be the type of understanding in which the individual has reflected on his schema and rationalised his thinking as to how it fits together coherently. This is an individual thing and is akin to the use of the term by Piaget. on the other hand there is the ability to put mathematics in a formal context, using the correct notation and so on. This is a corporate thing where the individual has learnt to share the schemes of mature mathematicians in the topic under consideration and is close to the interpretation of Byers and Herscovics. They give an example which they claim shows an absence of formal understanding in which the student writes

$$f(x)=x^2=f'(x)=2x.$$

This is clearly a lack of formal understanding in the corporate sense because the student uses the wrong notation, but if the second ‘=‘ is interpreted as ‘implies’, then the student could possibly have an individual formal understanding without manifesting it in the correct symbolism.

At this stage I do not wish to refine the above explanations any further since my primary target is to see the categories as various facets of an ongoing process rather than a hard and fast partition into qualitatively different types. For the same reason I do not wish to discuss new categories of understanding. If we are not careful, the discussion will degenerate into a battle between disciples of different schools of thought, propounding slogans with eyes blinkered and ears firmly closed.

Non-understanding

Having suggested that understanding be seen as part of the dynamic development of the individual, what about non-understanding? According to the Great Debate this is the subject that we should be addressing. It is clearly more than just absence of understanding.

As an example, take the case of an eight-year old girl learning decimals who read 6.34 as “six point thirty four” and considered it bigger than 6.5 (“six point five”). The child was also taught to multiply by ten by ‘moving all digits one place to the left’. Not only did she not know her left from her right (how many eight-year- olds do?), but she considered the decimal point as a solid object through which nothing could pass, so the process suggested for multiplying by ten was physically impossible. The repercussions of these (and possibly other) confusions made the whole subject of decimals a very fraught one, not solved by mere explanations of the points which initially caused the confusion. The mental reverberations left such an imprint that they could not easily be erased or reformulated.

Skemp talks of understanding as being assimilation into an appropriate schema. The adjusting of a schema itself to take in new material he calls ‘restructuring’ the schema (a more graphic term than Piaget’s ‘accommodation’). So non-understanding in these terms can mean two qualitatively different things: assimilation into an inappropriate schema, or total rejection because no schema is available—suitable or unsuitable. The previous example gives cases of inappropriate schemes. A case of rejection may be cited from a university course teaching mathematics to scientists. They were told that a non-empty subset V of a vector space is called a subspace if given vectors u,v in V and scalars a,b, then $au+bv$ also lies in V. This is an indirect form of definition; it does not say what V is, but describes it by a property. Not having met this type of definition before, many students were perplexed. Despite being told that the definition would get two easy marks in the examination, many complained that, though they tried to learn it by rote, somehow it ‘wouldn’t go in’ and after a time they forgot.

Once again we can see non-understanding as part of the dynamic development of the individual. In an article written with Rolph Schwarzenberger [4] we gave a number of examples of inappropriate mental linkages which could lead to problems. For example, saying 's_n gets as close as we please to s' carries the hidden implication that s_n cannot equal s, since 'close' does not mean 'coincident with' in everyday language. Starting with a simple inappropriate linkage as this, the later ramifications can get out of all proportions compared with the initial misconception. The learner is not always able to articulate the reasons for his confusion, leading to the classic 'mental block'.

The release of a mental block with a sudden leap of insight is one of the most pleasant of mathematical experiences, as previously unconnected ideas resonate together. After struggling with a problem, it may have been left; then after a period of relaxation the answer suddenly comes seemingly from nowhere. Those subconscious processes have been at work again and the removal of inhibiting tension has led to a miraculous solution. (As a student, my leaps of understanding always came whilst browsing in record shops in mid-afternoon, but I always had a hard time explaining that to non-mathematicians!)

Such a distinct leap is not a feature of all understanding. The leap may be so small as to be unnoticeable, or the change in thought may simply seem to be smooth. Nevertheless, classification of understanding must take this very real process into account.

Conclusion

In this article the attempt has been made to see different types of understanding and, equally important, non- understanding, as different facets of the dynamic development of the mental processes of the learner. Division of understanding into different categories may well be helpful on certain occasions, but it can blind us to other possible factors. (A classic example of unhelpful classification is the psychological distinction between 'cognitive' (knowing) factors and 'affective' (emotional) factors. Any mathematics teacher looking at the furrowed brow of a mental block or the look of delight in a leap of understanding will know that many emotions are linked to cognitive factors, rather than in a separate category.) Any useful classification of mathematical understanding will only prove itself if it can unambiguously describe categories of mathematical understanding and non-understanding in a way that exhibits the realities of the situation: remembering, forgetting, mental blocks, leaps of insight, and various other phenomena to be found in mathematical thinking.

References

[1] V. Byers & N. Herscovics: Understanding School Mathematics (MT No. 81).

[2] R. R. Skemp: Relational Understanding and Instrumental Understanding (MT No. 77).

[3] J. K. Backhouse: Understanding School Mathematics—A Comment (MT No. 82).

[4] D. O. Tall & R. L. E. Schwarzenberger: Conflicts in the Learning of Real Numbers and Limits (MT No. 82).