* CHAPTER 1 *

About this Book

Mathematics is a subject with patterns that generate enormous pleasure for some and problems that cause impossible difficulties for others. The situation is exacerbated by different views of what mathematics is and how it should be taught. This book takes a journey from the early conceptions of the newborn child to the frontiers of mathematical research. Its purpose is to present an account that can be shared, not only by experts in a wide range of disciplines, but also by teachers and learners at all levels, in a manner appropriate for their needs. At its foundation is the most fundamental question of all:

> How is it that humans can learn to think mathematically in a way that is far more subtle than the possibilities available for other species?

By focusing on foundational issues and relating them to the long-term development of the subject, it becomes possible to express general ideas at all levels within a single framework, from the ways in which we make sense of the world around us through our perceptions and actions, to the development of more sophisticated ideas using language and symbolism.

Contrary to common belief, new levels of mathematical thinking are not necessarily built logically and consistently on previous experience. Some experiences at one level may be supportive at the next but others may be problematic. For instance, number bonds learnt in whole number arithmetic continue to be supportive in fractions and decimals but the experience of multiplying whole numbers sets an expectation that the product is always larger. This becomes problematic when multiplying fractions. Everyday experience tells us that 'taking something away leaves something smaller'. This works for whole numbers and fractions, but it becomes problematic when subtracting a negative number. Over time, supportive aspects encourage progress and give pleasure, while problematic aspects may cause frustration and anxiety that can severely impede learning in new contexts. As differing individuals respond in varying ways to their experiences, there arises a wide spectrum of attitude and progress in making sense of mathematics.

The foundational ideas in this framework prove to be applicable not only in the teaching and learning of mathematics, but also in the study of its historical development. Even expert mathematicians begin their lives as newborn children and need to develop their mathematical ideas to mature levels in their own cultural environment.

This chapter will lay out all the main ideas of the framework, which will then be considered in greater detail in the remainder of the book.

John, aged six, sat anxiously at the back of his class as his teacher called out the problems. His page had the numbers one to ten down the left-hand side ready for ten sums in the very first Key Stage One Test in the English National Curriculum. 'Four plus three,' called the teacher firmly. Her instructions were to ask a question every five seconds. John held out four fingers on his left hand and three on his right and began to count them, pointing at his four left-hand fingers with his right index finger, saying silently, 'one, two, three, four', switching to his right hand, pointing with his left index finger, 'five, six, sev…' 'Six plus two!' said the teacher. John panicked. He did not have time to write his first sum down and turned his attention to the second. Six plus two is: 'one, two, three, four, five, six, …'. Again his thought was interrupted as the teacher called: 'Four plus two!' John managed this one: the answer was six. He started to write it down, but now he didn't know which number question he was on and wrote it in the space beside the number two. 'Five take away two.' John wrote 'three' in the space beside the number three. So it went on, as he sometimes failed to complete the sum in the given five seconds and sometimes, when he completed the problem in time, he didn't know where to write the answer. He failed his Key Stage One test, feeling glumly that he would never do well in mathematics. It was just too complicated[1].

In the same school Peter, not yet five years old, was given a calculator that enabled him to type in a sum such as '4+3' on one line then, when he pressed the 'equals' key, the answer was printed on the next line as '7'. He and several friends were asked to use these calculators to type in a sum whose answer was '8'. His friends typed in sums such as $4+4$ or $7+1$ or $10-2$, all of which they could also do practically by counting fingers or objects.

Peter typed in the sum $1000000-999992$. He knew this was 'a million take away nine hundred and ninety nine thousand, nine hundred and ninety two'. But, of course, he had never counted a million. Just think how long it would take! He could start briskly with 'one, two, three, four, five …' and keep a moderate pace with 'one hundred and eighty seven, one hundred and eighty eight, one hundred and eighty nine, …' but he would be really struggling with 'one hundred and eleven thousand two hundred and seventy eight, one hundred and eleven thousand two hundred and seventy nine, one hundred and eleven thousand two hundred and eighty'!

Peter's ideas arose not directly from counting experiences, but from

[1] This episode was observed and videoed by Eddie Gray and myself during a study of how young children perform arithmetic operations in Gray (1993) and Gray & Tall (1994).

his knowledge of number relationships. He had clearly been given a great deal of support with number concepts outside school. Even so, his knowledge was exceptional. He knew about place value, that 10 represented ten, 100 is a hundred, 1,000 a thousand, and 1,000,000 a million. He knew about tens of thousands, hundreds of thousands and that a million was a thousand thousands. He knew that 9 and 1 makes 10, 39 and 1 makes 40, 99 and 1 makes a hundred, and 999,999 and 1 makes a million. For him it was straightforward to see that just as the sum 92 and 8 gives 100, the sum of 992 and 8 gives a thousand, and 999,992 and 8 is a million.[2]

Here we have two children in the same school at about the same age thinking very differently. Can we find a single theoretical framework that encompasses both? Do they both go through the same kind of development, where one happens to be more successful than the other? How can we formulate a single theory that enables us to improve the teaching and learning of mathematics in a world where some find mathematics an amazing thing of beauty while others find it a source of problematic anxiety?

To seek a unified theory of the development of mathematical thinking, this book focuses on two complementary aspects: the fundamental operation of the human brain and the long-term development of mathematical knowledge.

2. THE LONG-TERM DEVELOPMENT OF MATHEMATICAL IDEAS

Mathematical thinking uses the same mental resources that are available for thinking in general. At its foundation is the stimulation of links between neurons in the brain. As these links are alerted, they change biochemically and, over time, well-used links produce more structured thinking processes and more richly connected knowledge structures[3]. The strengthening of useful links between neurons provides new and more immediate paths of thought, so that processes that occur in time—such as counting to add numbers together to get $3+2$ is 5—are shortened to operate without counting, so that $3+2$ immediately outputs the result 5. This involves a *compression* of knowledge in which lengthy operations are replaced by immediate conceptual links.

The long-term development of mathematical thinking is consequently more subtle than the addition of new experiences to a fixed knowledge structure. It is a continual reconstruction of mental connections that

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[2] This episode was recorded by Eddie Gray as part of the same study in the same school.

[3] The term 'knowledge structure' may have various connotations in cognitive science, philosophy and other disciplines. Here I refer broadly to the relationships that exist in a particular context or situation, including various links between concepts, processes, properties, beliefs and so on.

evolve to build increasingly sophisticated knowledge structures over time.

Geometry begins with the child playing with objects, recognizing their properties through the senses and describing them using language. Over time, the descriptions are made more precise and used as verbal definitions to specify figures that can be constructed by ruler and compass and eventually the properties of figures can be related in the formal framework of Euclidean geometry. For those who study mathematics at university, this may be further generalized to different forms of geometry, such as non-Euclidean geometries, differential geometry and topology. (More advanced topics mentioned in this first chapter will feature in later chapters, where they will be given elementary explanations for the general reader.)

The learning of arithmetic follows a different trajectory, starting not with a focus on the *properties* of physical objects, but on *actions* performed on those objects, including counting, grouping, sharing, ordering, adding, subtracting, multiplying, dividing. These actions become coherent mathematical operations and symbols are introduced that enable the operations to be performed routinely with little conscious effort. More subtly, the symbols themselves may be seen not only as operations to be performed but also compressed into mental number concepts that can be manipulated in the mind.

Young children are introduced to counting physical objects to develop the concept of number and to learn to calculate with numbers. As they learn to count, they will find that $7+2$ calculated by counting 2 after 7 to get 'eight, *nine*' is much easier than $2+7$ by counting 7 after 2 as 'three, four, five, six, seven, eight, *nine*.' Initially it may not be evident that addition by counting is independent of order, but when this is related to the visual layout of objects placed in various ways, properties of arithmetic emerge, such as addition and multiplication being independent of order of operation and multiplication being distributive over addition. These observations may be formulated as 'rules' of arithmetic. At a more advanced level, the whole numbers may be formulated in terms of a list of axioms (the Peano Postulates) from which familiar properties of arithmetic may be deduced as theorems.

Measurement also develops out of actions: measuring lengths, areas, volumes, weights and so on. These quantities can be calculated practically using fractions or to any desired level of accuracy using decimals. Numbers can be represented as points on a number line, and formulated at university level as an axiomatic system (a complete ordered field).

Algebra builds on the generalized operations of arithmetic with symbolic manipulations following the rules observed in arithmetic. Algebraic functions may be visualized as graphs, and later algebraic structures may be formulated in various axiomatic systems (such as groups, rings, fields).

Likewise, concepts in the calculus can be expressed visually and dynamically as the changing slope of a graph and the area under a graph, which may be approximated by numerical calculations or expressed precisely through the symbolic formulae for differentiation and related techniques for integration. At university these ideas may be expressed axiomatically in the formal theory of mathematical analysis.

Vectors are introduced as physical quantities with magnitude and direction, written symbolically as column vectors and matrices, and later reformulated axiomatically as vector spaces.

Probability begins by reflecting on the repetition of physical and mental experiments to think how to predict the likely outcome, then performing specific calculations to calculate the probability numerically, later formulating the principles axiomatically (as a probability space).

These developments incorporate three distinct forms of knowledge. Geometry studies objects and their properties, leading to mental imagery described in language that grows in increasingly subtle ways. A second form of knowledge grows out of actions that are formulated using symbolism. The highest level in both cases involves the formal definition of axiomatic systems and deduction of properties by mathematical proof.

The first two types of knowledge are developed in school and continue into a wide range of applications. The third flowers in the formal approach to pure mathematics encountered at university. The framework presented in this book takes the development of these three forms of knowledge as a foundation for the growth in sophistication of mathematical thinking from the activities of the child to the frontiers of mathematical research. (Figure 1.1.)

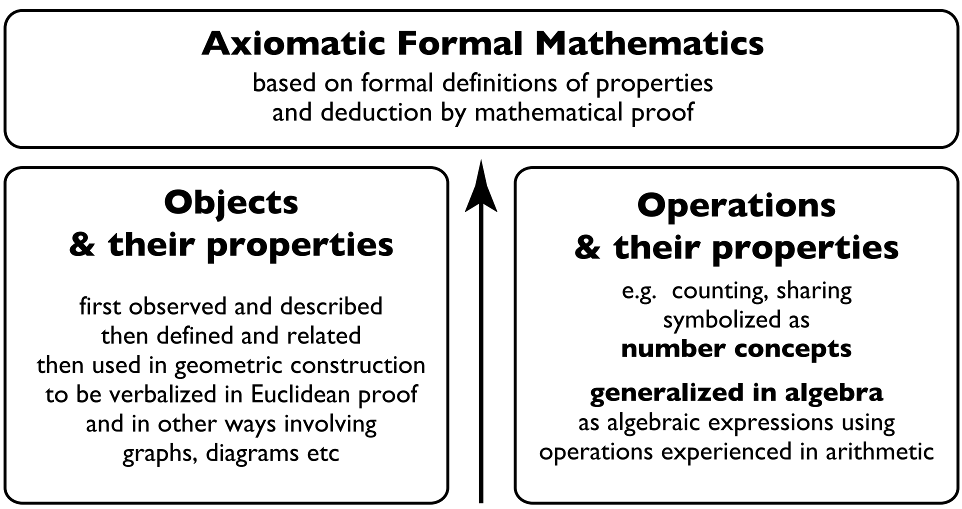

Figure 1.1: An initial outline of three forms of knowledge in mathematics

We are already privileged to have many frameworks available to provide an overview of human development in general and mathematics in particular. The father figure of modern developmental psychology, Jean Piaget[4], formulated a stage theory for the long-term development of the child through the pre-language *sensori-motor* stage, a *pre-operational* stage in which children develop language and mental imagery from a personal viewpoint, a *concrete-operational* stage where they develop stable conceptions of the world shared with others, and a *formal-operational* stage developing the capacity for abstract thought and logical reasoning.

Jerome Bruner[5] classified three modes of human representation and communication: *enactive* (action-based, using gestures), *iconic* (image-based using pictures and diagrams) and *symbolic* (including language and mathematical symbols).

Efraim Fischbein[6] focused on the development of mathematics and science, and formulated three different approaches, which he called *intuitive*, *algorithmic* and *formal*.

Each of these frameworks presents a long-term development from physical perception and action, through the development of symbolism and language and on to deductive reasoning. They also formulate different ways of building specific concepts. Bruner and Fischbein differ in detail, but both see a broad conceptual development in which the enaction and iconic imagery of Bruner relates to the intuition of Fischbein while Bruner's symbolic mode of operation includes two special forms of symbolism in arithmetic and logic (relating respectively to mathematical algorithms and formal proof).

Piaget complements his global stage theory by formulating several ways in which new concepts are constructed. The first is *empirical abstraction* through playing with objects to become aware of their properties (for instance, to recognize a triangle as a three-sided figure and to distinguish this from a square or a circle).

The second is *pseudo-empirical abstraction* through focusing on *actions* on objects. This plays a major role in arithmetic where operations such as counting and sharing lead to concepts such as number and fraction.

He also formulates *reflective abstraction* where operations at one level become mental objects of thought at a higher level. This has proved to be fruitful in describing how addition becomes sum, repeated addition becomes product, a generalized operation in arithmetic becomes an expression in algebra, and so on. Reflective abstraction is essentially a

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[4] References on Piaget's Stage Theory are numerous. See, for example Baron *et al* (1995), pp. 326-329.
[5] Bruner (1966), pp. 10,11.
[6] Fischbein (1987).

succession of higher-level extensions of pseudo-empirical abstraction.

By analogy, there is a fourth type of abstraction that generalizes empirical abstraction of the properties of objects, conceiving mental objects such as points with no size, and straight lines having no width that can be extended as far as desired in either direction. This may be termed *platonic abstraction* as it forms platonic mental objects by focusing on the essential properties of figures. (Figure 1.2.)

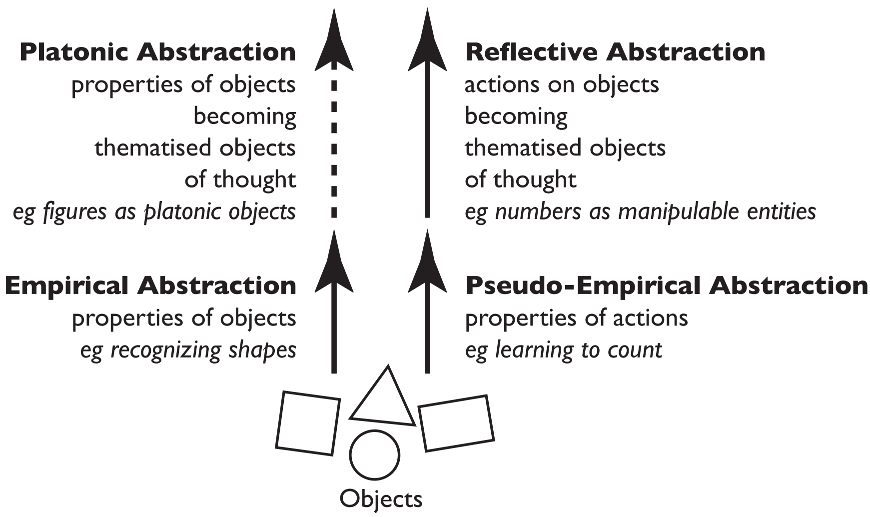

Figure 1.2: Piagetian and Platonic Abstraction

These four types of abstraction belong naturally to two long-term developments, one building from the properties of objects (empirical and platonic abstraction), the second from actions on objects (pseudo-empirical and reflective abstraction). These two developments relate directly to the first two forms of long-term development in mathematical thinking formulated earlier. The first focuses on the structure of objects, the second on actions that become operations. I shall refer to these as *structural abstraction* and *operational abstraction*. (Figure 1.3.)

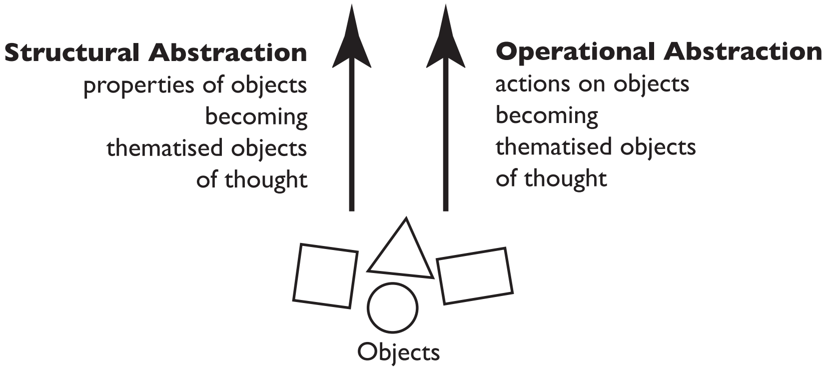

Figure 1.3: Long-term Abstraction

These ideas relate to the vision of Pierre van Hiele's *Structure and Insight*[7] in geometry, and Anna Sfard's formulation of operational and structural conceptions[8] in general mathematical thinking, which will evolve into essential aspects of the wider framework developed in this text.

While elementary recognition of shape and number is found in other species, only *Homo sapiens* develops sophisticated mathematical ideas such as the theorem of Pythagoras, or the idea that there are an infinite number of primes. This intellectual development arises through the development of language and symbolism, which Terence Deacon characterizes by recognizing *Homo Sapiens* as *The Symbolic Species*[9].

Mathematical thinking begins in human sensori-motor perception and action and is developed through language and symbolism. In *Philosophy in the Flesh*[10], George Lakoff and Mark Johnson formulate the idea of an 'embodied concept' as 'a neural structure that is actually part of, or makes use of, the sensori-motor system of our brains.' This analysis is consonant with a combination of Bruner's enactive mode operating 'through action' and iconic mode that 'depends upon visual or other sensory organization and upon the use of summarizing images.'[11]

In *Where Mathematics Comes From*[12], Lakoff and Núñez take the argument further to declare that 'all human thought is embodied through our sensori-motor experience'; this applies in particular to mathematics.

This classification of all human thought into a single category of embodiment can be usefully enhanced by a subdivision into subcategories that operate in clearly different ways. The term 'sensori-motor' already refers to two different aspects of the brain: the *sensory* part

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[7] Van Hiele (1986).
[8] Sfard (1991).
[9] Deacon (1997).
[10] Lakoff & Johnson (1999).
[11] Bruner (1966), pp. 11-12.
[12] Lakoff & Núñez (2000).

relating to how we perceive the world through our senses and the *motor* part relating to how we operate on the world through our action. This relates directly to the distinction made in this text between the sensory appreciation of shape and space focusing on the structural properties of objects and the operational motor activities such as counting and sharing that lead on to arithmetic and algebra.

In his earlier book on *Women, Fire and Dangerous Things*[13], Lakoff refers briefly to two different aspects of embodiment that he terms *conceptual embodiment* and *functional embodiment*. The former refers to the use of mental images and the latter to 'the automatic, unconscious use of concepts without noticeable effort as part of normal functioning.'

This distinction is not used, as far as I know, in any other work of Lakoff. Yet it resonates strongly with the contrast I noted above between a *structural* focus on properties of objects and an *operational* focus on actions symbolized as mathematical operations.

I shall use the term 'conceptual embodiment' to refer to the use of mental images, both static and dynamic, that arise from physical interaction with the world and become part of increasingly sophisticated human imagination. This includes the use of physical embodiments such as Dienes' blocks to relate to mental conceptions of numbers and arithmetic.[14] It also extends to the drawing of geometrical figures that become mental pictures described verbally in Euclidean geometry, the representation of functions and graphs as static images on paper, and dynamic visual images in general, as visualized using computer graphics or solely within the mind.

Meanwhile, the manipulation of symbols in arithmetic and algebra has a functional aspect in which symbols are imagined as being shifted around mentally on the page. This functional embodiment is intimated in phrases such as 'turn upside down and multiply', 'change sides, change signs', 'put over a common denominator and add', or 'shift all the terms in x on one side and all the numbers on the other.'

This gives two different ways in which mathematical thinking grows: the use of mental images supported by language to enable us to refine and develop more sophisticated meanings, and the use of symbolism in arithmetic and algebra to formulate problems as operational equations, to solve them by calculation and symbolic manipulation. These two forms of growth occur throughout schooling before the later development of a formal axiomatic approach arises in the work of pure mathematicians. This overall development is based on three fundamental human attributes: *input through the senses* that recognizes properties of objects, *output through actions* that become routine operations, and *language*

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[13] Lakoff (1987) pp. 12–13.
[14] Dienes (1960).

(together with symbolism) which supports both, to develop increasingly sophisticated ways of thinking about mathematical ideas.

4. SYMBOLS AS PROCESS AND CONCEPT

The symbols that occur in arithmetic and algebra are used in special ways. Not only do they specify operations that can be performed as a sequence of steps, they also operate as mental entities that can themselves be operated upon. This offers a mode of operation that is different from the usual linguistic analysis for speaking about numbers.

Number words are often interpreted as adjectives or nouns, such as 'three' as an adjective in 'the three musketeers' and a noun in 'three is a prime number.' In English, words freely function as various parts of speech, for instance, the term 'abstract' can be an adjective in 'an abstract idea', a noun in 'an abstract taken from a book' or a verb 'to abstract ideas from a concrete situation.' Actions are often transformed into nouns, such as the way in which the word 'running' in 'John is running' becomes 'Running is good for your health.' The participle 'running' becomes a noun using the linguistic device that is called a 'gerund'.

However, this analysis into various parts of speech fails to capture the subtle ways in which we think about the process of counting and the concept of number. Numbers are not only used as adjectives or nouns. An expression in arithmetic such as '$3+4$' operates flexibly as an instruction to calculate the result in 'what is $3+4$?' and also as a noun, the name of the result of the calculation $3+4$, which is 7. The symbol $3+4$ operates both as a *process* (addition) and a *concept* (the sum).

Throughout the development of symbolism in arithmetic and algebra, the child learns to carry out an operation, to practice it until it becomes routine, and then to use it as a thinkable concept. A young child spends many months grasping the process of pointing and counting to find the number of elements in a set is independent of the sequence of counting and this becomes the related concept of number.

Likewise, an algebraic expression, such as $2x+6$, may be interpreted both as a process of evaluation (twice the value of x plus 6) and also as a concept of algebraic expression that may itself be operated on. For instance, it can be factorized to give the product $2(x+3)$. As a process, $2(x+3)$ involves a different sequence of steps (double the result of adding the value of x and 3). However, in algebraic manipulation, the expressions $2x+6$ and $2(x+3)$ are interchangeable, so they may be considered as two different ways of writing the same thing. This gives a new flexibility in using symbols that occurs naturally and unconsciously for experts, but may need to be learnt explicitly by the novice.

Symbols that operate dually as both process and concept in this way give rise to a new part of speech in the language of mathematics, that

Gray and Tall named a *procept*.[15] As the child relates various ways of calculating the same result, different symbols such as $7+3$, $3+7$, $13-3$ may then be reconsidered as being different ways of writing the same procept. The procept here is the number 10 and all other possible ways that an individual thinks about it to manipulate it flexibly in arithmetic. Over time it grows in richness to encompass many other connections such as 5×2, $20\div 2$, $(-5)\times(-2)$ and even $-10i^2$. Flexible use of such symbolism to derive new relationships and to build a rich structure of flexible alternatives is called *proceptual thinking*. It manifests itself in early arithmetic as symbols are decomposed and recomposed to perform calculations. For example, the sum $7+6$ might be calculated by realizing that $7+3$ is 10 and the 6 can be seen as $3+3$, so that $7+6$ is $10+3$, which is 13. Later a student may factorize the expression $(2x+3)^2-(x+2)^2$ by recognizing the whole expression as the difference between two squares A^2-B^2 and writing the solution $(A-B)(A+B)$ *in one operation* as $(x+1)(3x+5)$. A procedural thinker operating step by step is faced with a more lengthy sequence of operations, first multiplying out the expression to get

$$(2x+3)^2-(x+2)^2=4x^2+12x+9-x^2-4x-4$$

then simplifying the expression to get

$$3x^2+8x+5$$

and then factorizing this into a product of two factors using a fairly complex algorithm.

Proceptual thinking is important not only in deriving facts in arithmetic, it is also essential in the flexible manipulation of algebra, and in the long-term development of powerful mathematical thinking.[16]

5. COMPRESSION OF KNOWLEDGE

The manner in which a process carried out in time can eventually be conceived as a mental concept independent of time is an example of a more general mental process to think of complicated situations in simple ways.

Compression of knowledge occurs when a phenomenon of some kind is conceived in the mind in a simpler or more efficient manner. This occurs through making more direct mental connections in the brain and is enhanced by using language to give the concept a name and to be able to share ideas about its properties and relationships to other concepts.

[15] Gray & Tall (1994).
[16] Gray, Pitta, Pinto & Tall (1999).

Compression of knowledge occurs in several different ways. We are able to recognize things through our perception of similarities and differences, to *categorize* concepts in a whole variety of ways, giving a name to identify the category, such as 'dog' or 'triangle'. This is a structural abstraction of the properties of a concept, drawing them together into a single named entity.

A second method involves the practicing of a sequence of actions as a *procedure* so that they can be performed with little mental effort. The further compression of a process (such as addition) being compressed into a mental concept (such as sum) is an operational abstraction termed the *encapsulation* of a process as a concept.

A third method occurs as individuals use increasingly sophisticated language to specify concepts through *definition*. This is, of course, a special case of categorization. However, instead of starting with a concept and categorizing its properties, the situation is reversed by specifying the definition and deducing all other properties from it.

In the framework developed in this book, mathematical thinking is seen to use categorization, encapsulation and definition in a variety of ways to compress ideas into more flexible forms.

The development of geometry begins with the categorization of objects through visual and tactile experience. Language enables this categorization to become more refined through a succession of structural abstractions as properties are recognized, described, defined and then used to prove properties in geometry.

Symbolic thinking in arithmetic and algebra begins with operations using numbers to count objects, then fractions to measure quantities, and more sophisticated representations using signed numbers, finite and infinite decimals. Each stage involves the encapsulation of an arithmetic process as a number concept, and there is a growing divergence (which Gray and Tall[17] termed 'the proceptual divide') between those who remain entrenched at best in the procedures of counting and those that develop more flexible proceptual thinking.

Operations in arithmetic may be seen to have properties that may be recognized, described, and then defined as rules of arithmetic. This also leads to the numbers constructed having properties that enable us to speak of odd numbers, even numbers, prime numbers, and to consider profound relationships, such as the idea that every whole number can be factorized uniquely into primes. The development of these ideas again involves recognizing, describing, defining and deducing properties of symbolic constructs. The world of operational symbolism therefore also involves a structural abstraction of the properties of procepts as the use of

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[17] Gray & Tall (1994).

symbols becomes increasingly sophisticated.

At the formal level, a new form of abstraction occurs which takes mathematical thinking onto a new level. The mental processes of structural abstraction (operating on objects to discover their properties) and operational abstraction (operating on objects to discover the properties of the operations) are extended to *formal abstraction* (operating on formal definitions to deduce new formal properties). This gives three forms of abstraction in mathematics in which each new form of abstraction builds on and incorporates earlier forms. (Figure 1.4.)

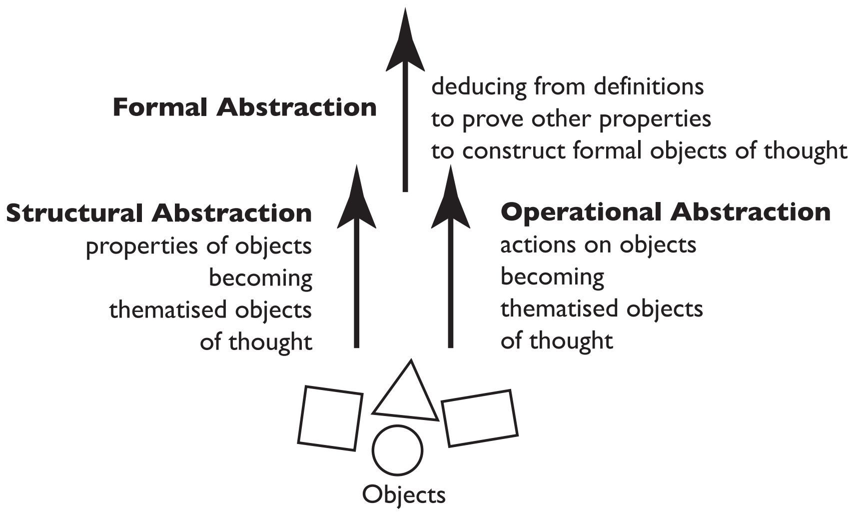

Figure 1.4: Three forms of abstraction

The introduction of formal abstraction involves a significant change in meaning. Whereas structural abstraction and operational abstraction build from perceptual ideas that become conceptualized as mathematical concepts, formal abstraction builds essentially on definitions formulated linguistically. Subconsciously there may continue to be links with perception and action, but formally, it offers a new, *universal* approach to mathematics in which the theorems proved depend only on definition and proof and not on the particular context.

6. THREE WORLDS OF MATHEMATICS

The previous discussion highlights three essentially different ways in which mathematical thinking develops[18]:

> *Conceptual embodiment* builds on human perceptions and actions developing mental images that are verbalized in increasingly sophisticated

[18] This was first described in Gray & Tall (2001).

ways and become perfect mental entities in our imagination.

Operational symbolism grows out of physical actions into mathematical procedures. While some learners may remain at a procedural level, others may conceive the symbols flexibly as operations to perform and also to be operated on through calculation and manipulation.[19]

Axiomatic formalism builds formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof.

Each of these ways of working develops in sophistication over time, using increasingly subtle forms of language. They are more than different modes of operation. Each one has a quality of its own in a world that develops in its own special way. One is based on (conceptual) embodiment, one on (operational) symbolism, and the third on (axiomatic) formalism, as each one grows from earlier experience.

Embodiment includes both perception and action where actions operate on objects to tease out their properties, later developing formal aspects in Euclidean proof. The overlap of embodiment and symbolism focuses on embodied actions on objects, such as counting, which develop into symbolic operations with numbers. Algebraic manipulation develops symbolic formal proof based on the rules of arithmetic. Both lead on to formal mathematical proof from set-theoretic definitions. (Figure 1.5.)

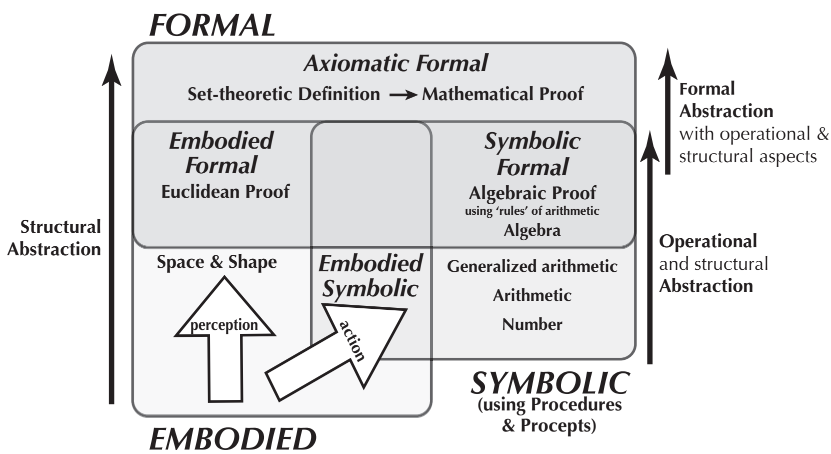

Figure 1.5: Preliminary outline of the development of the three worlds of mathematics

To be able to refer to these links between worlds, I will often compress the names to 'embodiment', 'symbolism' and 'formalism' to allow the term 'embodied symbolism' to refer to the transition between conceptual

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[19] Initially the world of operational symbolism was named as 'proceptual symbolism' to represent the desirable form of flexible symbolic thinking. It is now referred to as 'operational symbolism' to include all forms of operations in arithmetic and algebra, to include both flexible (proceptual) and rote-learnt procedural.

embodiment and operational symbolism, 'embodied formalism' to refer to Euclidean proof, and 'symbolic formalism' to refer to algebraic proof using the 'rules of arithmetic'.

For some time I wondered which way round I should use the pairs of words. For instance, should I speak of 'embodied formalism' or 'formal embodiment'? The answer to this question lies in the manner in which the link is approached. Looking at embodiment becoming increasingly formal, it seemed more appropriate to use the term 'formal embodiment', likewise, as symbolism becomes more formal, then the term 'formal symbolism' would be more appropriate. However, if one looks at the final picture, one can look from above to see formalism subdivided into three distinct forms, 'embodied formalism', 'symbolic formalism' and 'axiomatic formalism'.

I also realized the need for a certain flexibility in the use of the term 'formal'. As a mathematician, 'formal' refers to mathematics based on set-theoretic definitions and formal proof. As an educator, I realize that the term 'formal' is used in a different way to refer, for example, to the 'formal operational' stage of Piaget. Even in figure 1.5 I have used the term 'formal' to cover the three forms of 'embodied formal', 'symbolic formal' and 'axiomatic formal'. The convention I have adopted is to be flexible about the terminology to use it as seems appropriate at any given time. When I use the term 'formal' on its own without any qualifier, I will use it in the mathematical sense referring to formal set-theoretic mathematics.

The picture of the three-world development in figure 1.5 also applies to the development of long-term reasoning and proof. I will distinguish three stages of development across the full range of mathematics. The initial stages involving practical experience of space and shape and calculations in arithmetic I shall call 'practical mathematics'. This involves initial experiences in recognizing and describing the properties of figures that occur simultaneously without necessarily realizing that one property may imply another.

The next broad stage will be termed 'theoretical mathematics'. In geometry this includes Euclidean definition and proof where the term 'theoretical' was applied by van Hiele to cover the use of definitions and Euclidean proof. In symbolic mathematics it includes the shift to algebra and proof of identities in algebra based on 'the rules of arithmetic'. Theoretical mathematics includes the more sophisticated levels of embodiment and symbolism that are entitled embodied formal and symbolic formal.

The third stage will be termed 'formal mathematics' to refer to the 'formalism' in mathematics, based on set-theoretic definitions and deductions. (Figure 1.6.)

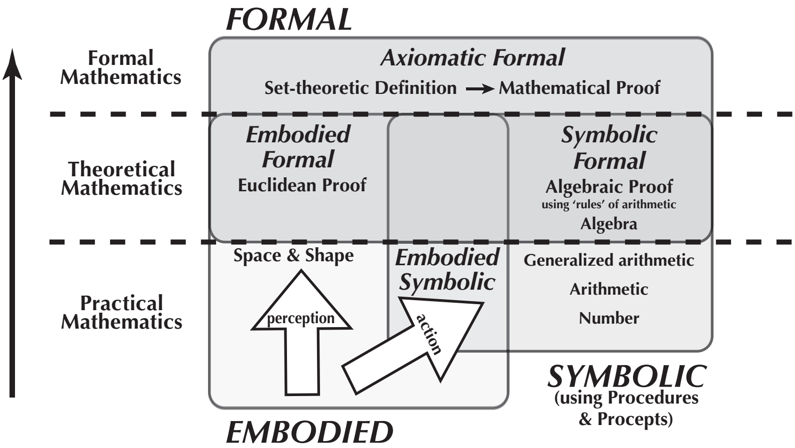

Figure 1.6 practical, theoretical and formal mathematics

Transitions from one world to another combine aspects of both worlds. For instance, the shift from embodiment to symbolism involves embodied actions on objects, such as counting, which lead to symbolic concepts such as number. Embodied actions, such as starting with a set of six objects subdivided into four and two, then moving the objects around, enable the learner to *see* properties of arithmetic, such as $4+2$ is the same as $2+4$, embodying the general property that addition is commutative.

The transition has two subtly different aspects. *Embodied compression* of operations into mental concepts focuses on the visible *effect* of the operations, which embodies general properties of the operations, such as the idea that addition is commutative. *Operational compression* focuses on the operations themselves as the individual practices them to learn specific number facts. Flexible ability in arithmetic requires both an appreciation of the general properties of operations and also the specific details of calculation. These subtle difficulties will play an important role in the flexible understanding of arithmetic and its generalization into algebra.

In the literature, the terms 'embodiment', 'symbolism' and 'formalism' and the corresponding adjectives 'embodied', 'symbolic' and 'formal' are used with a variety of meanings, which will be discussed in detail in chapter six. In this text, they will be used with the meanings given here.

Conceptual embodiment builds from human perceptions and actions, becoming increasingly refined and supported by language to rise to a level of mental thought experiment and formal embodiment. Operational symbolism in arithmetic and algebra builds from operations that may be compressed into symbols to be manipulated to solve problems and the properties of the operations are used to act as a basis for formal development of symbolism. Out of the experiences, axiomatic formal mathematics develops from set-theoretic definitions and formal proof.

The transition to (axiomatic) formal mathematics not only shifts mathematical thinking to a new level where theorems are proved that do not depend on specific embodiments and specific methods of calculation. It will be shown that formal mathematics can lead to proving 'structure theorems' that specify the *structure* of an axiomatic system, giving it new forms of embodiment and symbolism, deduced formally from the axioms. The growth in sophistication from embodiment to symbolism and on to formalism therefore leads back into human embodiment and operational symbolism, completing the circle and underlining the integration of the three worlds of mathematics into a single overall structure.

Different individuals cope with their journeys through the worlds of mathematics in different ways. For instance, in the world of operational symbolism, some students may learn to operate procedurally with the operations but may be less successful in developing flexible ways of dealing with the symbols as manipulable concepts. In the world of axiomatic formalism, some may build 'naturally' from embodied and symbolic experience, some may build 'formally' from the written definitions, while others may attempt to pass examinations by learning proofs procedurally by rote.

It will be part of our journey through the three worlds to consider how different individuals develop a spectrum of approaches to mathematical thinking. It will also allow us as individuals to stand back and reflect on our own personal ways of thinking so that we may become aware of the nature of our own personal growth and reflect on the different ways in which individual development occurs.

7. ATTRIBUTES THAT WE ALL SHARE

In our discussion so far, three distinct worlds of mathematics have been proposed, developing in different ways. These worlds are not arbitrary. They develop based on distinct essential features that we all share.

The first of these is our sensory capacity for *recognition* to see patterns, similarities and differences that we express in language to categorize objects such as 'dog', 'cat', and 'triangle'.

The second builds on our motor capacity for *repetition* that enables us to practice sequences of actions until we can perform them automatically as sequential operations with little conscious thought.

The third is our fundamentally human ability for *language*. This enables us to give names to phenomena, to talk about them and refine their meaning, so that they become *thinkable concepts* that we can talk about and make mental connections to build up sophisticated *knowledge* *structures*. Language, including the use of mathematical symbols, raises our mathematical thinking to successively higher levels.

Language enhances recognition by enabling us to *categorize* objects and phenomena, to give them names, to talk about them and refine their meaning, to compress knowledge into thinkable concepts that we can use to build more sophisticated knowledge structures.

Language enhances repetition through the ability to give a name to the process, to *encapsulate* it as a thinkable concept that can be mentally manipulated in its own right. Now a process that is performed as a sequence of actions in time is compressed into a single entity that can operate at a higher level of thought. Complex ideas are then expressed in ways that are both sophisticated and simple.

Christopher Zeeman expressed this succinctly, saying:

> Technical skill is mastery of complexity while creativity is mastery of simplicity. [20]

Mathematical thinking requires technical skill to make calculations and manipulate symbols. It can enable the individual to solve routine problems and perform well on standardized examinations. Creative mathematical thinking requires more. It requires knowledge structures connected together in compressed ways that make complex ideas essentially simple.

8. ATTRIBUTES BUILT ON EXPERIENCE

Intellectual development depends on how we use our experiences to cope with new situations. Learning at one stage can affect how we think at the next. A child will learn that when something is taken away, what is left is less. If you start with five apples and take away three, then only two are left. This experience serves the child well in everyday life. It is even taken as a common notion in Euclid, that 'the whole is greater than the part'. Yet this property that we all share becomes problematic in mathematics when we attempt to take away a negative number. Here starting with 5 and taking away –2 gives 7. Taking away a negative number gives *more*. Likewise, early experience of arithmetic with whole numbers tells us that multiplication gives a bigger result, and this causes great difficulty when the product of two fractions can be smaller than either of them.

In *Metaphors We Live By*,[21] Lakoff and Johnson theorized that our thinking involves metaphors, using ideas from previous experience to refer to a new experience in a different context. This enables the biological brain to re-use existing connections to make sense of new phenomena.

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[20] Zeeman (1977).
[21] Lakoff & Johnson (1980).

However, the term 'metaphor' is part of a sophisticated framework to explain a subtle aspect of thought from an expert viewpoint. In developing a framework for mathematical thinking from child to adult it is essential to look at the development *as it appears to the learner*, for it is this view that is directly involved in learning. I therefore sought another possible way of talking about previous experience that could be used in conversation not only from a top-down expert viewpoint, but also from a bottom-up development that could be of value to teachers and learners.

9. SET-BEFORE AND MET-BEFORE

As I mused on the word 'metaphor', I imagined it being said as 'met-afore', using the old English word 'afore' to relate it to experiences that had been met before in the life of the child. Initially this was an amusing joke that did not raise many laughs in others. Then I changed the word to 'met-before' and the new form not only sounded different, the play on words from 'metAphor' to 'metBefore' enabled the term to be used easily in conversation[22]. It became possible to say to a learner: 'What have you met before that makes you think that?' The term 'met-before' also proved amenable in talking to other experts, who seemed to take it up immediately and use it in their own conversation. It operated in the way that new words operate, first as a name, with its properties to be described and then to be defined, at least in the sense of a dictionary definition. A working definition of a 'met-before' is 'a structure we have in our brains *now* as a result of experiences we have met before.'

A met-before can be supportive in a new situation, or it can be problematic[23]. For instance, the met-before '2+2 is 4' is supportive not only in its original context of counting objects or fingers, but throughout the development of number systems to real numbers and even complex numbers. The met-before 'take away leaves less' works for whole numbers, even for (positive) fractions, but it is problematic with negative numbers, where taking away a negative number gives more. It is also problematic in the theory of infinite cardinal numbers where two sets are defined to have the same cardinal number (allowing us to say they are the same size) if their elements can be placed in one-one correspondence. The set of natural numbers and its subsets of even numbers and odd numbers, all have the same cardinal number using the mapping from n to $2n$ to

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[22] The term met-before is a play on words that works well in English. It translates less well in other languages where other terminology may be necessary.

[23] The idea of problematic met-before has a long history in mathematics and science education where it arises as an 'epistemological obstacle'. (Bachelard, 1938). However, the earlier usage often referred to intuitive ideas that cause difficulty in later theoretical applications. Here the term met-before applies to any earlier experiences that affect current thinking.

$2n-1$. Taking away the even numbers from the natural numbers leaves the odd numbers which are the same size as the full set.

In this way, a met-before (take away leaves less) can be supportive in some contexts (whole numbers, lengths, areas) yet problematic in others (negative numbers, infinite cardinal numbers). The manner in which individuals deal with these aspects and the resulting emotional effects plays a major role in individual development of mathematical thinking.

As the term 'met-before' was used in conversation, it led naturally to the introduction of the term 'set-before' to describe the fundamental attributes that we all share. A working definition of a set-before is 'a mental structure that we are born with, which may take a little time to mature as our brains make connections in early life.'

At this point I realized that the capacities of *recognition* and *repetition* are set-befores that we all share, based on our human capacities for perception and action. Meanwhile *language* is a further set-before specific to *Homo Sapiens* that enables us to develop more sophisticated thinking.

This offers a global long-term framework for the development of mathematical thinking, based on the three set-befores of recognition, repetition and language, with three distinct ways of forming mathematical concepts through categorization, encapsulation and definition, building on met-befores.

10. BLENDING KNOWLEDGE STRUCTURES

The journey to develop powerful mathematical thinking involves compressing knowledge into thinkable concepts and connecting them together in knowledge structures. One further construct is required: the *blending*[24] of different knowledge structures into a new knowledge structure, perhaps leading to a newly created thinkable concept.

Our biological brains evoke thinkable concepts by a selective binding of neural structures involving a range of senses and perceptions. An apple conjures up aspects of vision, touch, and smell. A red apple may offer the further promise of a sweet taste. This thinkable concept is a blend of neural structures.[25]

Likewise, a mathematical concept evokes a range of different cognitive structures, blending together different experiences to produce a single mental construct.

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[24] See, for example, Lakoff & Núñez (2000), Fauconnier & Turner (2002).

[25] See, for example, Fauconnier and Turner (2002), who build a detailed theory of blending domains of knowledge where the blend contains elements from both domains and also new emergent properties that arise from the blend. In this text, the notion of blending will refer to different forms of mathematical representation that have some elements that correspond and others which may be problematic yet have the potential to lead to new emergent properties.

The real number system is a blend of embodiment, symbolism and formalism in which each contributes different aspects to our understanding of number (figure 1.7). The number line allows us to see numbers as points on a horizontal line in order from left to right. If we point at the number zero and slide a finger along to the number 1 we may imagine we are moving continuously through all the numbers from 0 to 1. However, if we think of a number as an infinite decimal, it is impossible to imagine the decimal expressions running through all the possible decimals between 0 and 1 in a finite time.

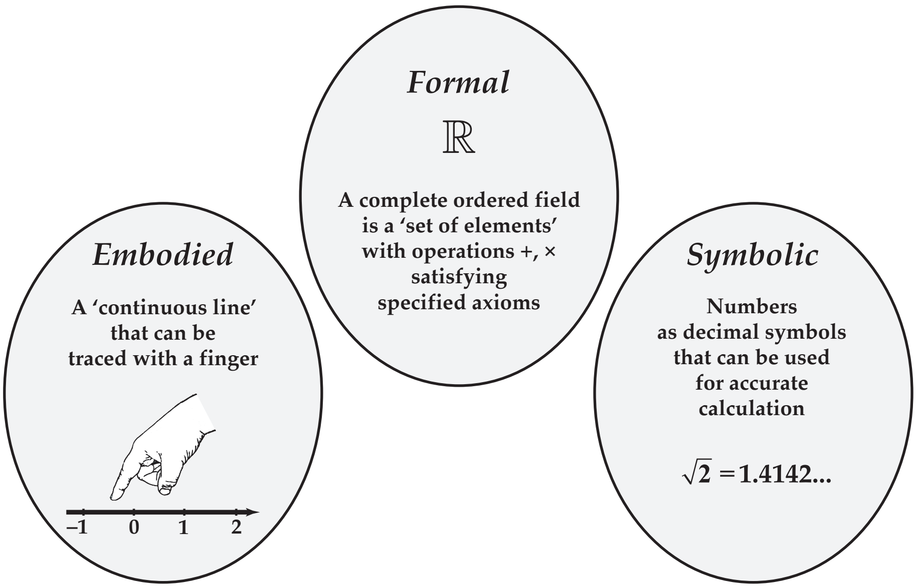

Figure 1.7 The real numbers as a blend

Blending ideas together from different contexts usually involves some aspects that are common and some that are in conflict. This leads to a divergence between those who focus on the power of the common aspects and those who are concerned about the differences.

Blending occurs when a particular context is generalized, for instance, from the counting numbers to the wider system of positive and negative integers. This will be termed an *extensional blend*. In this case the arithmetic of counting numbers is supportive, but the met-before 'takeaway makes less' becomes problematic.

The whole development of number—from whole number to fraction, to positive and negative numbers, to finite and infinite decimals represented as points on a number line—is a succession of extensional blends, broadening one number system to a larger one with richer properties. For some learners the increasing flexibility and generality prove to be extremely powerful. For others, the subtle changes in meaning and the complications of the operations often become problematic.

Blending offers creativity as mathematical thinking develops in history as well as in the individual. By blending the arithmetic of numbers with the geometric transformations of the plane, a whole new concept of complex number was created that extended the real number system envisaged on the real line to the whole of the complex plane. Historically this took several centuries and continues to be problematic for many students, though it proves supportive for modern mathematics and applications in areas such as engineering.

11. Emotional aspects of mathematics Learning

Making sense of new mathematics is a challenge that blends together some aspects that are supportive and others that are problematic. A major aspect of our study of the spectrum of outcomes of mathematical learning relates to the accompanying emotional reactions.

A learner may begin with the goal of making sense of a particular mathematical topic. Struggling with a problem that is resolved by a sudden insight can be accompanied by a deep sense of pleasure. A student accustomed to solving problems may persist when faced with a difficulty to seek once more the sweet taste of success. However, recurring difficulties can affect a learner's attitude that may in the longer term develop into mathematical anxiety.

When faced with lack of understanding and the fear of possible failure, students and their teachers may change their goal from conceptual understanding to that of learning procedures to pass examinations and to acquire techniques to use in applications. Procedural learning can be an alternative goal that gives pleasure in success. However, this success may be limited, enabling the student to solve routine problems but without further reflection to make sense of the relationships, it may lead to less flexible forms of mathematical thinking.

As individuals take personal routes through their development of mathematical thinking, human emotions play a significant role in supporting or inhibiting progress. While supportive met-befores encourage generalizations that give pleasure and power, problematic met-befores cause conflict in new situations, acting as a challenge to some and a source of anxiety to others.

This applies not just to learners but *to all of us*, including teachers, mathematicians, experts who build theories and, in particular, *to readers of this book*. The mathematical journey that we take in this text begins with young children and extends to the boundaries of mathematical research. It is evident that some topics will be unfamiliar to particular readers, be they teachers of young children with little or no experience of university mathematics or research mathematicians unfamiliar with the cognitive development of young children. I have therefore graded the expectations of mathematical knowledge so that the initial chapters are readable for anyone familiar with school mathematics including geometry and algebra, with a gentle introduction to the very different formal world of definition and proof. Thereafter, as the mathematics becomes more sophisticated, I shall attend to its special characteristics appropriate at this level while giving a sense of the ideas for the more general reader.

As the framework becomes more sophisticated, what matters is not the *detail* of the mathematics encountered at higher levels, for that requires specialist knowledge. Even mathematicians with an expertise in one area share ideas that are not understood by specialists in other areas. To formulate a broadly shared framework of the development of mathematical thinking requires an overall grasp of the general principles involved, evolving from human perception and action and developing sophisticated ideas through the use of language and symbolism, taking account of how we develop differently as individuals. This involves formulating *how* the ideas develop and *why* they present aspects that are supportive for some and problematic for others.

Using the material in this book with mixed groups, I have found that some primary school teachers are clearly scared of algebra or of the calculus, but that they benefited from realizing how their conceptions depended on their met-befores, not on any innate stupidity. Talking about more advanced mathematics in a relaxed manner allowed them to become aware of the origins of their fears that in turn helped them empathize with the difficulties experienced by the children they teach.

On the other hand, expert mathematicians who participated were often able to empathize with the detail of the cognitive development required to reach a sophisticated level of mathematical thinking.

Overall, the framework has helped individuals with different forms of expertise to have a clearer grasp of their roles within the whole development. In particular, it enabled participants to realize how mathematics has the potential to become simpler rather than more complicated, through building flexible connections between compressed concepts operating more fluently in new contexts.

12. CRYSTALLINE CONCEPTS

As I reflected on the increasing complication of mathematical thinking and my claim that true mathematical thinking should become not only more powerful but more simple, I realized that the whole edifice could be integrated using a single underlying idea. The thinkable concepts of mathematics are not just compressed at the whim of the thinker, to build creations of the human mind that are totally at the behest of their creator.

The material is not a malleable piece of metal that can be softened and beaten into any shape. It is tightly organized into a specific structure that is a consequence of the mathematics itself.

A thinkable concept that has a necessary structure as a consequence of its context will be said to be *crystalline*. The term does not signify that the concept necessarily has the physical features of a crystal, such as faces of a particular symmetrical shape, but that it has strong bonds within it that cause it to have inevitable properties in its given context.

This notion binds each of the developments in the three worlds of mathematics into a single overall framework. Each world builds from complicated situations, where phenomena may be imagined to have a combination of properties that are steadily linked together and seen to have necessary consequences that are implied by the context. In each world of mathematics, crystalline concepts emerge that have a network of related properties and, at its apex, mathematical thinking involves rich blends of formalism interrelated with embodiment and symbolism.

Even though each world constructs sophisticated mental objects in different ways, the objects themselves—as platonic figures in geometry, numbers in arithmetic, and defined concepts in formal mathematics—all grow as structures that need to be recognized, described, defined and related through appropriate forms of proof.

In the embodied world of Euclidean geometry, the phenomena are initially figures drawn on paper or sketched in sand. As their properties are observed and described, verbal definitions are used as a basis for constructing figures and proving theorems to develop the crystalline structures of Euclidean concepts.

Actions beginning in the embodied world are transformed into operations in the symbolic world enshrined in the crystalline structures of an increasingly sophisticated range of procepts.

In the world of axiomatic formal mathematics, a complex structure is seen as having properties that can be described, and then carefully defined as the basis of a formal theory whose crystalline structure is deduced by mathematical proof. In all three worlds we see a long-term structural abstraction in mathematical thinking through recognition, description, definition and deduction.

We therefore see the whole development of mathematical thinking as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts.

This development varies enormously in different individuals depending on how they cope with the long-term evolution of ideas as supportive met-befores lead to generalization in new contexts and problematic met-befores inhibit progress.

13. A BRIEF OVERVIEW

This opening chapter has outlined the major ideas that underpin a theory of how humans learn to think mathematically as they mature through three mental worlds of mathematics. In the following chapters, individual aspects will be considered in greater detail.

Chapter two will consider the early learning of the young child constructing ideas of shape and arithmetic.

Chapter three studies the general development of mathematical thinking in terms of set-befores and met-befores.

Chapter four considers the roles of compression, connection and blending into crystalline concepts.

Chapter five speaks of the emotions related to success and failure in mathematical thinking as the learner faces successive developments that may be powerful and pleasurable in some circumstances yet problematic in others.

Chapter six introduces the three worlds of mathematics in detail to see how the growing child in today's society is taught to develop mathematical thinking, blending together embodiment and symbolism before some go on to encounter the fundamental change in thinking required to shift to the axiomatic formal world of mathematics.

Chapter seven considers the subtle relationships between embodiment and symbolism as embodied compression focusing on the effect of operations on visible objects gives more general insight into their properties than operational compression that focuses on specific calculations. It transpires that embodiment provides human meaning in simple cases but specific embodiments may have problematic aspects that impede generalization. This leads to a discussion of the complementary nature of embodiment and symbolism in longer-term learning.

Chapter eight studies how knowledge structures may be used in problem solving at all levels and leads to a consideration of the long-term development of mathematical proof.

Chapter nine turns to the historical development of mathematical thinking in the light of the framework of embodiment, symbolism and formalism, in particular to take a more detailed look at the historical development of formal proof.

Chapter ten returns to individual development to consider the transition from school mathematics to formal mathematics at university, which may involve a 'natural' approach based on previous embodied and symbolic experiences, or a 'formal' route focused on making sense of the logical deductions in the theorems. Others, who find the ideas problematic may find it necessary to focus on the alternate goal of learning proofs procedurally to pass examinations.

Chapter eleven considers the teaching and learning of calculus where the formal limit concept often proves to be problematic. The three-world framework suggests that a natural approach blending embodied properties of graphs and related symbolism can provide a sound foundation, appropriate both for practical applications and also for later development in formal mathematical analysis.

Chapter twelve studies how expertise in formal mathematics may lead to formal knowledge structures becoming conceived as rich crystalline concepts. Furthermore, certain theorems, called 'structure theorems', will be shown to prove properties that lead to more sophisticated forms of embodiment and symbolism. This integrates mathematical thinking at the highest level as an intimate blend of all three worlds of mathematics. Individual mathematicians may work in specialisms that privilege embodied thought experiment, operational calculations, formal proof, or a blend of different aspects.

In chapter thirteen, the blend of three worlds is illustrated by considering the mathematics of the infinitely large and infinitely small. These ideas were seen as problematic at various stages in history. However, a structure theorem may be proved in the formal world that reveals a blend of visual magnification and operational symbolism that makes sense of long-standing conflicts

Chapter fourteen considers the further expansion of mathematical thinking through research, blending natural embodiment, symbolic manipulation, and formal proof, as appropriate for the particular context.

Chapter fifteen reflects on the overall framework and relates it to other theories of mathematical thinking. This reveals a long-term evolution of our understanding of how humans learn to think mathematically. It reveals a unified framework based on the *sensori motor language* of mathematics through blending conceptual embodiment, operational symbolism and axiomatic formalism to form crystalline concepts that are the essence of sophisticated mathematical thinking.

It also reveals how supportive and problematic met-befores affect the emotional and cognitive development of individuals. This includes not only children in school and students at university learning mathematics, but all of us, including mathematicians at the frontiers of research and theoreticians like myself who develop theories of how humans learn to think mathematically. As we consider the theoretical frameworks available to us, we need to reflect deeply on how our own beliefs are subtly shaped by our own personal experiences.

The book closes with an appendix tracing the evolution of this theory to reveal its origins in the insights of others to whom I am forever in debt.

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