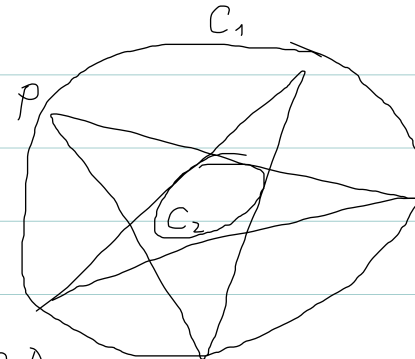
Damian Testa: Poncelet's Porism



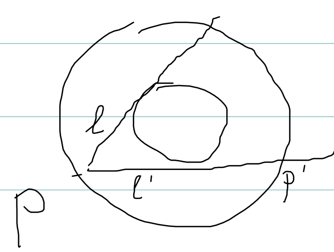
Start with two plane conics, and point p on one of the conics. Starting at p draw successive tangents to the other one. If we eventually get back to p, then you get back to q on the first conic for any such q in the same number of steps. This is called the Poncelet's Theorem.

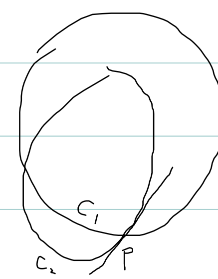
What does this have to do with the moduli of abelian varieties?

Let $E = \{(p, l) \mid p \in C_1,\ l$ line through p & tangent to $C_2\}$.

$E \xrightarrow{2:1} C_1$, $(p, l) \mapsto p$ is a 2:1 cover branched at points of intersection of C_1 and C_2, where the tangents merge

Note that $C_1 \cong \mathbb{P}^1$





We also have a map

$E \to E$, $(p, l) \mapsto (p', l')$

Here, l' is tgt line to C_2 at p different from l.

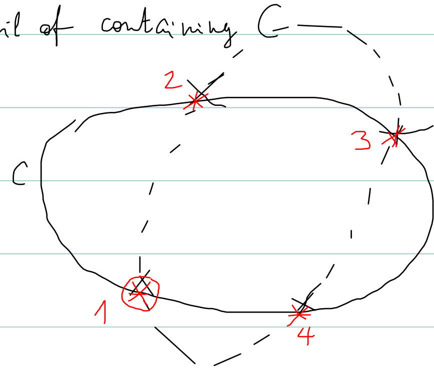
E is an elliptic curve, since there are 4 ramification points.

A self-map from elliptic curve to an elliptic curve is an automorphism with no fixed points

Therefore τ is a translation, if is a point of finite order.

So, two conics in plane $\rightsquigarrow$ genus one curve and a translation with some order.

Now, fix one conic C and a general pencil of containing C (i.e. a linear family of conics)



Let Y be the double cover of $\mathbb{P}^2$ branched over C.

Each conic in the pencil produces a double cover of itself branched at four points.

This yields a one-parameter family of genus one curves with four points marked.

$\rightsquigarrow \tilde{Y} \to \mathbb{P}^1_t$ this is a family of elliptic curves with 2-torsion points marked

($\mathbb{P}^1_t$: base of the pencil)

$\rightsquigarrow X_1(2)$ this requires work, which can be found in Barth, Michel (that's what $\sim$ means).

This is a moduli space of elliptic curves with level 2-structure

$C_m \subset \widetilde{Y}$, $\quad C_m = \{(t,p) \mid p \text{ has order } m\}$

C_m is a moduli space of elliptic curves with level-2 structure and a point of order m.

To come:

HS 1: $\mathcal{A}_g$ and level structures

HS 2: Classifying where moduli spaces lie in classification

HS 4: Explicit examples of moduli spaces ($g = 2$)

HS 6: Degenerations

Marco Streng

<u>Defn</u> (Elliptic curve via Weierstrass equation).

<u>Defn</u> A (smooth geom. irred. proj. alg.) pointed curves of genus 1.

There are proj. group varieties of genus 1.

$m: E \times E \to E$, $\iota: E \to E$.

<u>Defn</u> If $k = \mathbb{C}$, a compact analytic manifold of form V/Λ with V a 1-dim'l vector space, Λ a lattice of rank 2.

<u>Defn</u> (Complex torus as $T = V/\Lambda$ with $V \simeq \mathbb{C}^g$, $\operatorname{rank}_{\mathbb{Z}} \Lambda = 2g$)

Topologically, $T \cong (\mathbb{R}/\mathbb{Z})^{2g} \cong (\text{torus})^g$

<u>Defn</u> Abelian variety over a field k is a projective algebraic irreducible group variety.

<u>Defn</u> If $k = \mathbb{C}$, a complex torus that is isomorphic to a projective variety.

Defn A Riemann form on $T = V/\Lambda$ is an $\mathbb{R}$-bilinear form
$E : V \times V \to \mathbb{R}$ s.t.

- E is alternating $E(v,u) = -E(u,v)$
- $E(\Lambda \times \Lambda) \subseteq \mathbb{Z}$
- $(u,v) \mapsto E(iu,v)$ is symm. & positive definite

Thm T is algebraic iff T is projective iff exists Riemann form on T

Defn A pair $T = (T, E)$ is called a polarized abelian variety

Remark: there exists a purely algebraic defn of polarization.

Lemma Let $\Lambda = \mathbb{Z}^{2g}$ and $E : \Lambda \times \Lambda \to \mathbb{Z}$ alternating.
Then exist basis $e_1, \dots, e_g, v_1, \dots, v_g$ of Λ
and nonnegative integers $d_1 | \dots | d_g$ s.t. E is given w.r.t. to this basis by

$$\begin{pmatrix} 0_{g\times g} & D \\ -D & 0_{g \times g} \end{pmatrix}$$

where

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{pmatrix}.$$

i.e. $E(e_i, e_j) = E(v_i, v_j) = 0$

$E(e_i, v_i) = -E(v_i, e_i) = d_i$

$E(e_i, v_j) = E(v_i, e_j) = 0$ for $i \neq j$.

We call such a basis symplectic. The numbers $d_1, \dots, d_g$ are uniquely determined. We call $(d_1, \dots, d_g)$ the type of the polarization E.

<u>Def</u> E is <u>principal</u> if its type is $(1, \dots, 1)$.

<u>Eg</u> If $g=1$, then any T has a unique principal polarization

$$T = \mathbb{C}/\tau\mathbb{Z}+\mathbb{Z} \qquad \operatorname{Im}\tau > 0$$

$$E = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$$

§ Jacobians

Let C be any curve over $\mathbb{C}$ (equiv. a Riemann surface)

We will assoc a p.p.a.v. $J(C)$ to C

$$H^0(C, \omega_C)^\vee / H_1(C, \mathbb{Z})$$

called the Jacobian of C.

via $\quad H_1(C,\mathbb{Z}) \hookrightarrow H^0(\omega_C)^\vee$ $\leftarrow$ dual

$$[\gamma] \longmapsto \left(\omega \mapsto \int_\gamma \omega\right)$$

Fix $P_0 \in C$. Then we get $\quad C \longrightarrow J(C)$

$$P \mapsto \left(\omega \mapsto \int_{P_0}^{P} \omega\right)$$

This is injective for $g \geq 1$.

We also have the Abel-Jacobi map

$$\operatorname{Pic}^0(C) \xrightarrow{\sim} J(C)$$

$$\left[\Sigma((P_i) - (Q_i))\right] \mapsto \left(\omega \mapsto \Sigma \int_{P_i}^{Q_i} \omega\right)$$

Remark There exists a purely algebraic construction

Idea (say k is alg. closed).

$$C^{\mathrm{Sym}\, g} \longrightarrow \mathrm{Pic}^0(C)$$

$$\{P_1, \dots, P_g\} \longmapsto [P_1 + \dots + P_g - D]$$ fixed divisor of degree g.

by contracting subvarieties, can make it 1 - 1.

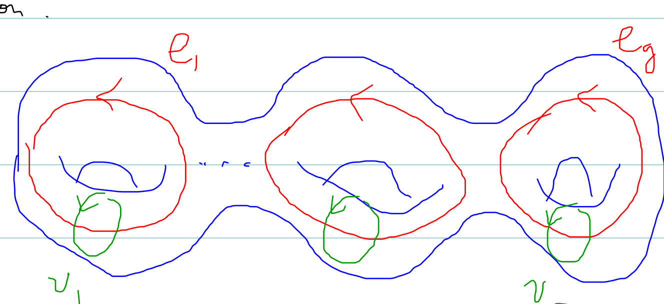
Polarization on $J(C)$: on $\Lambda = H_1(C, \mathbb{Z})$ there is an oriented intersection paring

for $u, v \in \Lambda$ let $E(u,v) = v \cdot u = - v \cdot u$

and extend by $\mathbb{R}$-linearity.

Fact: This is a polarization.

Symplectic basis:



Type $(1, \dots, 1)$

Hint to polarization in algebraic setting

$$\begin{array}{ccc} J(C) & \dashleftarrow\dashrightarrow & C^{\mathrm{Sym}\, g} \longrightarrow \mathrm{Pic}^0(C) \\ & & \uparrow \\ & & C^{\mathrm{Sym}\, g-1} \end{array}$$

• Diane

$A = V/L$, $V \cong \mathbb{C}^g$, $L \cong \mathbb{Z}^{2g}$ discrete

A is projective iff exists polarization, i.e. Riemann form:

$\mathbb{R}$-bilinear form $E: V \times V \to \mathbb{R}$ s.t.

- $E(u,v) = -E(v,u)$
- $E(L,L) \subseteq \mathbb{Z}$
- $(u,v) \mapsto E(iu,v) \overset{E(iv,u)}{=}$ is symmetric, pos. def., $E(iu,u) > 0$ for $u \neq 0$.

$H: V \times V \to \mathbb{C}$, $(u,v) \mapsto E(iu,v) + iE(u,v)$ is Hermitian, pos. def.,

satisfies $H(v,u) = \overline{H(u,v)}$ (Check all of this!)

$$\begin{array}{ccc} E(iu,iv) & = & E(u,v) \\ \parallel & & \parallel \\ E(iiv,u) & = & E(-v,u) \end{array}$$

Recall: given E we can choose a symplectic basis for L

$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, $d_1 | d_2 | \ldots | d_g$

s.t. $E(\alpha_i, \beta_i) = d_i$, all other pairings zero.

$$E(u,v) = u^T \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} v$$
$$D := \mathrm{diag}(d_1, \ldots, d_g)$$

Principal polarization: $d_1 = \ldots = d_g = 1$.

Note: we can describe V/L, E by giving $d_1, \ldots, d_g$, $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\} \subseteq \mathbb{C}^g$

Lemma Set $e_i := \frac{1}{d_i}\alpha_i$. Then $\{e_1, \ldots, e_g\}$ is a basis for V.

Pf: Let W be the real span of $\{e_1,\dots,e_g\}$. Now $E|_W = 0$, but

$Q(u) := E(iu, u) > 0$ for $u \neq 0$. So $u \in W \Rightarrow iu \notin W$.

$W \cap iW = \{0\}$, so $V = W \oplus iW$. $\boxtimes$

Let Ω be the $2g \times g$ matrix with rows $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ in the basis $e_1,\dots,e_g$; $\left(\frac{D}{\tau}\right)$, e.g., if $g=1$, $\Omega = \binom{d_1}{\tau}$ for $L = \mathbb{Z} + \tau\mathbb{Z}$.

Prop. τ is a $g \times g$ symmetric matrix with $\mathrm{im}(\tau)$ pos. def.

Defn The Siegel upper half space is $\boxed{\mathbb{H}_g = \{\tau \in M(g,\mathbb{C}) \mid \tau = \tau^t, \mathrm{im}(\tau) > 0\}}$.

Note. Given $\tau \in \mathbb{H}_g$, we construct $\Omega = \binom{D}{\tau}$, $L_\tau = \mathbb{Z}[\text{rows of } \Omega]$

$A = \mathbb{C}^g / L_\tau$ is an abelian variety with polarization of type $(d_1,\dots,d_g)$ given by $\mathrm{im}(\tau)$.

Pf: Let $\tau = R + iS$. Consider the $\mathbb{R}$-basis $e_1,\dots,e_g, ie_1,\dots,ie_g$ of V, and compare with the basis $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$.

This has a change of basis matrix $[I]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} D & R \\ 0 & S \end{pmatrix}$

E in basis $\mathcal{B}_1$ is $\begin{pmatrix} D & R \\ 0 & S \end{pmatrix}^{-1} \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \begin{pmatrix} D & R \\ 0 & S \end{pmatrix} = \begin{pmatrix} 0 & S^{-1} \\ -S^{-t} & S^{-t}(R - R^T) S^{-1} \end{pmatrix}$

$= \begin{pmatrix} 0 & S^{-1} \\ -S^{-t} & 0 \end{pmatrix}$, because

$S^{-t}(R - R^t) S^{-1} = 0 \Rightarrow R = R^T$ See Debarre's Complex Tori...

since E is skew-symmetric and so $S^{-t} = S^{-1} \Rightarrow S = S^t$, i.e. $\tau = \tau^t$. $\boxtimes$

Summary so far: $\mathbb{H}_g \longrightarrow \{$ab. vars with polarization of type $(d_1,\dots,d_g)\}$
$\tau \longmapsto L \qquad \begin{pmatrix} D \\ \tau \end{pmatrix}$.

<u>Problem</u>: different τ can give isomorphic polarized ab. var.

<u>Eg</u> $\mathbb{C}/\langle 1, 1+i\rangle = \mathbb{C}/\langle 1, 2+i\rangle$.

Issue: chose a symplectic basis, write $\Lambda = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$

<u>Defn</u> $Sp(\Lambda, \mathbb{Z}) = \{h \in GL(2g, \mathbb{Z}),\ h^t \Lambda h = \Lambda\}$

If $D = I$, $Sp(\Lambda, \mathbb{Z}) = Sp(2g, \mathbb{Z})$

Point: $h = [I]^{\mathcal{B}_1}_{\mathcal{B}_2}$ if $\mathcal{B}_1, \mathcal{B}_2$ are different symplectic bases.

$Sp(\Lambda, \mathbb{Z}) \circlearrowright \mathbb{H}_g$.

$h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then $h\begin{pmatrix} \tau \\ D \end{pmatrix} = \begin{pmatrix} \alpha\tau + \beta D \\ \gamma\tau + \delta D \end{pmatrix}$.

Multiply by $(\gamma\tau + \delta D)^{-1} D$ to obtain $\begin{pmatrix} (\alpha\tau + \beta D)(\gamma\tau + \delta D)^{-1} D \\ D \end{pmatrix}$

i.e. $h.\tau = (\alpha\tau + \beta D)(\gamma\tau + \delta D)^{-1} D$.

<u>Eg</u> $g = 1, D = 1$: $\quad \tau \mapsto \dfrac{\alpha\tau + \beta}{\gamma\tau + \delta}$

<u>Defn</u> $A_{d_1,\dots,d_g} := \mathbb{H}_g / Sp(\Lambda, \mathbb{Z})$

= moduli space of abelian varieties with polarization of type $(d_1, \dots, d_g)$

!

$A_g(n) := A_{\underbrace{n,\dots,n}_{g \text{ times}}}$, $A_g := A_g(1)$, also $\mathbb{H}_g/\Gamma_g(n)$ for $\Gamma_g(n) := \{h \in Sp(2g, \mathbb{Z}) \mid h \equiv I \bmod n\}$.

- David Holmes. Compactification of Siegel Modular Varieties.

Recall: principally polarized level 1 case

$\mathbb{H}_g = \{ \tau \in M_g(\mathbb{C}) \mid \tau^t = \tau, \ \mathrm{Im}(\tau) > 0 \}$

$Sp_{2g}\mathbb{Z} \circlearrowright \mathbb{H}_g$ by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau := (A\tau + B)(C\tau + D)^{-1}$.

$A_g := Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$.

Def A compactification of A_g is a complex proj. variety with a dense open subset $\cong A_g$ as alg. variety.

Why? - Michael's Talk

- degeneration of abelian varieties

Eg $\dim = 1$: recall $j: A_1 \xrightarrow{\sim} \mathbb{A}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$ - Satake compactification.

Topological view: Cayley transform $\varphi: \mathbb{H}_1 \longrightarrow \mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \} \subset \mathbb{C}$

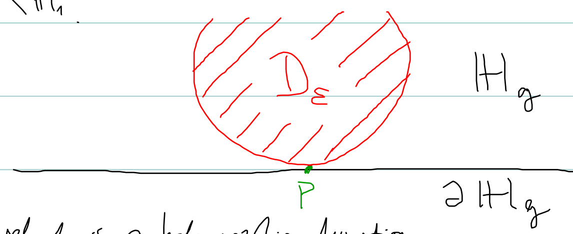
$\tau \mapsto \frac{\tau - i}{\tau + i}$

Add in $\varphi(\mathbb{Q}) \cup \{1\}$.

Def A partial compactification of $\mathbb{H}_g$ is a top. space $\overline{\mathbb{H}}_g$ containing $\mathbb{H}_g$ as a dense open subset, s.t. $Sp_{2g}(\mathbb{Z})$ acts on $\overline{\mathbb{H}}_g$ and $Sp_{2g} \backslash \overline{\mathbb{H}}_g$ is compact.

Eg Set $\overline{\mathbb{H}}_1 := \varphi(\mathbb{H}_1 \cup \mathbb{Q}) \cup \{i\}$ as a set. DON'T take subspace topology from $\mathbb{C}$: the quotient will not be hausdorff.

Put the horocycle \ Satake topology on $\overline{\mathbb{H}}_1$. Let $p \in \overline{\mathbb{H}}_1 \setminus \mathbb{H}$. A system of open nbhds of p is $\{\{p\} \cup D_\varepsilon \mid \varepsilon > 0\}$ where D_ε is an open disk of radius ε touching $\partial \mathbb{H}_1$ at p. Then $\overline{\mathbb{A}}_1^S = Sp_2\mathbb{Z} \setminus \overline{\mathbb{H}}_1$.



- Dimension > 1.

Def A modular form of weight k and level 1 is a holomorphic function $F : \mathbb{H}_g \to \mathbb{C}$ s.t. for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$ we have

$$F(M\tau) = \det(C\tau + D)^k F(\tau).$$

Facts: - Elements of $Sp_{2g}(\mathbb{Z})$ which have fixed pts are torsion

- the order of all torsion el'ts of $Sp_{2g}(\mathbb{Z})$ is bounded.

Fix $k \geq 1$. Then exists $n > 0$ s.t. the $M(nk)$ of modular forms of weight nk is equal to the global sections of very ample line bundle on A_g.

$$A_g \hookrightarrow \mathbb{P}(M(nk))^\vee$$

Def The Satake compactification is the closure of A_g in $\mathbb{P}(M(nk))^\vee$.

- Partial compactifications:

Step 1. Cayley transform: $\varphi : \mathbb{H}_g \to \mathrm{Sym}^g(\mathbb{C})$ $\quad \mathbb{D}_g := \varphi(\mathbb{H}_g)$

$$\tau \mapsto (\tau - Ii)(\tau + Ii)^{-1}$$

Step 2. A real affine hyperplane $H \subset \mathrm{Sym}^g \mathbb{C}$ is

- a supporting hyperplane if $\mathbb{D}_g \cap H = \emptyset$, but $\overline{\mathbb{D}}_g \cap H \neq \emptyset$,
- rational if it is spanned by rational vectors

Step 3. $\overline{\mathbb{H}}_g = \mathbb{H} \cup \{\text{boundary components}\}$

Let H be a rational supp. hyperplane. Let $\overline{F} := H \cap \overline{\mathbb{D}}_g$, let L be the smallest affine space containing $\overline{F}$.

Then the interior of $\overline{F}$ in L is a boundary component.

Satake compactification

- Normal
- singular along boundary
- $\overline{A}_g = \bigsqcup_{n \le g} A_n$

Toroidal compactification

- resolution of singularities of Satake compactification
- not unique.

Classification stuff. Surfaces

§0. Introduction.

Classification up to birational isomorphism:

C	$\mathbb{P}^1$	Ell	K_C ample
g	0	1	2

<u>Defn</u> X – smooth proj. surface, then X is <u>minimal</u> if for all birational morphisms $X \longrightarrow Y$ we have $X \xrightarrow{\sim} Y$.

Except for some cases, X is minimal iff K_X is nef
(Recall: nef is $K_X . C \geq 0$ for all curves $C \subset X$)

<u>Defn</u> Say X is normal proj. variety over $\mathbb{C}$ of dim'n n, X° is the nonsingular locus, $\omega_{X^\circ} := \bigwedge^n \Omega_{X^\circ}$ is invertible on $X^\circ \longleftrightarrow K_{X^\circ}$ on X°, extend to X via Hartog (take closure)

<u>Defn</u> <u>Kodaira dimension</u>

$$\kappa(X) := \begin{cases} \max\limits_{m \geq 1} \{\dim \varphi_{mK_X}(X)\} \\ -\infty \text{ if } \underbrace{h^0(mK_X)}_{=: P_m} = 0 \text{ for all } m \geq 1 \end{cases}$$

Fact: $\kappa(X) = d$ iff $P_m \sim m$.

§1. Calculating the canonical divisor.

Recall: $\mathbb{H}_g := \{\tau \in \mathrm{Sym}^{g\times g}(\mathbb{C}) \mid \mathrm{im}(\tau) > 0\}$

$\Gamma_g := Sp(2g, \mathbb{Z})$ or $\Gamma_g(n) := \{\gamma \in \Gamma_g \mid \gamma \equiv 1 \bmod n\}$

$\Gamma_g \circlearrowright \mathbb{H}_g$ by $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\gamma\cdot\tau = (A\tau + B)(C\tau + D)^{-1}$.

$\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$, $\mathcal{A}_g(n) := \Gamma_g(n) \backslash \mathbb{H}_g$.

Defn: $F : \mathbb{H}_g \to \mathbb{C}$ is a modular form of weight k and level $\Gamma_g(n)$ if

- F is holomorphic
- $F(M\tau) = \det(C\tau + D)^k F(\tau)$ for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)$

- Line bundles on $\mathcal{A}_g(n)$.

  Assume: Γ_g acts freely on $\mathbb{H}_g$.

  Define action $\Gamma_g \circlearrowright \mathbb{H}_g \times \mathbb{C}$ as $\gamma(\tau, z) = (\gamma\tau, \det(C\tau + D)^{-1} z)$.

This makes $\Gamma_g \backslash \mathbb{H}_g \times \mathbb{C}$ into a line bundle over $\mathcal{A}_g$.

Γ_g-invariant section is a Γ_g-invariant function on $\mathbb{H}_g \times \mathbb{C}$ with values in $\mathbb{C}$.

Let $\tilde{F} : \mathbb{H}_g \times \mathbb{C} \to \mathbb{C}$ be a modular form of weight 1.

$\tilde{F}(\tau, z) := F(\tau) z$.

Then $\widetilde{F}$ is Γ_g-invariant. Indeed,

$$\widetilde{F}(M(\tau,z)) = \widetilde{F}(M\tau,\ \det(C\tau+D)^{-1}z)$$
$$= F(M\tau)\cdot\det(C\tau+D)^{-1}z$$
$$= \det(C\tau+D)\,F(\tau)\,\det(C\tau+D^{-1})z$$
$$= F(\tau)\,z$$
$$= \widetilde{F}(\tau,z).$$

Similarly, $\Gamma_g \circlearrowright \mathbb{H}_g\times\mathbb{C}$

$$\gamma(\tau,z) = (\gamma\tau,\ \det(C\tau+D)^{-k}z)$$

$\Gamma_g \circlearrowright \mathbb{H}_g\times\mathbb{C}$ defines a line bundle L on A_g.

<u>Punchline</u>: sections of $L^{\otimes k}$ correspond to weight k modular forms $\mathbb{H}_g \longrightarrow \mathbb{C}$

<u>Aim</u>: Want a top diff. form $d\tau$ on $\mathbb{H}_g$,

let F be a weight $(g+1)$ modular form,

Prove: $F\,d\tau$ is Γ_g-invariant.

Conclusion: $K_{A_g} = L^{\otimes(g+1)}$.

<u>Case</u> $g=1$: $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\gamma.(F d\tau) = F|_\gamma\, d(\gamma\tau)$

$$= F|_\gamma\, d\left(\frac{a\tau+b}{c\tau+d}\right) = \ldots (\text{proof})$$
$$= F\,d\tau$$

$g>1$ case: Exercise in multivariate calculus with worries about commuting matrices.

Here, for
$$\tau = \begin{pmatrix} \tau_{11} & & \tau_{1g} \\ & \ddots & \\ \tau_{g1} & & \tau_{gg} \end{pmatrix}$$
$$d\tau := \bigwedge_{i \le j} d\tau_{ij} = d\tau_{11} \wedge d\tau_{12} \wedge \ldots \wedge d\tau_{gg}.$$

- What about boundary?

Fact: Let $\overline{A}_g$ be a toroidal compactification s.t. $D := \overline{A}_g \setminus A_g$ is a normal crossing divisor.

and $\operatorname{rank} \underbrace{\operatorname{Pic}(\overline{A}_g)}_{= \langle L, D \rangle} = 2$.

Claim $F d\tau$ has a simple pole along D, where F is a weight $(g+1)$ cusp form not vanishing too often.

$g=1$ $\mathbb{H} \xrightarrow{\varphi} \mathbb{C}$
$$z \mapsto q := e^{2\pi i z}$$
$$\varphi(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})$$
$$dz = \frac{1}{2\pi i} \frac{dq}{q}.$$
$$\operatorname{div}(dz)|_\infty = -(\infty)$$

Conclusion: $K_{\overline{A}_g} = (g+1)L - D$

<u>Thm</u> (Freitag) This extends to a non-free action.

Also true for $A_g(n)$ of general type (i.e. all $n \geq n_0$ for

g	1	2	3	4	5	6	7	...
n_0	7(?)	4	3	2	2	2	1	...)

- Level (2,4,8) structure on genus 2 curves
  (or rather on their jacobians)

We will be working with p.p. ab. surfaces.

k - field of char $\neq 2$.

C - genus 2 curve

$J(C)$ - its Jacobian

$J(C)[2]$ - it's 2-torsion with symplectic structure.

Level 2 structure - choice of symplectic basis.

If C is of genus 2, then it is hyperelliptic with canonical map $\varphi : C \xrightarrow{2:1} \mathbb{P}^1$ ramified at exactly 6 pts.

It has an affine model $y^2 = f(x)$ with $\deg f = 6$, f having no repeated roots.

What is $J(C)$? $\quad \mathrm{Sym}^2(C)$ "=" $J(C)$

$\downarrow$

unordered pairs of pts. in C
with rep'ns allowed: $p+q$, $p,q \in C$

Recall that, as a group, $J(C) = \mathrm{Div}^0(C)/\mathrm{Div}^L(C)$.

$= \{\text{line bundles of deg } 0\}/\cong$

We have $\varphi : \mathrm{Sym}^2(C) \longrightarrow J(C)$

$$p+q \mapsto \mathcal{O}(p+q)(-K_C) = \mathcal{O}_C(p+q-K).$$

where K is the canonical divisor.

What are the fibers of φ:

$\mathcal{L}$ – line bundle of degree zero,

$$\mathbb{P}(H^0(\mathcal{L}(K)) = \varphi^{-1}(\mathcal{L})$$

$$H^0(\overbrace{\mathcal{L}(K)}^{\deg 2 \text{ by R.R.}}) \qquad h^0(\mathcal{L}(K)) - h^1(\mathcal{L}(K)) = 2 - 1 = 1$$

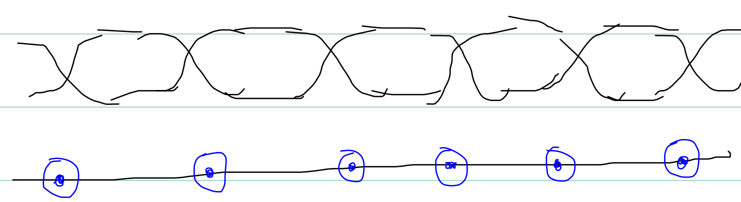
Typically, $h^0(\mathcal{L}(K)) = 1$, $h^1(\mathcal{L}(K)) = 0$

So, φ is an "isomorphism" away from $\varphi^{-1}(\mathcal{O}_C)$ and replaces $\mathcal{O}_C$ by $\mathbb{P}^1$.

So $\mathrm{Sym}^2(C) = Bl_{\mathcal{O}_C}(J(C))$.

Remark (J. Cremona): This is how CompSci views Jacobians of genus 2 curves for the purposes of cryptography

§ Level-2 structures:



$f(x) = y^2$, If p is ram. pt., then $2p \in \mathrm{Sym}^2(C)$

$\varphi(2p) = \mathcal{O}_C(2p - K) = \mathcal{O}_C$, since $2p \sim K$.

Say $p \neq q$ are ram. pts. What is $\mathcal{O}_C(p+q-K)$?

$\mathcal{O}_C(2(p+q-K)) = \mathcal{O}(K+K-2K) = \mathcal{O}_C$.

If $p \neq q$, then $p+q$ "is" a 2-torsion pt not of order 1.
6 branch pts $\Rightarrow$ 6 ram pts $\Rightarrow$ there are $\binom{6}{2} = 15$ points on $J(C)$ of order 2. These are the only ones.
Choosing $p=q$ yields identity on $J(C)$, i.e., we have 6 different representatives for the sole el't of order 1.

§ Symplectic structure:

$$\langle (p,q), (p',q') \rangle := \tau \bmod 2 \in \mathbb{Z}/2\mathbb{Z}$$

where $\tau = \#$ coincidences among (p,q) and (p',q')

$$= \#(\{p,q\} \cap \{p',q'\})$$

This defines a symplectic structure on $J(C)[2]$.

What is a symplectic basis?

We have $\underbrace{v_1 \quad v_2}_{\text{Lagrangian subspace}}$ $\underbrace{v_3 \quad v_4}_{\text{Ditto}}$, where $\langle v_1, v_3 \rangle = \langle v_1, v_4 \rangle = 0$

$\langle v_1, v_2 \rangle = 1$

$\langle v_3, v_4 \rangle = 1$

$J(C)[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$ 12 23 | 45 56

Q symmetric basis iff ordering of branch pts.

Moduli space of abelian surfaces (p.p.) with level-2 structure
– roots of poly $f(x)$ up to action of GL_2.

§ Igusa Quartic:

$Q_4 \subset \mathbb{P}^4$

$Q_4 < \mathbb{P}^5_{x_0, \dots, x_5}$

$\parallel$

$V(\sigma_1(x), 4\sigma_4 - \sigma_2^2)$, where σ_i are the elem. sym. functions,

i.e., $\prod_i (t + x_i) = \sum \sigma_{6-i}(x) t^i$.

We have $p \in V \rightsquigarrow$ ab. surface with level-2 structure

Q_4 is singular along 15 lines:

$x_0 = x_2, \quad x_1 = x_3, \quad x_0 + x_1 + x_4 = 0, \quad x_0 + x_1 + x_5 = 0$

$=: Q_4^{sing}$

tangent space to Q_4 at P

$\mathcal{T}_{Q_4, P} \cong \mathbb{P}^3 \supset \overbrace{(Q_4 \cap \mathcal{T})}^{/\!/ \, Q_P} = \text{quartic in } \mathbb{P}^3$

Q_P is singular at P and at the 15 intersection pts of $\mathcal{T}_{Q_4, P}$ with sing pts of Q_4

Therefore, Q_4 is a Kummer surface.

Thus $Q_p \xleftarrow{2:1} A_p$, where A_p is a p.p.a.v. of dim 2
with $Q_p = A_p / \{\pm 1\}$ with framing of 2-torsion subgroup.

<u>Remark</u> (from Miles Reid): something about tropes, 16 conics are called tropes.

More level structures on ab. surfaces
Can also put a $(2,4,8)$-level structure

$$\Theta_{0000}^2 = \Theta_{00}^2 + \Theta_{01}^2 + \Theta_{10}^2 + \Theta_{11}^2$$
$$\dots = \dots \pm \dots \pm \dots \pm \dots$$

These are also the equations for the cuboid problem (Euler)