MULTIGRADED HILBERT SCHEMES

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These are lecture notes for my lectures at the School on *Hilbert schemes, McKay correspondence, and singularities*, held in Paris, 16–18 December 2019.

The goal of these lectures is to introduce the multigraded Hilbert scheme, originally introduced by Haiman and Sturmfels in [HS04], and explain some of what is known about it.

These are lecture notes, and thus are not polished. Please let me know of any typos or mathematical errors!

1. Lecture 1: Definitions

The multigraded Hilbert scheme parameterizes all ideals in a polynomial ring that are homogeneous and have a fixed Hilbert function with respect to a grading by an abelian group. Examples include the Grothendieck Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ of subschemes of $\mathbb{P}^n$ with Hilbert polynomial P, the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{A}^n)$ of points in affine space, Nakamura's G-Hilbert scheme that arises in the McKay correspondence for abelian G [Nak01], and the toric Hilbert scheme [PS02].

We now introduce these Hilbert schemes more formally.

Let $S = R[x_1, \dots, x_n]$ be the polynomial ring with coefficients in a commutative ring R. We can view S as the monoid algebra $R[\mathbb{N}^n]$. A grading of S by an abelian group A is induced from a semigroup homomorphism $\deg\colon \mathbb{N}^n \to A$. This induces a direct sum decomposition $S \cong \oplus_{a \in A} S_a$, where S_a is the R-module spanned by monomials of degree a. This decomposition satisfies $S_a S_b \subseteq S_{a+b}$.

Example 1.1.
(1) $\deg\colon \mathbb{N}^n \to \mathbb{Z}$ given by $\mathbf{u} \mapsto |\mathbf{u}| = \sum_{i=1}^n u_i$. This is the standard grading on the polynomial ring: $\deg(x_i) = 1$ for $1 \leq i \leq n$.
(2) $\deg\colon \mathbb{N}^2 \to \mathbb{Z}/3\mathbb{Z}$ given by $\deg((1,0)) = 1 \bmod 3$, $\deg((0,1)) = 2 \bmod 3$, so $\deg(x_1) = 1 \bmod 3$, and $\deg(x_2) = 2 \bmod 3$. Note that we have $S_0 = R[x_1^3, x_1x_2, x_2^3] = R[x_1, x_2]^{S_3}$.
(3) $\deg\colon \mathbb{N}^n \to 0$, given by $\mathbf{u} \mapsto 0$ for all $\mathbf{u} \in \mathbb{N}^n$.

Definition 1.2. A homogeneous ideal I in S is *admissible* if $(S/I)_a = S_a/I_a$ is a locally free R-module of finite rank, constant on $\mathrm{Spec}(R)$, for all $a \in A$. This means that S_a/I_a is an R-module with $(S_a/I_a) \otimes_R R_P \cong R_P^h$ for all primes $P \subseteq R$, where h does not depend on the choice of prime P.

Example 1.3. (1) When K is a field, and $S = K[x_1, \ldots, x_n]$ has the standard grading, then any homogeneous ideal is admissible.

(2) Consider the standard grading on $S = \mathbb{Z}[x_1, x_2]$. Then $I = \langle x_1 + 3x_2 \rangle$ is admissible, as $(\mathbb{Z}[x_1, x_2]/I)_d$ is locally free of rank 1 forall $d > 0$. Indeed,

$$(\mathbb{Z}[x_1, x_2]/\langle x_1 + 3x_2 \rangle)_d \otimes \mathbb{Z}_{\langle p \rangle} \cong (\mathbb{Z}_{\langle p \rangle}[x_1, x_2]/\langle x_1 + 3x_2 \rangle)_d \cong \mathbb{Z}_{\langle p \rangle}$$

is a free $\mathbb{Z}_{\langle p \rangle}$-module with basis x_2^d.

However $J = \langle 2x_1 + 4x_2 \rangle$ is not admissible, as

$$\mathbb{Z}[x_1, x_2]/\langle 2x_1 + 4x_2 \rangle \otimes \mathbb{Z}_{\langle 2 \rangle} \cong \mathbb{Z}_{\langle 2 \rangle}[x_1, x_2]/\langle 2x_1 + 4x_2 \rangle$$

is not a free $\mathbb{Z}_{\langle 2 \rangle}$-module, since it is torsion: $2(x_1 + 2x_2) = 0$, but $x_1 + 2x_2 \neq 0$.

(3) Under the trivial grading $\deg\colon \mathbb{N}^2 \to 0$, $I = \langle x^2, xy \rangle \subseteq \mathbb{C}[x, y]$ is not admissible, as $\mathbb{C}[x, y]/I$ is a free $\mathbb{C}$-module, but is not of finite rank. The ideal $J = \langle x^2, y^2 \rangle$ is admissible, as $\mathbb{C}[x, y]/J$ is free of rank 4.

Definition 1.4. The *Hilbert function* of an admissible ideal I in S is

$$h_I\colon A \to \mathbb{N}$$

given by

$$h_I(a) = \mathrm{rk}_R(S/I)_a.$$

Informally, given a function $h\colon A \to \mathbb{N}$, the multigraded Hilbert scheme Hilb_S^h parameterizes all admissible ideals I in S with $h_I = h$.

We now give a form formal definition. Fix a commutative ring K. We construct a functor $H_S^h\colon K$-algebras $\to$ Sets given by setting, for a commutative ring R,

$$H_S^h(R) = \{ \text{ homogeneous ideals } I \subseteq R[x_1, \dot{,} x_n] \text{ such that } (R[x_1, \ldots, x_n]/I)_a$$
$$\text{is a locally free } R-\text{module of rank } h(a) \text{ for all } a \in A\}.$$

Recall that a scheme Z over K represents a functor $F\colon K-\text{algebras} \to$ Sets if $F \cong \mathrm{Hom}(-, Z)$, so there is a natural bijection between $F(R)$ and $\mathrm{Hom}(\mathrm{Spec}(R), Z)$. We say that F is the functor of points of Z, and that Z represents F.

We say that a grading $\deg\colon \mathbb{N}^n \to A$ is *positive* if $\deg^{-1}(0) = \mathbf{0}$. The standard grading on the polynomial ring is positive, but the other two gradings of Example 1.1 are not.

Theorem 1.5. *(Haiman-Sturmfels [HS04]). There is quasi-projective scheme* Hilb_S^h *over* K *that represents* H_S^h*. This scheme is projective if the grading is positive.*

Example 1.6. Let $\deg\colon \mathbb{N}^2 \to \mathbb{Z}$ be given by $\deg((1,0)) = 1$, and $\deg((0,1)) = 2$. Consider the function $h\colon \mathbb{Z} \to \mathbb{N}$ given by $h(0) = h(1) = h(2) = 1$, and $h(d) = 0$ for all other d. Then admissible ideals with Hilbert function h have the form $\langle ax^2 + by, x^3, xy, y^2\rangle \subseteq R[x,y]$, for $a, b \in R$ with $ab \neq 0$, so $\mathrm{Hilb}_S^h \cong \mathbb{P}^1$.

Example 1.7. Let $\deg\colon \mathbb{N}^n \to 0$ be the trivial map, and let $h\colon 0 \to \mathbb{N}$ to be given by $h(0) = d$ for some $d \in \mathbb{N}$. Then $\mathrm{Hilb}_S^h \cong \mathrm{Hilb}^d(\mathbb{A}^n)$ is the Hilbert scheme of d points in $\mathbb{A}^n$. When K is a field, admissible ideals in $K[x_1,\dots,x_n]$ with Hilbert function h are ideals $I \subseteq K[x_1,\dots,x_n]$ with $\dim_K K[x_1,\dots,x_n]/I = d$.

Example 1.8. Let A be a finite abelian group, and let $h\colon A \to \mathbb{N}$ be given by $h(a) = 1$ for all $a \in A$. Then Hilb_S^h is Nakamura's G-Hilbert scheme [Nak01] for $G = A$. This is a key player in the McKay correspondence.

Example 1.9. Let $A = \mathbb{Z}^r$, and let $\deg\colon \mathbb{N}^n \to A$ be a grading with $\deg(\mathbf{e}_i) \in \mathbb{N}^r$ for $1 \leq i \leq n$. Let A^+ be the submonoid of $\mathbb{N}^r$ generated by $\{\deg(\mathbf{e}_1),\dots,\deg(\mathbf{e}_r\}$, and let $h\colon A \to \mathbb{N}$ be given by $h(a) = 1$ if $a \in A^+$ and $h(a) = 0$ otherwise. Then Hilb_S^h is the toric Hilbert scheme of Peeva and Stillman [PS02].

Example 1.10. Let $\deg\colon \mathbb{N}^n \to \mathbb{Z}$ be the standard grading, and fix $P \in \mathbb{Q}[t]$ with $P(m) \in \mathbb{N}$ for all $m \in \mathbb{N}$. Fix $D \gg 0$. Define $h\colon \mathbb{Z} \to \mathbb{N}$ by

$$h(d) = \begin{cases} \dim_K(S_d) = \binom{n+d-1}{d} & \text{for } 0 \leq d \leq D-1 \\ P(d) & d \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Then Hilb_S^h is $\mathrm{Hilb}_P(\mathbb{P}^{n-1})$, which is Grothendieck's Hilbert scheme parameterizing subschemes of $\mathbb{P}^{n-1}$ with Hilbert polynomial P.

This is not the standard description of this Hilbert scheme. A more usual version of the Hilbert function H_P takes a scheme Z to the set of flat families

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ Z \end{array}$$

where $\mathcal{X} \subseteq Z \times \mathbb{P}^{n-1}$, with the Hilbert polynomial of every fiber $\mathcal{X}_z := \mathcal{X} \times_Z \kappa(z)$ for $z \in Z$ equal to P.

To see that this is the same scheme, first note that we can restrict to affine schemes $Z = \mathrm{Spec}(R)$ by standard Yoneda arguments (see [EH00, Proposition VI.2]. So we need only consider $\mathcal{X} \subseteq \mathbb{P}^n_R$. There is a bijection between such X and *saturated* homogeneous ideals $I \subseteq R[x_0,\dots,x_n]$ with Hilbert polynomial P, where an ideal is saturated if $I = I^{sat} = (I : \mathfrak{m}^\infty) := \{f \in S : fx_i^m \in I \text{ for some } m > 0\}$. There is a uniform bound on the degree $D \gg 0$ for which saturated ideals with Hilbert polynomial P have $h_I(d) = P(d)$ for all $d \geq D$. An explicit description of this bound is given by the Gotzmann number associated to P; see [Got78], [Bru98, Chapter 4, §3], and Lecture 3 below. This means that there is a map from $X \subseteq \mathbb{P}^n_R$ with the Hilbert polynomial of fibers equal to P, and ideals with the Hilbert function h given in (1), with the map given by taking $I \subseteq R[x_0, \dots, x_n]$ to $I_{\geq D}$. Furthermore, if $I_d = J_d$ for $d \gg 0$, then $I^{sat} = J^{sat}$, so the map is an injection, and all ideals with Hilbert function h have Hilbert polynomial P.

The fact that Hilb^h_S represents the functor H^h_S means that we have a universal family

$$\begin{array}{c} \mathcal{U} \\ \downarrow \\ \mathrm{Hilb}^h_S \end{array}$$

where $\mathcal{U} \subseteq \mathrm{Hilb}^h_S \times \mathbb{A}^n$ with the property that if

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ B \end{array}$$

is a family with $\mathcal{F} \subseteq B \times \mathbb{A}^n$ invariant under the $\mathrm{Hom}(A, \mathbb{G}_m)$-action corresponding to the grading and every fiber has Hilbert function h, then there is a unique morphism $\phi \colon B \to \mathrm{Hilb}^h_S$ with $\mathcal{F} = \phi^*(\mathcal{U})$.

Example 1.11. With the grading and Hilbert function of Example 1.6, the universal family is $\mathrm{Proj}(\mathbb{Z}[a, b, x, y]/\langle ax^2 + by, x^3, xy, y^2\rangle) \subseteq \mathbb{P}^1 \times \mathbb{A}^2$. Here the grading for the Proj has $\deg(a) = \deg(b) = 1$, and $\deg(x) = \deg(y) = 0$.

2. Lecture 2: Constructions

We now discuss the construction of the multigraded Hilbert scheme. We will show that Hilb^h_S is a subscheme of a product of Grassmannians.

We first roughly sketch this construction. If I is an admissible ideal with $h_I = h$, then I_a defines a point in the Grassmannian $\mathrm{Gr}(h(a), \mathrm{rk}\, S_a)$. Thus I corresponds to a point in the (possibly infinite) product $\prod_{a \in A} \mathrm{Gr}(h(a), \mathrm{rk}(S_a))$. In fact finitely many a suffice to determine I, and if this selection of a is large enough, then any ideals with prescribed Hilbert function in these degrees has Hilbert function h. Equations in this product come from the conditions $x^u I_a \subseteq I_{a+\deg(x^u)}$ for monomials x^u.

Example 2.1. Let $S = \mathbb{C}[x, y]$ have the standard grading $\deg(x) = \deg(y) = 1$. Set $h(0) = 1$, $h(1) = h(2) = 2$, $h(3) = 1$, and $h(d) = 0$ for all $d \geq 4$. If

$h_I = h$, then $I_d = S_d$ for $d \geq 4$, and $I_1 = 0$, so I is determined by I_2 and I_3. Since $h(2) = 2$, while $\dim_{\mathbb{C}}(S_2) = 3$, and $h(3) = 1$ while $\dim_{\mathbb{C}}(S_3) = 4$, I_2 corresponds to a point in $\mathrm{Gr}(1,3) \cong \mathbb{P}^2$, while I_3 corresponds to a point in $\mathbb{P}^{3\vee}$. Let the coordinates on $\mathbb{P}^2$ be denoted by a_0, a_1, a_2, corresponding to the monomials x^2, xy, y^2, and the coordinates on $\mathbb{P}^{3\vee}$ be denoted by b_0, b_1, b_2, b_3, corresponding to the monomials x^3, x^2y, xy^2, y^3. The equations come from the fact that $a_0x^2 + a_1xy + a_2y^2 \in I$ implies that $a_0x^3 + a_1x^2y + a_2xy^2 \in I$ and $a_0x^2y + a_1xy^2 + a_2y^3 \in I$, so

$$a_0b_0 + a_1b_1 + a_2b_2 = a_0b_1 + a_1b_2 + a_2b_3 = 0.$$

Thus Hilb^h_S is the subscheme of $\mathbb{P}^2 \times \mathbb{P}^{3\vee}$ cut out by the ideal $\langle a_0b_1 + a_1b_1 + a_2b_2, a_0b_1 + a_1b_2 + a_2b_3\rangle$. The universal family is defined by the ideal

$$\langle a_0x^2+a_1xy+a_2y^2, b_1x^3-b_0x^2y, b_2x^3-b_0xy^2, b_3x^3-b_0y^3, b_2x^2y-b_1xy^2, b_3x^2y-b_1y^3,$$

$$b_3xy^2 - b_2y^3, a_0b_1 + a_1b_1 + a_2b_2, a_0b_1 + a_1b_2 + a_2b_3\rangle$$

in $\mathbb{C}[a_0, a_1, a_2, b_0, b_1, b_2, b_3, x, y]$, which defines a subscheme of $\mathbb{P}^2 \times \mathbb{P}^{3\vee} \times \mathbb{A}^2$.

We now provide some more detail on this construction.

Given a multigrading $\deg\colon \mathbb{N}^n \to A$ and a Hilbert function $h\colon A \to \mathbb{N}$, we consider the following conditions on a finite set $D \subseteq A$:

1. **(g)**. Every monomial ideal with Hilbert function H is generated by monomials of degrees belonging to D.
2. **(h)** Every monomial ideal I generated in degrees in D satisfies: if $h_I(a) = h(a)$ for all $a \in D$, then $h_I(a) = h(a)$ for all $a \in A$.
3. **(h′)** Every monomial ideal I generated in degrees in D satisfies: if $h_I(a) = h(a)$ for all $a \in D$, then $h_I(a) \leq h(a)$ for all $a \in A$.
4. **(s)** Every monomial ideal I with $h_I = h$ has the property that the syzygy module of I is generated by syzygies $x^u x^{v'} = x^v x^{u'} = \mathrm{lcm}(x^u, x^v)$ among generators x^u, x^v of I with $\deg(\mathrm{lcm}(x^u, x^v)) \in D$.

A set $D \subseteq A$ is called *supportive* if it satisfies g and h', and *very supportive* if it satisfies g, h, and s.

Given a finite set D of degrees, and $h\colon D \to \mathbb{N}$, we construct a subscheme of $\prod_{a\in D} \mathrm{Gr}(h(a), \mathrm{rk}\, S_a)$ as follows. Write $L = (L_a)_{a\in D}$ for an element of $\prod_{a\in D} \mathrm{Gr}(h(a), \mathrm{rk}\, S_a)$. The equations come from

$$x^u L_a \subseteq L_{a+\deg(x^u)}$$

for all monomials x^u with $a, a+\deg(x^u) \in D$. These equations are quadratic in the Plücker coordinates of these two Grassmannians. We call this subscheme $\mathrm{Hilb}^h_{S_D}$.

Theorem 2.2. *(Haiman-Stumfels) If D is supportive then Hilb_S^h is a subscheme of $\mathrm{Hilb}_{S_D}^h$. If D is very supportive then $\mathrm{Hilb}_S^h \cong \mathrm{Hilb}_{S_D}^h$. Finite (very) supportive sets always exist.*

The idea of the proof here is a generalization of Gröbner theory to this setting.

Example 2.3. When deg is the standard grading, and

$$h(d) = \begin{cases} \dim_K(S_d) = \binom{n+d-1}{d} & \text{for } 0 \leq d \leq D-1 \\ P(d) & d \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $D = D(P)$ is the Gotzmann bound, then Gotzmann's theorems imply that $\{D\}$ is a supportive set, and $\{D, D+1\}$ is a very supportive set.

When $|A| < \infty$, then A is a (very) supportive set.

Challenge: Give explicit descriptions of (very) supportive sets in more generality. The existence proof given in [HS04] is nonconstructive.

3. Lecture 3: What is known, and open problems

We begin with a summary of what is known in this area.

(1) Let K be a commutative ring. Write $S_K = K[x_1, \dots, x_n$. Then for any grading and Hilbert function, we have $\mathrm{Hilb}_{S_K}^h \cong \mathrm{Hilb}_{S_{\mathbb{Z}}}^h \times_{\mathbb{Z}} \mathrm{Spec}(K)$.

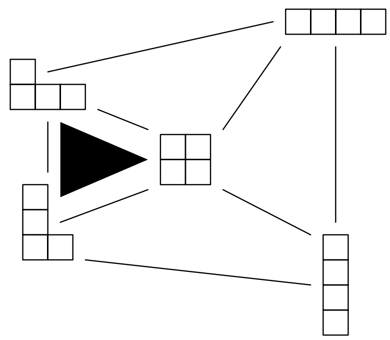
(2) Since $\mathrm{Hilb}_P(\mathbb{P}^{n-1})$ is a special case, all known pathologies of Hilbert schemes occur in multigraded Hilbert schemes. For example, there are non-reduced components [Mum62]. In addition, "Murphy's Law" holds [Vak06]: every singularity type defined over $\mathbb{Z}$ occurs in some multigraded Hilbert scheme. Here by a singularity type we mean the equivalence relation where $(X, p) \sim (Y, q)$ if there is a smooth morphism $(X, p) \to (Y, q)$. The general philosophy is that whatever bad things can occur, you should expect.

(3) Fogarty [Fog68] proved that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth and irreducible. In addition, Nakamura's G-Hilbert scheme is smooth and irreducible when $G \subseteq \mathrm{SL}(2, \mathbb{C})$ (and is a crepant resolution of $\mathbb{C}^2/G$. We have the following generalization of these results.

   Theorem 3.1. *(M-Smith [MS10]) Let $S = K[x, y]$, where K is a commutative ring, let $\deg\colon \mathbb{N}^2 \to A$ be any grading, and let $h\colon A \to \mathbb{N}$ be any Hilbert function. Then Hilb_S^h is smooth and irreducible.*

   The proof involves a combinatorial understanding of the tangent space, inspired by work of Haiman [Hai98], and an explicit description of Białnicki-Birula cells inspired by work of Evain [Eva04].

FIGURE 1. The T-graph of $\mathrm{Hilb}^4(\mathbb{A}^2)$.



(4) One consequence (realised in conversations with Rob Silversmith) is the following (work-in-progress).

The *spine* of the T-graph of $\mathrm{Hilb}^d(\mathbb{A}^d)$ is independent of characteristic.

The torus $T = \mathbb{G}_m^n \cong (K^*)^n$ of $\mathbb{A}^n$ acts on Hilb_S^h for any multigraded Hilbert scheme: $t \cdot I = I|_{x_i = t_i x_i}$. Fixed points of this T-action are monomial ideals. The closure of a one-dimensional T-orbit adds either one or two fixed points, so we can construct a graph where the vertices the the T-fixed points, and the edges are the one-dimensional T-orbits.

One thing that makes the study of T-graphs of multigraded Hilbert schemes challenging is that while they only have finitely vertices, they can have infinitely many edges.

Example 3.2. Let $\mathrm{Hilb}_S^h = \mathrm{Hilb}^4(\mathbb{A}^2)$. The T-graph has five fixed points, corresponding to the five monomial ideals I in $S = K[x, y]$ with $\dim_K S/I = 4$, or equivalently to the five partitions of 4. The edges are shown in Figure 1, which is taken from [HM12]. The shaded triangle represents an infinite number of edges joining the ideals $\langle x^3, xy, y^2\rangle$ and $\langle x^2, xyy^3\rangle$. These form a variety that has the ideal $\langle x^2, y^2\rangle$ in its closure.

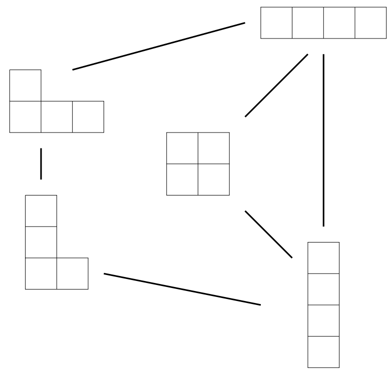
If an ideal lives on a one-dimensional T-orbit, it is homogeneous with respect to a $\mathbb{Z}^n/\mathbf{c}$-grading, so every ideal on a T-edge lives in a smaller multigraded Hilbert scheme. for $S = K[x, y]$, Hilb_s^h is smooth and irreducible for the $\mathbb{Z}^n/\mathbf{c}$-grading. This result does not have any field assumptions. It also has a unique "max" and "min" monomial ideals (being the limit points of the torus action for generic points in the multigraded Hilbert scheme).

Definition 3.3. The *spine* of the T-graph of $\mathrm{Hilb}^d(\mathbb{A}^d)$ has one edge connecting the max and min vertex of the T-graph of each Hilb_S^h.

This is characteristic independent!

Example 3.4. When $d = 4$, the spine is the graph shown in Figure 2.

FIGURE 2. The spine of the T-graph of $\mathrm{Hilb}^4(\mathbb{A}^2)$.



(5) One of the few global results about the Hilbert scheme is Hartshorne's proof that the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ of subschemes of $\mathbb{P}^n$ with a give Hilbert polynomial is always connected [Har66]. This is not the case in general for multgraded Hilbert schemes! There exists a toric Hilbert scheme ($h(a) = 1$ for all $a \in \deg(\mathbb{N}^n) \subseteq A = \mathbb{Z}^d$) that are disconnected. However the smallest known example has $n = 26$.

Idea of construction:

Write $\deg(x_i) = \mathbf{a}_i \in \mathbb{Z}^d$.

(a) (Sturmfels) A monomial ideal I with $h_I = h$ induces a triangulation of $\mathrm{pos}(\mathbf{a}_i : 1 \leq i \leq n) := \{\sum_{i=1}^n \lambda_i \mathbf{a}_i : \lambda_i > 0 \text{ for } 1 \leq i \leq n\}$. The corresponding simplicial complex is the Stanley-Reisner complex of the radical of I: $\Delta(\sqrt{I})$. A cone $\mathrm{pos}(\mathbf{a}_i : i \in \sigma)$ with $\sigma \subseteq \{1, \dots, n\}$ is in the triangulation if and only if there is no monomial in I with support in σ.

Example 3.5. Let $S = \mathbb{C}[x_1, x_2, x_2, x_4]$, with $\deg(x_1) = (1, 0, 0)$, $\deg(x_2) = (1, 1, 0)$, $\deg(x_3) = (1, 0, 1)$, and $\deg(x_3) = (1, 1, 1)$. Then $I_1 = \langle x_1 x_4 \rangle$ and $I_2 = \langle x_2 x_3 \rangle$ both have Hilbert function $h(a) = 1$ if $a \in \deg(\mathbb{N}^4)$, and $h(a) = 0$ otherwise. The corresponding two triangulations are shown in cross-section in Figure 3.

(b) The two triangulations in Example 3.5 are connected by a *bistellar flip*. This is a flip in the sense of Mori theory of the corresponding toric varieties. See [DLRS10] for more on bistellar flips of triangulations.

Each irreducible component of a toric Hilbert scheme is a not-necessarily-normal toric variety, and there are torus-fixed points

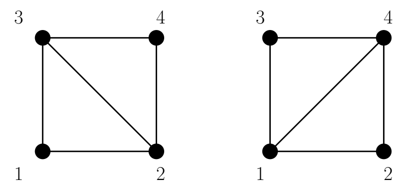


FIGURE 3. The two triangulations of Example 3.5

in the intersection of any two components. This means that the Hilbert scheme Hilb^h_S is connected if and only if its T-graph is connected. In [MT02] Maclagan and Thomas showed that two monomial ideals I, J in Hilb^h_S are connected by an edge in the T-graph if and only if either $\Delta(\sqrt{I}) = \Delta(\sqrt{J})$, or the two triangulations are connected by a bistellar flip.

(c) In [San05], Santos showed that there is a configuration of 26 integer vectors in $\mathbb{R}^6$ (so corresponding to a map $\deg\colon \mathbb{N}^{26} \to \mathbb{Z}^6$ with disconnected bistellar flip graph, and triangulations of the form $\Delta(\sqrt{I})$ in two different connected components. This shows that this toric Hilbert scheme is disconnected.

Find a smaller disconnected example!

(6) When the grading is positive ($\mathrm{rk}\, S_0 = 1$), every multigraded Hilbert scheme is a multigraded Hilbert scheme of points [HM12, Theorem 1.1].

This is a consequence of the construction of the Hilbert scheme.

For example, when $\mathrm{Hilb}^h_S = \mathrm{Hilb}_P(\mathbb{P}^{n-1})$, the set $\{D, D+1\}$ is a very supportive set for D sufficiently large. There is then a bijection between the two sets:

(a) $\{$admissible ideals with Hilbert function $h\}$, and
(b) $\{$admissible ideals with Hilbert function h in degrees $\leq d+1$, and Hilbert function 0 in degrees $\geq D+2\}$.

The bijection takes an ideal I from the first set to $I + S_{D+2}$, and an ideal J from the second set to $\langle J_{\leq D+1}$.

This means that every positively graded multigraded Hilbert scheme also parameterizes collections of points (supported at the origin) invariant under a group action (dual to the grading group) with prescribed multiplicity of representations in the action on the coordinate ring of the affine scheme.

We conclude with some open questions.

(1) When are multigraded Hilbert schemes irreducible? Smooth? How about when $A = \mathbb{Z}^n/\mathbf{c}$? We expect this nice behaviour to be rare, but it might be tractable to understand when this occurs.
(2) Find a smaller example of a disconnected multigraded Hilbert scheme.

(3) Give explicit conditions for Hilb_S^h to be nonempty.

For example, if S has the standard grading, for any n, and $h\colon \mathbb{Z} \to \mathbb{N}$ is given by $h(1) = 0$, $h(2) = 3$, and any other choices for other $d \in \mathbb{Z}$, then Hilb_S^h is empty, as $h(1) = 0$ implies that every variable is in any admissible ideal I with this Hilbert function§, and so we should have $h(2) = 0$.

In the standard graded case the condition for Hilb_S^h to be nonempty is given by Macaulay's theorem. Write

$$h(d) = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_j}{j},$$

where $m_d > m_{d-1} > \cdots > m_j \geq j \geq 1$. This expression is unique.

Define

$$h(d)^{\langle d \rangle} = \binom{m_d + 1}{d+1} + \binom{m_{d-1}+1}{d} + \cdots + \binom{m_j + 1}{j+1}.$$

Then $h(d+1) \leq h(d)^{\langle d \rangle}$. This bound is sharp, and is achieved for all d by the lexicographic ideal, which is a canonical smooth point on the Hilbert scheme. See [Bru98, Chapter 4, §2] for more details on Macaulay's theorem.

The question is thus: What is the multigraded version of this story?

One difficulty in generalizing is given by the fact that lexicographic ideals do not exist for general multigradings: the set of monomials consisting of the lexicographically largest $\mathrm{rk}(S_a) - h(a)$ monomials in each degree a is not always the set of monomials in a monomial ideal, as it may not be closed under multiplication by variables. However in the case $n = 2$, in [MS10] it is shown that there is always a "lex-most" ideal (the largest in a partial order induced by the lexicographic order). Does this result extend to larger n?

(4) Give explicit constructions of supportive and very supportive sets.

In the standard graded case the Macaulay description for $h(d)$ stabilizes for $d \gg 0$, and the Gotzmann number is the number j of binomial coefficients appearing in this description. Gotzmann's regularity and persistence theorems [Got78] [Bru98, Chapter 4, §3] imply that the set $\{j\}$ satisfies conditions g and h', and the set $\{j, j+1\}$ satisfy g, h, and s. Generalize this!

4. Exercises

4.1. **Basics of multigradings.**

(1) Consider the grading of $S = \mathbb{C}[x, y]$ by $\deg(x) = 1 \bmod 7$, $\deg(y) = 6 \bmod 7$. What is H_I for $I = \langle x^4, xy, y^4 \rangle$? What about $I = \langle x^3, y^3 \rangle$? Is the ideal $\langle x^5, x^2y^2, xy^5 \rangle$ admissible?
(2) Consider the grading of $S = \mathbb{C}[x, y]$ by $\deg(x) = 2$, $\deg(y) = 3$. Check that the zero ideal is admissible. What is $H_I(d)$ for $d \geq 0$?
(3) Consider the grading of $S = \mathbb{C}[x, y, z]$ by $\deg(x) = (1, 2, 3)$, $\deg(y) = (1, 0, 1)$, and $\deg(z) = (0, 3, 4)$. Check that the zero ideal is admissible. What is $H_I(\mathbf{u})$ for $\mathbf{u} \in \mathbb{Z}^3$?

4.2. **Examples of Multigraded Hilbert Schemes.**

(1) Consider the grading of $S = \mathbb{C}[x_1, x_2]$ by A given by $\deg(x_1) = 2$, $\deg(x_2) = 3$. Consider $h \colon \mathbb{Z} \to \mathbb{N}$ given by $h(0) = h(2) = h(3) = h(4) = h(5) = h(6) = 1$, and $h(d) = 0$ for all other d. What is Hilb_S^h?
(2) With the same grading as in the previous question, consider $h \colon \mathbb{Z} \to \mathbb{N}$ given by $h(0) = h(1) = 1$, $h(2) = h(3) = 2$, $h(4) = h(5) = 3$, $h(6) = 2$, and $h(d) = 0$ for all other d. What is Hilb_S^h?

4.3. **Construction.**

(1) Verify the description of the universal family in the case $S = \mathbb{C}[x, y]$ has the standard grading, and $h(0) = 1$, $h(1) = h(2) = 2$, $h(3) = 1$, and $h(d) = 0$ otherwise.
(2) Consider the grading $\deg(x) = 2$, $\deg(y) = 2$ of $\mathbb{C}[x, y]$, and the Hilbert function $h(0) = h(2) = h(3) = h(4) = h(5) = h(7) = 1$, $h(6) = h(8) = h(9) = h(10) = 2$, $h(11) = 0$, $h(12) = 2$, and $h(d) = 0$ otherwise.
   (a) Find all monomial ideals with Hilbert function h.
   (b) Find a supportive set for h. Find a very supportive set for h.
   (c) Compute Hilb_S^h.
(3) Consider the grading $\deg(x) = 1$, $\deg(y) = 2$ on $\mathbb{C}[x, y]$. Set $h(4) = h(6) = 2$. Compute $\mathrm{Hilb}_{S_{4,6}}^h$.

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