TROPICAL GEOMETRY

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Abstract. This is lecture notes and exercises for my course at the LMS Undergraduate Summer School at Manchester in July 2017.

1. Lecture 1

Definition 1.1. The tropical semiring is

$$\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot),$$

where $\oplus$ is the usual minimum, and $\odot$ is the usual addition.

For example,

$$3 \oplus 5 = 3, \quad 5 \odot 7 = 12$$

$$3 \odot (5 \oplus 7) = 3 \odot 5 \oplus 3 \odot 7$$

$$0 \odot 3 = 3 \quad \infty \oplus 7 = 7.$$

Thus 0 is the multiplicative identity (playing the role of 1 in the usual real numbers) and ∞ is the additive identity (playing the role of 0 in the usual real numbers).

In fact (check!) $\overline{\mathbb{R}}$ obeys the associative and distributive rules. Subtraction is the only part of the ring axioms that is missing. (Technically: addition is a monoid, not a group).

In these lectures: We will redo some algebra and geometry here with $\overline{\mathbb{R}}$ replacing fields such as $\mathbb{R}$ or $\mathbb{C}$.

Why?

Tropical mathematics is about 40 years old, and has been reinvented several times.

One place it naturally occurs is in optimization. For example, suppose the trains in a region are set up so that there are two routes from town A to town E: via town B and D, or via C and D, with travel times as in the diagram. Then the shortest travel time from A to E is $\min(a+b, c+d) + e$, which equals $(a \odot b \oplus c \odot d) \odot e$.

In the last fifteen years there have been increasing connections to algebraic geometry and related fields. We'll touch lightly on this in these lectures. There have also been new connections to other areas,

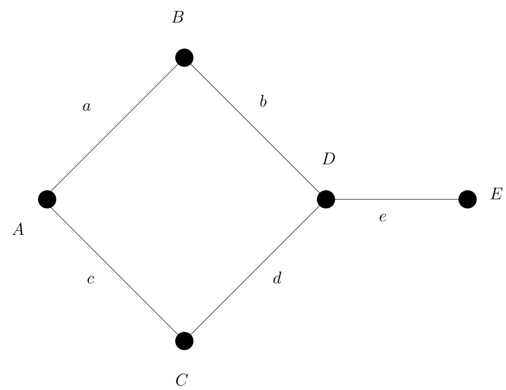

FIGURE 1. A sample train network

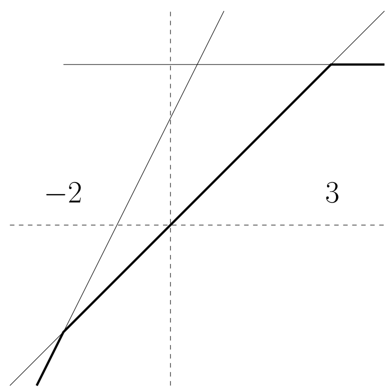

FIGURE 2. The graph of $\min(2x + 2, x, 3)$

such as economics, biology, and physics. In addition, there is a lot of interplay with (polyhedral) combinatorics.

Polynomials in one variable

Note: A tropical polynomial in one variable is a piecewise linear function $p : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Example 1.2. Let $p = 2 \odot x^2 \oplus 0 \odot x \oplus 3$. This is the function $\min(2x + 2, x, 3)$, and has the graph shown in Figure 2.

Note that $p = 2 \odot (x \oplus -2) \odot (x \oplus 3)$.

Definition 1.3. The roots of a tropical polynomial p are the real numbers where the graph of p is not differentiable.

Example 1.4. The roots of $2 \odot x^2 \oplus 0 \odot x \oplus 3$ are $x = -2$ and $x = 3$.

Exercise 1.5. What are the roots of:

(1) $x^2 \oplus x \oplus 2$?

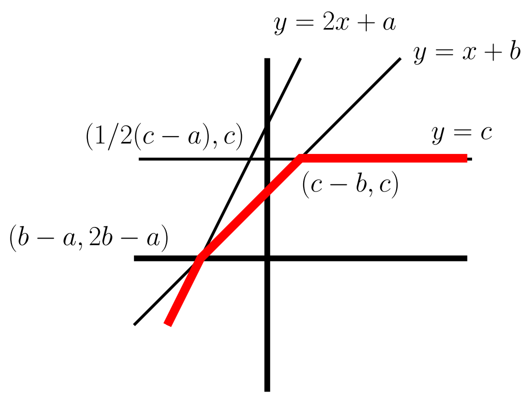

FIGURE 3. The graph of a tropical quadratic

(2) $x^2 \oplus 2 \odot x \oplus 4$?
(3) $x^2 \oplus 3 \odot x \oplus 4$?

Exercise 1.6. Show that tropical polynomials factor into linear factors as *functions* but not necessarily as polynomials.

Quadratic formula:
Let $p = a \odot x^2 \oplus b \odot x \oplus c$, where $a, b, c \in \mathbb{R}$.
The roots of p are: $\begin{cases} b - a, c - b & \text{if } 2b \leq a + c \\ (c - a)/2 & \text{if } b > a + c \end{cases}$

Exercise 1.7. Work out the tropical cubic formula. What about the quartic formula, or the quintic?

Connection with "usual" equations.

Definition 1.8. A valuation on a field K is a function $\text{val}\colon K \to \mathbb{R} \cup \{\infty\}$ such that

(1) $\text{val}(ab) = \text{val}(a) + \text{val}(b)$,
(2) $\text{val}(a + b) \geq \min(\text{val}(a), \text{val}(b))$, and
(3) $\text{val}(a) = \infty$ if and only if $a = 0$.

Example 1.9. (1) Any K, $\text{val}(a) = 0$ if $a \neq 0$.
(2) $K = \mathbb{Q}$. $\text{val}_p(p^n a/b) = n$ if p does not divide a or b. For example, $\text{val}_2(12) = 2$, and $\text{val}_3(5/6) = -1$.
(3) $K = \mathbb{C}(t)$. $\text{val}(f/g) = \text{lowdeg}(f) - \text{lowdeg}(g)$, where $\text{lowdeg}(f)$ is the lowest exponent of a power of t that occurs in f.

Exercise 1.10. Show that for any valuation we have $\text{val}(1) = 0$, $\text{val}(-a) = \text{val}(a)$, and $\text{val}(a+b) = \min(\text{val}(a), \text{val}(b))$ if $\text{val}(a) \neq \text{val}(b)$.

Definition 1.11. Fix a field K with a valuation val. The *tropicalization* of a polynomial $f = \sum_{i=0}^m a_i x^i \in K[x]$ is

$$\operatorname{trop}(f) = \oplus_{i=0}^m \operatorname{val}(a_i) \odot x^i = \min(\operatorname{val}(a_i) + ix).$$

Example 1.12. Let $f = 4x^2 - 33x + 8 \in \mathbb{Q}[x]$, where $\mathbb{Q}$ has the 2-adic valuation. Then $\operatorname{trop}(f) = 2 \odot x^2 \oplus 0 \odot x \oplus 3$.

Definition 1.13. A field K is algebraically closed if every nonconstant polynomial f in $K[x]$ has a root in K. In other words, there is $a \in K$ with $f(a) = 0$.

An example of an algebraically closed field is $\mathbb{C}$.

Theorem 1.14. *Fix $f \in K[x]$, where K is an algebraically closed field with a valuation. The valuations of the roots of f equal the roots of* $\operatorname{trop}(f)$.

Example 1.15. Let $f = 4x^2 - 33x + 8 = (4x - 1)(x - 8)$. Then f has roots $1/4$ and 8, which have 2-adic valuations -2 and 3. We've already seen that the roots of $\operatorname{trop}(f) = 2 \odot x^2 \oplus x \oplus 3$ are -2 and 3.

2. Exercises

(1) Show that tropical multiplication distributes over addition. More generally, check all the axioms to show that $\overline{\mathbb{R}}$ is a semiring.
(2) Check that the tropical semiring is isomorphic as a semiring to the one where addition is given by maximum instead of minimum.
(3) Show that every tropical polynomial $f \in \overline{\mathbb{R}}[x]$ factors "as a function". For example $x^2 \oplus 3 \odot x \oplus 0 = (x \oplus 0)^2$ as a function.
(4) What is the tropical cubic formula? What about the quartic formula? Quintic?
(5) What is the 3-adic valuation of the following numbers:
   (a) 9;
   (b) $6/12$;
   (c) $6/5$;
   (d) $-7/18$.
(6) Show that for any valuation we have $\operatorname{val}(1) = 0$, $\operatorname{val}(-a) = \operatorname{val}(a)$, and $\operatorname{val}(a + b) = \min(\operatorname{val}(a), \operatorname{val}(b))$ if $\operatorname{val}(a) \neq \operatorname{val}(b)$.
(7) For each of the following polynomials f in $\mathbb{Q}[x]$, where $\mathbb{Q}$ has the 2-adic valuation, compute the tropicalization of f, then compute the roots of $\operatorname{trop}(f)$, and verify Theorem 1.14 for these examples. You will secretly work over the *algebraic closure* of $\mathbb{Q}$ for this question; ask after you have done the first one if you don't know what this means.

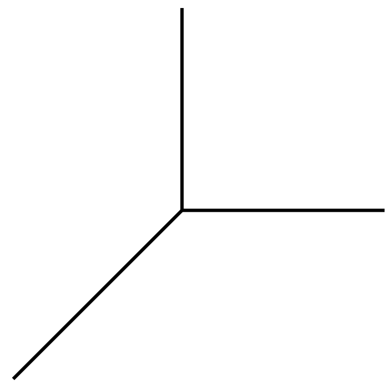

FIGURE 4. The tropical line

   (a) $f = x^2 - 3x + 2$;
   (b) $f = x^2 - 4x + 3$;
   (c) $f = x^3 - 7x^2 + 14x - 8$;
   (d) $f = x^2 - 5x + 2$;
(8) Prove Theorem 1.14.

3. Lecture 2

In the first lecture we studied tropical polynomials in one variable; now we will expand to more variables.

We write $\overline{\mathbb{R}}[x_1, \dots, x_n]$ for the *semiring of tropical polynomials* in the variables $x_1, \dots, x_n$. For $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{N}^n$ we write $x^{\mathbf{u}}$ for the product $x_1^{u_1} \odot x_2^{u_2} \odot \dots \odot x_n^{u_n} = u_1 x_1 + \cdots + u_n x_n$. An element of $\overline{\mathbb{R}}[x_1, \dots, x_n]$ is then a polynomial of the form $\bigoplus_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} \odot x^{\mathbf{u}}$, where $c_{\mathbf{u}} \in \overline{\mathbb{R}}$, and all operations are tropical operations.

Fix a tropical polynomial $f = \bigoplus c_{\mathbf{u}} x^{\mathbf{u}} \in \overline{\mathbb{R}}[x_1, \dots, x_n]$. Note that as a function we have $f = \min(c_{\mathbf{u}} + \mathbf{x} \cdot \mathbf{u})$, so, f restricted to $\mathbb{R}^n$ is a piecewise linear function from $\mathbb{R}^n$ to $\mathbb{R}$.

Definition 3.1. The tropical hypersurface $V(f)$ of a tropical polynomial $f = \sum c_{\mathbf{u}} x^{\mathbf{u}}$ is the locus in $\mathbb{R}^n$ where the graph of f is not differentiable.

Example 3.2. $f = x \oplus y \oplus 0 = \min(x, y, 0)$. This is illustrated in Figure 4

Example 3.3.

$$f = 0 \oplus 2 \odot x \oplus 3 \odot y \oplus 4 \odot xy.$$

This is illustrated in Figure 5.

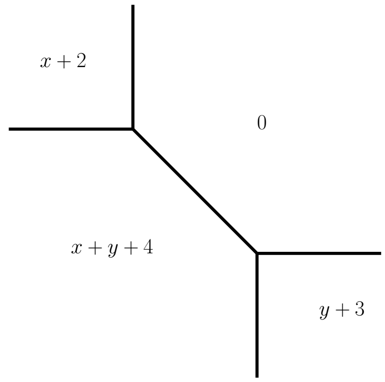

FIGURE 5. The tropical hypersurface for Example 3.3

Exercise 3.4. Draw the tropical hypersurface for $f = 0 \oplus 1 \odot y \oplus 2 \odot xy \oplus 7 \odot x^2$.

Write $S = K[x_1, \dots, x_n]$ for the polynomial ring in n variables with coefficients in K. Recall that an ideal $I \subset S$ is a subset of S satisfying that if $f, g \in I$, then $f + g \in I$, and if $f \in I$, $h \in S$, then $hf \in I$. We write $I = \langle f_1, \dots, f_r \rangle$ for the smallest ideal containing $f_1, \dots, f_r \in S$, and call this the ideal generated by I. This is $I = \{\sum_{i=1}^r g_i f_i : g_i \in S\}$.

Definition 3.5. An affine variety is a subset of K^n of the form

$$V(I) = \{a \in K^n : f(a) = 0 \text{ for all } f \in I\}.$$

For $I = \langle f_1, \dots, f_r \rangle$, $V(I) = \{a \in K^n : f_1(a) = \cdots = f_r(a) = 0\}$.

Example 3.6. The variety $V(x + y + z + 3, x + 2y + 3z)$ is a line in K^3.

Definition 3.7. The tropicalization of $X = V(I)$ is

$$\text{trop}(X) = \cap_{f \in I} V(\text{trop}(f)) \subseteq \overline{\mathbb{R}}^n.$$

Warning: In the intersection of Definition 3.7 it is not sufficient to take just the generators of I.

Definition 3.8. For $\mathbf{v}_1, \dots, \mathbf{v}_r \subset \mathbb{R}^n$, the set

$$\text{pos}(\mathbf{v}_1, \dots, \mathbf{v}_r) = \{\sum_{i=1}^r \lambda_i \mathbf{v}_i : \lambda_i \geq 0\}$$

is the cone generated by $\mathbf{v}_1, \dots, \mathbf{v}_r$.

Example 3.9. Let $I = \langle x+y+z+1, x+2y+3z+4\rangle \subseteq \mathbb{C}[x,y,z]$, where $\mathbb{C}$ has the trivial valuation. Then $X = V(I)$ is the line $(2,-3,0) + \operatorname{span}(1,-2,1)$ in $\mathbb{C}^3$. Write $\mathbf{e}_0 = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 \in \mathbb{R}^3$. The tropicalization is

$$\operatorname{trop}(X) = \operatorname{pos}(\mathbf{e}_0) \cup \operatorname{pos}(\mathbf{e}_1) \cup \operatorname{pos}(\mathbf{e}_2) \cup \operatorname{pos}(\mathbf{e}_3).$$

Note that $\operatorname{trop}(x+y+z+1) = \operatorname{trop}(x+2y+3z+4) = x \oplus y \oplus z \oplus 0$, but $\operatorname{trop}(X) \subsetneq V(x \oplus y \oplus z \oplus 0)$, as $(0,1,1) \in V(x \oplus y \oplus z \oplus 0)$, but $(0,1,1) \notin V(\operatorname{trop}(y+2z+3))$, and $y+2z+3 \in I$.

Definition 3.10. A *polyhedron* is a subset of $\mathbb{R}^n$ of the form

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$

where A is a $d \times n$ matrix, and $b \in \mathbb{R}^d$. In other words, a polyhedron is the intersection of finitely many half-spaces.

Example 3.11. Familiar examples of polyhedra are polygons in the plane, and the Platonic solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron) in $\mathbb{R}^3$.

Definition 3.12. The *face* of a polyhedron P induced by a vector $\mathbf{w} \in \mathbb{R}^n$ is $\operatorname{face}_{\mathbf{w}}(P) = \{x \in P \colon \mathbf{w} \cdot x \leq \mathbf{w} \cdot y \text{ for all } y \in P\}$.

Example 3.13. Let P be the square

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Then $\operatorname{face}_{(1,0)}(P)$ is the left edge with vertices $(0,0)$ and $(0,1)$. The set of all faces is the vertices, edges, and the whole square.

Definition 3.14. A *polyhedral complex* is a collection of polyhedra in $\mathbb{R}^n$ for which the intersection of any two is either empty, or a face of each. The support $|\Sigma|$ of a polyhedral complex Σ is the subset of $\mathbb{R}^n$ that is the union of all polyhedra in Σ.

Example 3.15. An example of a polyhedral complex is shown in Figure 6.

Theorem 3.16. *Let $I \subseteq K[x_1, \ldots, x_n]$, and let $X = V(I) \subseteq K^n$. Then $\operatorname{trop}(X) \cap \mathbb{R}^n$ is the support of a finite polyhedral complex. If I is a prime ideal, then all maximal polyhedra in this complex have the same dimension.*

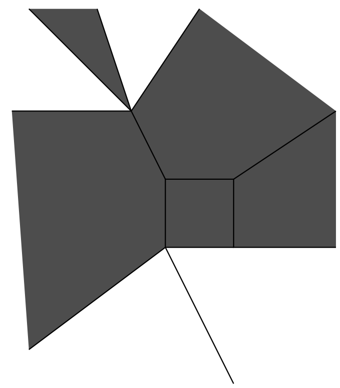

FIGURE 6. A polyhedral complex

4. EXERCISES

(1) Draw trop($V(f)$) for the following $f \in \mathbb{Q}[x, y]]$, where $\mathbb{Q}$ has the 2-adic valuation.
   (a) $f = 8x + 6y + 5/4$;
   (b) $f = 3x + 4y + 48$;
   (c) $f = 8x^2 + xy + 10y^2 + 6x + y + 1$;
   (d) $f = 16x^2 + 6xy + 7y^2 + 7x + 5y + 2$;
   (e) $f = 2x^2 + 3xy - 7y^2 + 5$;
   (f) $f = 64x^3 + x^2y + xy^2 + 64y^3 + 8x^2 + 1/2xy + 8y^2 + 2x + 2y + 1$.

(2) The goal of this exercise is to show the connection between tropical curves in the plane and triangulations of a certain point configuration.

Fix $d > 0$. Let $\mathcal{A}_d = \{(a, b) : a + b \leq d, a, b \geq 0\}$. Fix a polynomial $f = \sum_{(a,b)\in\mathcal{A}_d} c_{ab}x^a y^b$ with $c_{ab} \in K$, where K is a field with a valuation. The *regular triangulation* of $\mathcal{A}_d$ induced by f is obtained by taking the convex hull of the points $\{(a, b, \text{val}(c_{ab}) : (a, b) \in \mathcal{A}\}$ and taking the (projections of the) set of *lower faces*. These are the faces that you can see if you look from $(0, 0, -N)$ for $N \gg 0$.

Example: Let $d = 2$, so $\mathcal{A}_2 = \{(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)\}$. Let $f = 2x^2 + xy + 6y^2 + x + y + 64$, where $\mathbb{Q}$ has the 2-adic valuation. We form the convex hull of the points

$$\{(2, 0, 1), (1, 1, 0), (0, 2, 1), (1, 0, 0), (0, 1, 0), (0, 0, 6)\}.$$

The lower faces of this polytope are illustrated in Figure 7.

   (a) Draw the regular triangulation of $\mathcal{A}_2$ corresponding to the polynomial $f = 2x^2 + xy + 24y^2 + x + 10y + 1$.

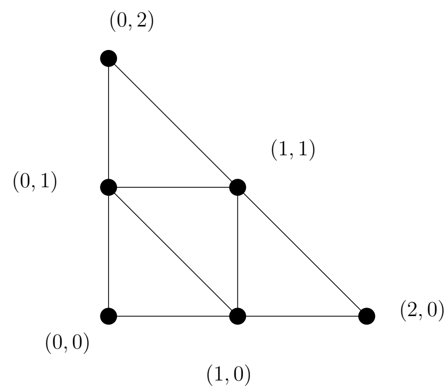

FIGURE 7.

(b) Draw the regular triangulation of $\mathcal{A}_1$ corresponding to the polynomial $f = 32x + 8y + 1024$.
(c) Draw the regular triangulation of $\mathcal{A}_3$ corresponding to the polynomial $f = 8x^3+2x^2y-2xy^2+8y^3+10x^2+xy+6y^2-2x+2y+8$.

The dual graph to a triangulation has a vertex for every triangle. There are two types of edges. The finite edges join two adjacent triangles, and have direction orthogonal to the common edge of the triangles. The infinite edges start at the triangles adjacent to the boundary of the large triangle $\text{conv}((d,0),(0,d),(0,0))$, and have direction orthogonal to the external edge. This is defined up to the lengths of the finite edges.

Example: In the example above, a dual graph for the regular triangulation is shown in Figure 8.

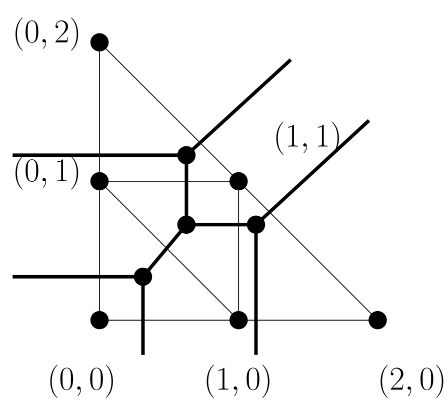

FIGURE 8.

(d) Draw a dual graph to the regular triangulation of $\mathcal{A}_2$ corresponding to $f = -2x^2 + xy + 8y^2 + x + 14y + 3$.
(e) Draw a dual graph to the regular triangulation of $\mathcal{A}_1$ corresponding to $f = 32x - 8y + 1024$.
(f) Draw a dual graph to the regular triangulation of $\mathcal{A}_3$ corresponding to $f = 8x^3 + 2x^2y - 2xy^2 + 8y^3 + 10x^2 + xy + 6y^2 - 2x + 2y + 8$.
(g) Let $f = \sum_{(a,b)\in\mathcal{A}_d} c_{ab}x^a y^b$ with $c_{d0}, c_{0d}, c_{00} \neq 0$. Show that the tropical curve defined by f is the image under $x \mapsto -x$ of a dual graph to the regular triangulation defined by f.
(h) Check the previous claim for the examples of the first question.

5. Lecture 3

Question 5.1. Let $f, g \in \mathbb{C}[x, y]$ with $\gcd(f, g) = 1$, and let $C = V(f) \subset \mathbb{C}^2$, $C' = V(g) \subseteq \mathbb{C}^2$. What can we say about $|C \cap C'|$?

Example 5.2.
(1) If $f = x + y$, $g = x - y$, then C and C' are two lines that intersect in the one point $(0, 0)$.
(2) If $f = y - x^2$, $g = y - x - 2$, then $|C \cap C'| = 2$, as they meet in the two points $(2, 4)$ and $(-1, 1)$.
(3) If $f = x^2 + y^2 - 1$, and $g = 1/4x^2 + 4y^2 - 1$, then $|C \cap C'| = 4$.
(4) If $f = y - x^3 + x$ and $g = y - 1/100x$, then $|C \cap C'| = 3$.

In all of these examples, we have $|C \cap C'| = \deg(f)\deg(g)$.

This is not always the case: $f = y - x$ and $g = y - x - 1$ have $V(f) \cap V(g) = \emptyset \subset \mathbb{C}^2$. However in this case they “intersect at infinity”. This can be made more precise by introducing the projective plane $\mathbb{P}^2$, but we will not go there today. Another potential counter-example is $f = y - x^2$, $g = y - 2x + 1$. In this case $C \cap C'$ is just one point: $(1, 1)$. However in this case the line $V(g)$ is tangent to the quadratic, and we say that it intersects “with multiplicity two”. This can also be made more precise, but again we will not go into all the details here.

The weaker form that we will consider here is:

Theorem 5.3. *(Bézout) If $f, g \in \mathbb{C}[x, y]$ with $\gcd(f, g) = 1$, then $|V(f) \cap V(g)| \leq \deg(f)\deg(g)$.*

Exercise 5.4. Prove this when $\deg(g) = 1$.

We'll now see how to approach this tropically.

We saw in the exercises for lecture 2 that $\mathrm{trop}(V(f))$ is determined by a triangulation (or more generally a subdivision). Under this, edges of the tropical variety are dual to edges of the subdivision. We assign a weight w_e to each edge/ray of the tropical variety by setting it equal to the *lattice length* of the edge of the subdivision. This is the number of lattice points (points with integral entries) in the edge minus one. For example the lattice length of an edge from $(0,0)$ to $(2,0)$ is 2.

For example, for $f = x^2 + y^2 + 0$, we have three rays, each of which have multiplicity two.

Note: If $x \in V(f) \cap V(g)$, then $\text{val}(f) \in \text{trop}(V(f)) \cap \text{trop}(V(g))$.

The plan is thus:

(1) Bound $|\text{trop}(V(f)) \cap \text{trop}(V(g))|$.
(2) For each $\mathbf{w} \in \text{trop}(V(f)) \cap \text{trop}(V(g))|$ bound the number of $x \in V(f) \cap V(g)$ with $\text{val}(x) = \mathbf{w}$.

Example 5.5. (1) Let $f = x + y + 1$, and $g = x + 4y + 2$ be polynomials in $\mathbb{Q}[x, y]$, where $\mathbb{Q}$ has the 2-adic valuation. Then $V(f) \cap V(g) = \{(-2/3, -1/3)\}$ which has valuation $(1, 0)$. This is the only intersection point of $\text{trop}(V(f))$ and $\text{trop}(V(g))$.
(2) Let $f = 2x^2 + xy + 2y^2 + x + y + 2$, and $g = x + 16y + 4$. Then $V(f) \cap V(g)$ is two points, as is $\text{trop}(V(f)) \cap \text{trop}(V(g))$, and these latter two points are the valuations of the first two.

The following example shows that this is not always so straightforward.

Example 5.6. Let $f = x+y+1$, and $g = x+2y+2$. Then $V(f) \cap V(g) = \{(0, -1)\}$, but $\text{trop}(V(f)) \cap \text{trop}(V(g))$ is the ray $\{(\lambda, 0) : \lambda \geq 1\}$.

Definition 5.7. Let $f, g \in K[x, y]$, and let $C = V(f)$, and $C' = V(g) \in K^2$. Fix $\mathbf{w} \in \text{trop}(C) \cap \text{trop}(C')$. Then $\text{trop}(C)$ and $\text{trop}(C')$ *intersect transversely* at $\mathbf{w}$ if, locally near $\mathbf{w}$, C looks like $\mathbf{w} + \text{span}(\mathbf{u})$, and C' looks like $\mathbf{w} + \text{span}(\mathbf{u}')$, where $\text{span}(\mathbf{u}, \mathbf{u}') = \mathbb{R}^2$.

Theorem 5.8. *If $f, g \in K[x, y]$ and $\text{trop}(V(f))$ and $\text{trop}(V(g))$ intersect transversely at $\mathbf{w}$, then $\mathbf{w} \in \text{trop}(V(f) \cap V(g))$, so there is $x \in V(f) \cap V(g)$ with $\text{val}(x) = \mathbf{w}$.*

The following example, however, shows that we cannot simply bound the number of tropical intersection points.

Example 5.9. Let $f = x + y + 1$, and $g = x^2 + 4y - 16$. Then $V(f) \cap V(g) = \{(2 \pm 2\sqrt{6}, -3 \mp 2\sqrt{6})\}$, which has size two. However both points have the same valuation in any extension of the 2-adic valuation to $\mathbb{C}$, which is $(1, 0)$.

The solution to this is to define a tropical multiplicity to each transverse intersection point.

Definition 5.10. Suppose $f, g \in K[x, y]$ satisfy that $\operatorname{trop}(V(f))$ and $\operatorname{trop}(V(g))$ meet transversely at $\mathbf{w} \in \mathbb{R}^2$, and, locally near $\mathbf{w}$, $\operatorname{trop}(V(f))$ looks like $\mathbf{w} + \operatorname{span}(\mathbf{u})$, and $\operatorname{trop}(V(g))$ looks like $\mathbf{w} + \operatorname{span}(\mathbf{u}')$. The *multiplicity* of $\operatorname{trop}(V(f)) \cap \operatorname{trop}(V(g))$ near $\mathbf{w}$ is then the absolute value of the determinant of the matrix with columns $|\mathbf{u}\mathbf{u}'|$ multiplied by the weights m_f, m_g of the edges of $\operatorname{trop}(V(f))$ and $\operatorname{trop}(V(g))$ containing $\mathbf{w}$.

Example 5.11. For the polynomials of Example 5.9 the multiplicity at the point $(1, 0)$ is is $(1)(1)|\det(\begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix})| = 2$.

Proposition 5.12. *If $f, g \in K[x, y]$ and $\operatorname{trop}(V(f))$ and $\operatorname{trop}(V(g))$ intersect transversely at $\mathbf{w}$, then $|\{x \in V(f) \cap V(g) : \operatorname{val}(x) = \mathbf{w}\}|$ equals the tropical multiplicity of $\operatorname{trop}(V(f)) \cap \operatorname{trop}(V(g))$ at $\mathbf{w}$.*

Definition 5.13. Let $\Sigma, \Sigma' \subset \mathbb{R}^2$ be the tropicalizations of two curves in K^2. The *stable intersection* of Σ and Σ' is the the limit as ϵ goes to zero of the intersection of Σ and $\epsilon \mathbf{v} + \Sigma'$ for a generic vector $\mathbf{v}$. This is independent of the choice of vector $\mathbf{v}$.

Example 5.14. The stable intersection of the two tropical varieties of Example 5.6 is the point $(1, 0)$.

The following theorem shows that the multiplicity of the origin in the stable intersection bounds the number of solutions with all coordinates nonzero. In the statement "general" is in the sense of algebraic geometry, meaning that there is a finite list of polynomials (hidden in the proof) in the coefficients such that we only consider f, g where at least one of these polynomials is nonzero.

Theorem 5.15. *Let $f = \sum a_{ij} x^i y^j$ and $g = \sum b_{ij} x^i y^j$ be polynomials in $K[x, y]$. If a_{ij}, b_{ij} are sufficiently general, then $|V(f) \cap V(g) \cap (K \setminus \{0\})^2|$ equals the multiplicity of the origin in the intersection of $\operatorname{trop}(V(f))$ and $\operatorname{trop}(V(g))$.*

For "non-general" f, g the number of solutions can only drop. In addition, given a pair f, g, we can always do a linear change of coordinates (one of the form $x' = ax + by$, $y' = cx + dy$) to get a pair where all solutions have all coordinate nonnegative. This means that to prove Bézout's theorem it suffices to bound the multiplicity of the origin in the stable intersection of the tropicalizations of two curves of given degrees.

6. Exercises

(1) For each of the following pairs of polynomials, compute $\text{trop}(V(f))$, $\text{trop}(V(g))$, and the stable intersection of $\text{trop}(V(f))$ and $\text{trop}(V(g))$. Verify that the sum of the multiplicities of the points in the stable intersection bounded by $\deg(f)\deg(g)$. Here $\mathbb{Q}$ has the 2-adic valuation.
   (a) $f = x + y + 1$, $g = x + 2y + 1$.
   (b) $f = x + y + 1$, $g = x + 3x + 5$.
   (c) $f = 2x^2 + xy + 2y^2 + x + y + 2$, $g = 4x^2 + y^2 + 4$.
   (d) $f = 8x^3 + 2x^2y + 2xy^2 + 8y^3 + 2x^2 + xy + 2y^2 + 2x + 2y + 8$, $g = x + 16y + 4$.

(2) Work out more details in the sketch proof of Bézout's theorem given at the end, assuming Theorem 5.15.

(3) Everything here generalizes to higher dimensions, such as for the common solutions to three polynomials in three variables. Do this! (You need to think about the multiplicity of an intersection point in the tropicalization, and what it means to be transverse. In higher dimensions, we put weights on the maximal faces, which are still dual to edges).

7. Challenge Problems

(1) Let $f = \sum_{i,j=0}^{n} a_{ij}x^iy^j, g = \sum_{i,j=0}^{n} b_{ij}x^iy^j \in \mathbb{C}[x,y]$, where $a_{ij}, b_{ij} \in \mathbb{C}$, and $a_{ij}, b_{ij} = 0$ if $i^2 + j^2 > n^2$. You may assume that a_{ij}, b_{ij} are otherwise generic, meaning that they do not satisfy any finite collection of polynomial equations. Let c_n be the multiplicity of the origin in the stable intersection of $\text{trop}(V(f))$ and $\text{trop}(V(g))$. What is $\lim_{n\to\infty} c_n/n^2$?

   To be concrete, when $n = 1$, f has the form $a_1x + a_2y + a_3$. When $n = 2$, f has the form $a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$.

(2) Consider the following list of n polynomials in $\mathbb{C}[x_1,\dots,x_n]$: $f_i = x_i^2 + a_ix_ix_{i+1} + b_ix_{i+1}^2 + c_ix_ix_{i+2} + d_ix_{i+1}x_{i+2}^2 + e_ix_{i+2}^2$, where $x_{n+1} = x_1$, $x_{n+2} = x_2$, and a_i, b_i, c_i, d_i, e_i are generic complex numbers (meaning that you may impose any polynomial inequalities you need on them). How many solutions are there to $f_1(x) = \cdots = f_n(x) = 0$?

8. What next?

(1) The more standard approach to proving Bézout's theorem can be found in many elementary algebraic geometry books, including the details about projective space and multiplicities. One

that I recommend is Hassett, *Introduction to algebraic geometry*, Cambridge, 2007.

(2) One place to learn more about tropical geometry is my book *Introduction to tropical geometry*, with Bernd Sturmfels (Graduate Studies in Mathematics, volume 161, AMS, 2015).

(3) An incomplete list of other mathematicians in the UK whose research touches tropical geometry is: Peter Butkovic (Birmingham), Alex Fink (QMUL), Jeff Giansiracusa (Swansea), Mark Gross (Cambridge), Milena Hering (Edinburgh), James Hook (Bath), Zur Izhakian (Aberdeen), Marianne Johnson (Manchester), Mark Kambites (Manchester), Johannes Nicaise (Imperial), Sergey Sergeev (Birmingham). Their research interests range from algebraic geometry to numerical analysis and optimization, via combinatorics, with areas in between.

(4) Tropical geometry touches many areas of algebraic geometry, so many other UK algebraic geometers work in related areas.

(5) There is also a thriving community world-wide. If you're considering doing a PhD, and are open to moving overseas, there are many exciting options.