FIBER FANS AND TORIC QUOTIENTS

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Abstract. The GIT chamber decomposition arising from a subtorus action on a polarized quasiprojective toric variety is a polyhedral complex. Denote by Σ the fan that is the cone over the polyhedral complex. In this paper we show that the toric variety defined by the fan Σ is the normalization of the toric Chow quotient of a closely related affine toric variety by a complementary torus.

1. Introduction

Let X_P be the quasiprojective toric variety defined by a full-dimensional polyhedron $P \subseteq \mathbb{Q}^n$, and write $(\Bbbk^*)^n = \operatorname{Spec} \Bbbk[\mathbb{Z}^n]$ for the dense algebraic torus of X_P. Given a lattice L of rank $d \leq n$, a surjective lattice map $\pi_{\mathbb{Z}} \colon \mathbb{Z}^n \to L$ induces a $\mathbb{Q}$-linear map $\pi \colon \mathbb{Q}^n \to \mathbb{Q}^d \cong L \otimes_{\mathbb{Z}} \mathbb{Q}$ and hence a linear projection of polyhedra $\pi \colon P \to Q$, where $Q = \pi(P)$. The map $\pi_{\mathbb{Z}}$ also induces an inclusion of the d-dimensional torus $T_L = \operatorname{Spec} \Bbbk[L]$ into $(\Bbbk^*)^n$, hence T_L acts on X_P. Note that every subtorus of $(\Bbbk^*)^n$ arises from a lattice L in this way.

The map π induces a cell decomposition of Q that we call the (*polarized*) *GIT chamber decomposition* arising from the action of T_L on X_P (see [BP90], [Tha94]). The chambers in Q correspond to different linearizations of the relatively ample bundle on X_P determined by the polyhedron P. By taking the cone over each cell in the GIT chamber decomposition of Q, we obtain a fan Σ in $\mathbb{Q}^d \oplus \mathbb{Q}$. It is natural to ask for an explicit description of the toric variety defined by this fan. The main result of this paper provides an answer to this question as follows (see Theorem 3.14 and Section 2.4 for relevant notation):

Theorem 1.1. *Let T_L be a subtorus acting on a toric variety X_P, and let Σ be the fan arising from the GIT chamber decomposition associated to the T_L-action on X_P as above. Then there is a polyhedral cone $P' := (\widetilde{P})^\vee$ and a lattice $L' := \operatorname{im}_{\mathbb{Z}}(\widetilde{\pi}^\vee)$ such that Σ is the fan of the toric Chow quotient $X_{P'}/T_{L'}$.*

The Chow quotient X_P/T_L of a *projective* toric variety X_P by a subtorus T_L was studied by Kapranov–Sturmfels–Zelevinsky [KSZ92]. We generalize this by constructing the *toric Chow quotient* X_P/T_L of an arbitrary quasiprojective toric variety X_P by a subtorus T_L (see Definition 3.10).

In order to prove Theorem 1.1, we consider the *fiber fan* associated to a linear projection of polyhedra $\pi \colon P \to Q$, denoted $\mathcal{N}(P,Q)$, generalizing the normal fan of a fiber polytope (see Billera–Sturmfels [BS92]). We show that the toric Chow quotient X_P/T_L is a not-necessarily normal toric variety whose fan is the fiber fan $\mathcal{N}(P,Q)$. In the special case where P is a polytope, so X_P is projective, this

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result is due to Kapranov–Sturmfels–Zelevinsky [KSZ92]. Note however that the statement of Theorem 1.1 is new even in the case where P is a polytope.

If P is a polyhedral cone, so X_P is affine, the statement of Theorem 1.1 can be streamlined, and a converse can be added as follows (see Theorem 3.15).

Theorem 1.2 (Duality for affine toric quotients)**.** *Let $P \subseteq \mathbb{Q}^n$ be an n-dimensional polyhedral cone and $\pi\colon P \to Q$ a surjective linear projection. Set $P' := P^\vee$ and $L' := \mathrm{im}_{\mathbb{Z}}(\pi^\vee)$. Then*

(i) *the fan of the toric Chow quotient X_P/T_L is equal to the GIT chamber decomposition of Q arising from the action of $T_{L'}$ on $X_{P'}$; and*
(ii) *the GIT chamber decomposition arising from the T_L-action on X_P is the fan of the toric Chow quotient $X_{P'}/T_{L'}$.*

Theorem 1.2, part (i), was partially known to Altmann–Hausen [AH06] in the case where P is a simplicial cone, though no proof was given.

Note that the term GIT chamber decomposition often refers to the chamber structure obtained by varying the polarizing line bundle among all T_L-linearized line bundles (see Dolgachev–Hu [DH98] and Ressayre [Res00]). Our work does not touch on this, nor on the recent work of Berchtold–Hausen [BH05] that goes beyond the space of all T-ample linearizations by working with divisorial sheaves. For another approach to toric GIT see [Hu02].

The main theorems are proved in Section 3.3. Section 2 contains the necessary polyhedral preliminaries, while Sections 3.1 and 3.2 contain background on toric GIT, and introduce the toric Chow quotient, respectively.

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2. Polyhedral Geometry

2.1. **Polyhedral Conventions.** Let $P \subseteq \mathbb{Q}^n$ be an n-dimensional convex polyhedron. For $\mathbf{w} \in (\mathbb{Q}^n)^*$ we denote by $\mathrm{face}_{\mathbf{w}}(P)$ the face of P minimizing $\mathbf{w}$. Given a face F of P, the *inner normal cone* $\mathcal{N}_P(F)$ is the set of $\mathbf{w} \in (\mathbb{Q}^n)^*$ such that $\mathrm{face}_{\mathbf{w}}(P) = F$. The *inner normal fan* $\mathcal{N}(P)$ of P is the polyhedral fan whose cells are the inner normal cones $\{\mathcal{N}_P(F)\}$ as F varies over the faces of P. Two polyhedra P and P' are *normally equivalent* if they have the same normal fan.

The *recession cone* $\mathrm{rec}(P)$ of a polyhedron P is $\{\mathbf{u} \in \mathbb{Q}^n : \mathbf{x}+\mathbf{u} \in P \text{ for all } \mathbf{x} \in P\}$. If $P = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \geq \mathbf{b}\}$ for some $r \times n$ matrix A and vector $\mathbf{b} \in \mathbb{Q}^r$, then $\mathrm{rec}(P) = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \geq 0\}$. The *support* of a polyhedral fan is the set of vectors lying in some cone of the fan. The fan $\mathcal{N}(P)$ is supported on the dual cone $\mathrm{rec}(P)^\vee = \{\mathbf{y} \in (\mathbb{Q}^k)^* : \mathbf{y}\cdot\mathbf{x} \geq 0 \ \forall\ \mathbf{x} \in \mathrm{rec}(P)\}$ of the recession cone $\mathrm{rec}(P)$ of P. Indeed, $\mathbf{y} \notin \mathrm{rec}(P)^\vee$ if and only if there exists $\mathbf{x} \in \mathrm{rec}(P)$ such that $\mathbf{y}\cdot\mathbf{x} < 0$, in which case $\mathbf{y}\cdot\mathbf{x}'$ is unbounded below for $\mathbf{x}' := \lambda\mathbf{x} \in \mathrm{rec}(P)$ with $\lambda \in \mathbb{Q}_{\geq 0}$.

A *polyhedral complex* is a collection of polyhedra such that the intersection of any two is a face of each. The support of a polyhedral complex is the union of the supports of the polyhedra. Note that a fan is a special case of a polyhedral complex. If $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are polyhedral fans with the same support, then their *common refinement* is the fan $\mathcal{F}$ whose cones are the intersections $\cap_{i=1}^r \sigma_i$, where σ_i is a cone in $\mathcal{F}_i$ for $1 \leq i \leq r$.

Let $\pi\colon \mathbb{Q}^n \to \mathbb{Q}^d$ be a linear map, and set $Q := \pi(P)$. We may assume that π is the projection onto the last d coordinates. We denote by π_{n-d} the projection onto the first $n-d$ coordinates $\mathbb{Q}^{n-d}$, and for $\mathbf{q} \in Q$ we set $P_{\mathbf{q}} := \pi_{n-d}(\pi^{-1}(\mathbf{q}) \cap P)$. For $\mathbf{w} \in (\mathbb{Q}^{n-d})^*$, we write $\widetilde{\text{face}}_{\mathbf{w}}(P_{\mathbf{q}})$ for the unique smallest face of P containing the preimage of $\text{face}_{\mathbf{w}}(P_{\mathbf{q}})$ under π_{n-d}.

2.2. **Fiber fans.** In this section we define the notion of a fiber fan $\mathcal{N}(P,Q)$ for a linear projection $\pi : P \to Q$ of polyhedra. This is the common refinement of the normal fans of the fibers $\{\pi^{-1}(\mathbf{q}) : \mathbf{q} \in Q\}$ of the map π. We first show that this definition makes sense.

Lemma 2.1. *For $\mathbf{q}, \mathbf{q}' \in Q$, set $\mathbf{q} \sim \mathbf{q}$ if $P_{\mathbf{q}}$ is normally equivalent to $P_{\mathbf{q}'}$. Then there are only finitely many equivalence classes for $\sim$.*

Proof. Write $P = \{\mathbf{x} : A\mathbf{x} \geq \mathbf{b}\}$, where A is an $r \times n$ matrix for some r, and $\mathbf{b} \in \mathbb{Q}^r$. Then for $\mathbf{q} \in Q$, $\pi^{-1}(\mathbf{q}) \cap P = \{(\mathbf{x}', \mathbf{q}) : \mathbf{x}' \in \mathbb{Q}^{n-d}, A(\mathbf{x}', \mathbf{q}) \geq \mathbf{b}\}$. Write A in block form as $(A'|A'')$, where A' is an $r \times (n-d)$ matrix, and A'' is an $r \times d$ matrix. Then $P_{\mathbf{q}} = \{\mathbf{x}' \in \mathbb{Q}^{n-d} : A'\mathbf{x}' \geq \mathbf{b} - A''\mathbf{q}\}$. Now for a given matrix A' there are only finitely many normal equivalence classes of polyhedra of the form $\{\mathbf{x}' \in \mathbb{Q}^{n-d} : A'\mathbf{x}' \geq \mathbf{c}\}$ as $\mathbf{c}$ varies, so it follows that there are only finitely many normal equivalence classes of fibers of π. $\square$

Definition 2.2. Pick $\mathbf{q}_1, \ldots, \mathbf{q}_r \in Q$ such that the $P_{\mathbf{q}_i}$ are representatives of the different normal equivalence classes of fibers of π. Let F be the Minkowski sum of the $P_{\mathbf{q}_i}$. The *fiber fan* $\mathcal{N}(P,Q)$ is the inner normal fan of F.

Lemma 2.3. *The recession cone of $P_{\mathbf{q}}$ is the same for all $\mathbf{q} \in Q$.*

Proof. Write $P = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \geq \mathbf{b}\}$ for some $r \times n$ matrix A and $\mathbf{b} \in \mathbb{Q}^r$. Decompose $A = (A'|A'')$, where A' is an $r \times (n-d)$ matrix, and A'' is an $r \times d$ matrix. Translating the polyhedron P by $\mathbf{a} \in \mathbb{Q}^n$ translates Q by $\pi(\mathbf{a})$, and $P_{\mathbf{q}} = (P + \mathbf{a})_{\mathbf{q}+\pi(\mathbf{a})}$, so we may assume $\mathbf{0} \in P$. Note that $P_{\mathbf{0}} = \{\mathbf{x} \in \mathbb{Q}^{n-d} : A'\mathbf{x} \geq \mathbf{b}\}$. We will show that $\text{rec}(P_{\mathbf{q}}) = \text{rec}(P_{\mathbf{0}})$ for all $\mathbf{q} \in Q$.

Let $\mathbf{v} \in \text{rec}(P_{\mathbf{q}})$. Then $\mathbf{u} + \lambda\mathbf{v} \in P_{\mathbf{q}}$ for all $\mathbf{u} \in P_{\mathbf{q}}$ and $\lambda > 0$. Since $\mathbf{0} \in P$, we have $b_i \leq 0$ for all i, where b_i is the ith component of $\mathbf{b}$. Suppose that $(\mathbf{v}, \mathbf{0}) \notin P$ for $\mathbf{0} \in \mathbb{Q}^d$. Then there is some i with $\mathbf{a_i} \cdot (\mathbf{v}, \mathbf{0}) < b_i \leq 0$, where $\mathbf{a_i}$ is the ith row of A. But then $\mathbf{a_i} \cdot (\mathbf{u} + \lambda\mathbf{v}, \mathbf{q}) = \mathbf{a_i} \cdot (\mathbf{u}, \mathbf{q}) + \lambda\mathbf{a_i} \cdot (\mathbf{v}, 0) < b_i$ for λ sufficiently large, which means that $(\mathbf{u} + \lambda\mathbf{v}, \mathbf{q}) \notin P$, so $\mathbf{u} + \lambda\mathbf{v} \notin P_{\mathbf{q}}$. Therefore we have $\{(\mathbf{v}, \mathbf{0}) \in \mathbb{Q}^n : \mathbf{v} \in \text{rec}(P_{\mathbf{q}})\} \subseteq P$ after all. The above argument shows that $\mathbf{a_i} \cdot (\mathbf{v}, \mathbf{0}) = \mathbf{a}'_{\mathbf{i}} \cdot \mathbf{v} \geq 0$ for each row $\mathbf{a}'_{\mathbf{i}}$ of A', so $\mathbf{v} \in \text{rec}(P_0)$.

For the opposite inclusion, note that since $P_{\mathbf{0}} = \{\mathbf{x} \in \mathbb{Q}^{n-d} : A'\mathbf{x} \geq \mathbf{b}\}$, the set $\{(\mathbf{v}, 0) : \mathbf{v} \in \text{rec}(P_0)\}$ lies in $\text{rec}(P)$. Thus if $\mathbf{v} \in \text{rec}(P_0)$, then $(\mathbf{u}, \mathbf{q}) + \lambda(\mathbf{v}, \mathbf{0}) \in P$ for all $\lambda > 0$ and $\mathbf{u} \in P_{\mathbf{q}}$. This means $(\mathbf{u} + \lambda\mathbf{v}, \mathbf{q}) \in P$ for all $\lambda > 0$ and $\mathbf{u} \in P_{\mathbf{q}}$, so $\mathbf{u} + \lambda\mathbf{v} \in P_{\mathbf{q}}$ for all $\lambda > 0$ and $\mathbf{u} \in P_{\mathbf{q}}$, giving $\mathbf{v} \in \text{rec}(P_{\mathbf{q}})$ as required. $\square$

If P and P' are two polyhedra with the same recession cone, then the normal fan of the Minkowski sum of P and P' is the common refinement of the normal fans of P and P'. Thus the fiber fan $\mathcal{N}(P, Q)$ is the common refinement of the normal fans of the fibers.

Remark 2.4. In the case that P is a polytope, the fiber fan $\mathcal{N}(P, Q)$ is the normal fan of the *fiber polytope* introduced by Billera and Sturmfels [BS92]. While the fiber fan of a linear projection of polyhedra is still the normal fan of a polyhedron, there is not a canonical choice of a polyhedron with that normal fan if the polyhedron being projected is not a polytope. This motivates our definition of the fiber fan, instead of a more general fiber polyhedron. Note that for applications to toric varieties, a fan suffices to define the variety.

2.3. **Duality for polyhedral cones.** This section establishes a duality result for polyhedral cones that is key for Theorem 1.2. Versions of Lemmas 2.5 and 2.6 are known to the experts, though we know of no proofs in the literature.

Let $C \subset \mathbb{Q}^n$ be a full-dimensional polyhedral cone. For $d < n$, let $\pi\colon \mathbb{Q}^n \to \mathbb{Q}^d$ be the projection onto the last d coordinates. For $\mathbf{v} \in \pi(C)$, consider the polyhedral slice $C_{\mathbf{v}} \subseteq \mathbb{Q}^{n-d}$. As $\mathbb{Q}^{n-d}$ is canonically isomorphic to $\ker(\pi)$, we may regard the normal fan $\mathcal{N}(C_{\mathbf{v}})$ as lying in the dual vector space $\ker(\pi)^* \cong (\mathbb{Q}^{n-d})^*$. Let $\pi^\vee\colon (\mathbb{Q}^n)^* \to \ker(\pi)^*$ denote the map dual to the inclusion of $\ker(\pi)$ in $\mathbb{Q}^n$. In the basis of $(\mathbb{Q}^n)^*$ dual to the standard basis of $\mathbb{Q}^n$ this is projection on the first $n-d$ coordinates. For $\mathbf{w} \in \pi^\vee(C^\vee)$ we consider the polyhedral slice $C^\vee \cap (\pi^\vee)^{-1}(\mathbf{w})$, and write $C^\vee_{\mathbf{w}} := \pi_d(C^\vee \cap (\pi^\vee)^{-1}(\mathbf{w})) \subseteq (\mathbb{Q}^d)^*$ for the isomorphic image of $C^\vee \cap (\pi^\vee)^{-1}(\mathbf{w})$ under the projection $\pi_d\colon (\mathbb{Q}^n)^* \to (\mathbb{Q}^d)^*$ onto the last d coordinates. There is a canonical isomorphism $\ker(\pi^\vee) \cong (\mathbb{Q}^d)^*$, so the normal fan $\mathcal{N}(C^\vee_{\mathbf{w}})$ can be regarded as living in either $\mathbb{Q}^d$ or in $(\ker(\pi^\vee))^*$ for all $\mathbf{w} \in (\mathbb{Q}^{n-d})^*$.

Let $\operatorname{relint}(F)$ denote the relative interior of a set F, which is the interior of F in its affine span.

Lemma 2.5. *Let $F \subset C$ be a face. If $\mathbf{v} \in \pi(\operatorname{relint}(F))$ then we have $\pi^\vee(\mathcal{N}_C(F)) = \mathcal{N}_{C_{\mathbf{v}}}(F_{\mathbf{v}})$.*

Proof. Write $C = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \geq \mathbf{0}\}$ where A is the $r \times n$ matrix whose rows $\mathbf{a}_1, \ldots, \mathbf{a}_r \in (\mathbb{Q}^n)^*$ form the facet normals of C. We may assume that F is cut out by the first p facet inequalities defining C, so $\mathcal{N}_C(F) = \mathbb{Q}_{\geq 0}\langle \mathbf{a}_1, \ldots, \mathbf{a}_p\rangle$ and $\pi^\vee(\mathcal{N}_C(F)) = \mathbb{Q}_{\geq 0}\langle \pi^\vee(\mathbf{a}_1), \ldots, \pi^\vee(\mathbf{a}_p)\rangle$. Note that

$$F \cap \pi^{-1}(\mathbf{v}) = \left\{ (\mathbf{u}, \mathbf{v}) : \begin{array}{ll} \mathbf{a}_i \cdot (\mathbf{u}, \mathbf{v}) = 0 & \text{for } i = 1, \ldots, p; \\ \mathbf{a}_i \cdot (\mathbf{u}, \mathbf{v}) \geq 0 & \text{for } i = p+1, \ldots, r. \end{array} \right\}$$

since F is cut out by the first p facet inequalities defining C. The assumption that $\mathbf{v} \in \pi(\operatorname{relint}(F))$ ensures that each $\geq$ above is a $>$, so

$$F_{\mathbf{v}} = \left\{ \mathbf{u} \in \mathbb{Q}^{n-d} : \pi^\vee(\mathbf{a}_i)\cdot\mathbf{u} = c_i \text{ for } i = 1, \ldots, p;\ \pi^\vee(\mathbf{a}_i)\cdot\mathbf{u} > c_i \text{ for } i = p+1, \ldots, r \right\}$$

where $c_i = -\mathbf{a}_i \cdot \mathbf{v}$ for $i = 1, \ldots, r$. This implies that the rational cone $\mathcal{N}_{C_{\mathbf{v}}}(F_{\mathbf{v}})$ is also generated by $\{\pi^\vee(\mathbf{a}_1), \ldots, \pi^\vee(\mathbf{a}_p)\}$ as claimed. □

Lemma 2.6. *Fix $\mathbf{v} \in \pi(C)$ and $\mathbf{w} \in \pi^\vee(C^\vee)$. If $F \subset C$ is a face then*

$$F = \widetilde{\operatorname{face}}_{\mathbf{w}}(C_{\mathbf{v}}) \iff \mathbf{v} \in \pi(\operatorname{relint}(F)) \text{ and } \mathbf{w} \in \pi^\vee\big(\operatorname{relint}(\mathcal{N}_C(F))\big).$$

Proof. Suppose $F = \widetilde{\mathrm{face}}_{\mathbf{w}}(C_{\mathbf{v}})$. Then the preimage of $\mathrm{face}_{\mathbf{w}}(C_{\mathbf{v}})$ under π_{n-d} lies in no proper face of F, so the intersection of $\mathrm{relint}(F)$ with this preimage is nonempty. This implies the set $\pi^{-1}(\mathbf{v}) \cap \mathrm{relint}(F)$ is nonempty, so $\mathbf{v} \in \pi(\mathrm{relint}(F))$. We have $\mathbf{w} \in \mathrm{relint}(\mathcal{N}_{C_{\mathbf{v}}}(\mathrm{face}_{\mathbf{w}}(C_{\mathbf{v}})))$ by definition. Applying Lemma 2.5 to the face $F_{\mathbf{v}} = \pi_n(F \cap \pi^{-1}(\mathbf{v})) = \mathrm{face}_{\mathbf{w}}(C_{\mathbf{v}})$ gives $\mathbf{w} \in \mathrm{relint}\, \pi^{\vee}(\mathcal{N}_C(F)) = \pi^{\vee}(\mathrm{relint}\, \mathcal{N}_C(F))$.

Conversely, suppose $\mathbf{v} \in \pi(\mathrm{relint}(F))$ and $\mathbf{w} \in \pi^{\vee}(\mathrm{relint}(\mathcal{N}_C(F)))$. The first assumption ensures that the face $F_{\mathbf{v}} \subseteq C_{\mathbf{v}}$ is nonempty and its preimage under π_{n-d} lies in no proper subface of F. Also, $\mathbf{w} \in \pi^{\vee}(\mathrm{relint}(\mathcal{N}_C(F))) = \mathrm{relint}\, \pi^{\vee}(\mathcal{N}_C(F)) = \mathrm{relint}\, \mathcal{N}_{C_{\mathbf{v}}}(F_{\mathbf{v}})$ by Lemma 2.5, hence $F_{\mathbf{v}} = \mathrm{face}_{\mathbf{w}}(C_{\mathbf{v}})$. Since the preimage of $F_{\mathbf{v}}$ under π_{n-d} lies in no proper subface of F, we have $F = \widetilde{\mathrm{face}}_{\mathbf{w}}(C_{\mathbf{v}})$. $\square$

We now present the main result of this section. See Figure 1 for an illustration.

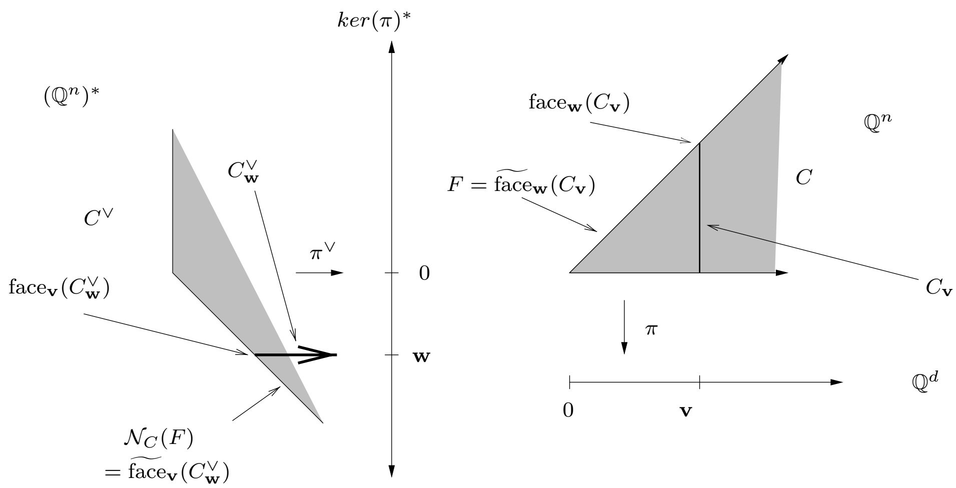

FIGURE 1. Figure for Proposition 2.7

Proposition 2.7. *For $\mathbf{v} \in \pi(C)$ and $\mathbf{w} \in \pi^{\vee}(C^{\vee})$ we have*

$$\mathcal{N}_C\big(\widetilde{\mathrm{face}}_{\mathbf{w}}(C_{\mathbf{v}})\big) = \widetilde{\mathrm{face}}_{\mathbf{v}}(C^{\vee}_{\mathbf{w}}).$$

Proof. Applying Lemma 2.6 to the face $F := \widetilde{\mathrm{face}}_{\mathbf{w}}(C_{\mathbf{v}})$ of C gives $\mathbf{v} \in \pi(\mathrm{relint}(F))$ and $\mathbf{w} \in \pi^{\vee}(\mathrm{relint}(\mathcal{N}_C(F)))$. The face $\mathcal{N}_C(F)$ of $C^{\vee}$ satisfies $\mathcal{N}_{C^{\vee}}(\mathcal{N}_C(F)) = F$, so $\mathbf{w} \in \pi^{\vee}(\mathrm{relint}(\mathcal{N}_C(F)))$ and $\mathbf{v} \in \pi(\mathrm{relint}(\mathcal{N}_{C^{\vee}}(\mathcal{N}_C(F))))$. Now apply Lemma 2.6 to the face $\mathcal{N}_C(F)$ of $C^{\vee}$ to deduce that $\mathcal{N}_C(F) = \widetilde{\mathrm{face}}_{\mathbf{v}}(C^{\vee}_{\mathbf{w}})$ as required. $\square$

Remark 2.8. Proposition 2.7 is the key duality result that is required in the proof of Theorem 7.2 of Craw–Maclagan–Thomas [CMT05].

The following definition is a variant for polyhedral cones of a notion for polytopes due to Billera–Sturmfels [BS92, §2].

Definition 2.9. For $\mathbf{v} \in \pi(C)$, the subdivision of $\pi^{\vee}(C^{\vee})$ consisting of the cells $\big\{\pi^{\vee}\big(\widetilde{\mathrm{face}}_{\mathbf{v}}(C^{\vee}_{\mathbf{w}})\big) : \mathbf{w} \in \pi^{\vee}(C)\big\}$ is the $\mathbf{v}$-*induced* $\pi^{\vee}$-*coherent subdivision* of $\pi^{\vee}(C^{\vee})$.

The duality result of Proposition 2.7 leads immediately to the following description of the normal fan of the polyhedral slice $C_{\mathbf{v}}$ for $\mathbf{v} \in \pi(C)$.

Corollary 2.10. *For $\mathbf{v} \in \pi(C)$, the inner normal fan of the polyhedron $C_{\mathbf{v}}$ is the $\mathbf{v}$-induced $\pi^\vee$-coherent subdivision of $\pi^\vee(C^\vee)$. In other words,*

$$\mathcal{N}(C_{\mathbf{v}}) = \big\{\pi^\vee\big(\widetilde{\mathrm{face}}_{\mathbf{v}}(C^\vee_{\mathbf{w}})\big) : \mathbf{w} \in \pi^\vee(C^\vee)\big\}.$$

In particular, for $\mathbf{v} = \mathbf{0}$ we have $\mathcal{N}(C_{\mathbf{0}}) = \pi^\vee(C^\vee)$.

Proof. By Lemma 2.5 and Proposition 2.7, we have

$$\mathcal{N}_{C_{\mathbf{v}}}\big(\mathrm{face}_{\mathbf{w}}(C_{\mathbf{v}})\big) = \pi^\vee\big(\mathcal{N}_C\big(\widetilde{\mathrm{face}}_{\mathbf{w}}(C_{\mathbf{v}})\big)\big) = \pi^\vee\big(\widetilde{\mathrm{face}}_{\mathbf{v}}(C^\vee_{\mathbf{w}})\big).$$

The set of cones $\mathcal{N}_{C_{\mathbf{v}}}(\mathrm{face}_{\mathbf{w}}(C_{\mathbf{v}}))$ taken over all vectors $\mathbf{w} \in \pi^\vee(C^\vee)$ equals the inner normal fan $\mathcal{N}(C_{\mathbf{v}})$ as required. For the second statement, observe that $\mathrm{face}_{\mathbf{0}}(C^\vee_{\mathbf{w}}) = C^\vee_{\mathbf{w}}$ for all $\mathbf{w} \in \pi^\vee(C^\vee)$, and the smallest face of the cone $C^\vee$ containing the set $C^\vee_{\mathbf{w}}$ is $C^\vee$ itself. □

Definition 2.11. Let C be a polyhedral cone in $\mathbb{Q}^n$, and let $\pi\colon \mathbb{Q}^n \to \mathbb{Q}^d$ be a linear map. The *fan induced by* $\pi(C)$ is the subdivision of $\pi(C)$ whose cells are the intersections $\cap_{\mathbf{w}\in\pi(F)}\pi(F)$ as $\mathbf{w}$ varies over the set $\pi(C)$, where the intersection is over faces F of C with $\mathbf{w} \in \pi(F)$.

The following lemma shows that the fan induced by $\pi(C)$ is actually a fan.

Lemma 2.12. *Let C be a polyhedral cone in $\mathbb{Q}^n$, and let $\pi\colon \mathbb{Q}^n \to \mathbb{Q}^d$ be a linear map. The fan induced by $\pi(C)$ is a fan with support $\pi(C)$.*

Proof. As $\mathbf{v}$ varies in $\pi^\vee(C^\vee)$, and $\mathbf{w}$ varies in $\pi(C)$, $\widetilde{\mathrm{face}}_{\mathbf{v}}(C_{\mathbf{w}})$ varies over all the faces of C. By Corollary 2.10 applied to the cone $C^\vee$, the collection of cones $\{\pi(\widetilde{\mathrm{face}}_{\mathbf{v}}(C_{\mathbf{w}})) : \mathbf{w} \in \pi(C)\}$ forms the normal fan of $C^\vee_{\mathbf{v}}$. Thus the cells of the subdivision induced by $\pi(C)$ are precisely the cones in the common refinement of the normal fans of $C^\vee_{\mathbf{v}}$ as $\mathbf{v}$ varies over $\pi^\vee(C^\vee)$, which is $\mathcal{N}(\mathcal{C}^\vee, \pi^\vee(C^\vee))$, and thus the subdivision induced by $\pi(C)$ is a fan. □

The proof of Lemma 2.12 contains the following fact.

Corollary 2.13. *The fiber fan $\mathcal{N}(C, \pi(C))$ is the fan induced by $\pi^\vee(C^\vee)$.*

2.4. **Duality for general polyhedra.** We now generalize the results of the previous subsection from polyhedral cones to general polyhedra.

Definition 2.14. For any polyhedron $P \subseteq \mathbb{Q}^n$, let $\widetilde{P} \subseteq \mathbb{Q}^n \oplus \mathbb{Q}$ be the polyhedral cone obtained as the closure in $\mathbb{Q}^n \oplus \mathbb{Q}$ of the set $\{(\lambda\mathbf{x}, \lambda) : \mathbf{x} \in P, \lambda \in \mathbb{Q}_{>0}\}$. Note the $\widetilde{P} \cap (\mathbb{Q}^n \oplus \{0\}) \cong \mathrm{rec}(P)$.

Given $\pi\colon \mathbb{Q}^n \to \mathbb{Q}^d$, we write $\widetilde{\pi}\colon \mathbb{Q}^n \oplus \mathbb{Q} \to \mathbb{Q}^d \oplus \mathbb{Q}$ for the map sending $(\mathbf{x}, \lambda)$ to $(\pi(\mathbf{x}), \lambda)$. The first projection $p_1\colon \mathbb{Q}^n \oplus \mathbb{Q} \to \mathbb{Q}^n$ fits into the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(\widetilde{\pi}) \arrow[r, "\widetilde{\iota}"] \arrow[d, "\cong"'] & \mathbb{Q}^n \oplus \mathbb{Q} \arrow[r, "\widetilde{\pi}"] \arrow[d, "p_1"'] & \mathbb{Q}^d \oplus \mathbb{Q} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(\pi) \arrow[r, "\iota"] & \mathbb{Q}^n \arrow[r, "\pi"] & \mathbb{Q}^d \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

and canonically identifies $\ker(\pi)$ with $\ker(\widetilde{\pi})$. Write $\pi^\vee\colon (\mathbb{Q}^n)^* \to \ker(\pi)^*$ and $\widetilde{\pi}^\vee\colon (\mathbb{Q}^{n+1})^* \to \ker(\widetilde{\pi})^*$ for the maps obtained by pullback via ι and $\widetilde{\iota}$ respectively.

These maps fit into the dual diagram

$$
\begin{tikzcd}
0 & \ker(\widetilde{\pi})^* \arrow[l] & (\mathbb{Q}^n \oplus \mathbb{Q})^* \arrow[l, "\widetilde{\pi}^\vee"'] & (\mathbb{Q}^d \oplus \mathbb{Q})^* \arrow[l] & 0 \arrow[l] \\
0 & \ker(\pi)^* \arrow[l] \arrow[u, "p_1^*", "\cong"'] & (\mathbb{Q}^n)^* \arrow[l, "\pi^\vee"'] \arrow[u, "p_1^*"] & (\mathbb{Q}^d)^* \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}
\tag{2.2}
$$

Since $P \subseteq \mathbb{Q}^n$ and $\widetilde{P} \subseteq \mathbb{Q}^n \oplus \mathbb{Q}$, the normal fans satisfy $\mathcal{N}(P) \subseteq (\mathbb{Q}^n)^*$ and $\mathcal{N}(\widetilde{P}) \subseteq (\mathbb{Q}^n \oplus \mathbb{Q})^*$ respectively. Note that $\mathcal{N}(\widetilde{P})$ consists of the faces of $(\widetilde{P})^\vee$.

Proposition 2.15. *After identifying* $\ker(\pi)^*$ *with* $\ker(\widetilde{\pi})^*$ *via the vertical map* p_1^* *from diagram (2.2), the following fans coincide:*

(1) *the fiber fan* $\mathcal{N}(P, \pi(P))$*;*
(2) *the fiber fan* $\mathcal{N}(\widetilde{P}, \widetilde{\pi}(\widetilde{P}))$*;*
(3) *the fan induced by* $\widetilde{\pi}^\vee\big((\widetilde{P})^\vee\big)$.

Proof. We first show that fans (1) and (2) coincide. For $\mathbf{q} \in \mathbb{Q}^d$, set $\widetilde{\mathbf{q}} := (\mathbf{q}, 1) \in \mathbb{Q}^d \oplus \mathbb{Q}$. Note that $(\widetilde{P})_{\widetilde{\mathbf{q}}} = P_{\mathbf{q}}$, since P is the intersection of $\widetilde{P}$ with $\mathbb{Q}^d \oplus \{1\}$. Since $\widetilde{P}$ is a cone, the polyhedron $(\widetilde{P})_{(\mathbf{q},j)}$ is normally equivalent to $(\widetilde{P})_{(\mathbf{q}/j,1)}$ for $j > 0$. It follows that the fan $\mathcal{N}(P, \pi(P))$ is equal to the common refinement of (the images under p_1^* of) the normal fans of fibers $(\widetilde{P})_{(\mathbf{q},j)}$ for $j > 0$ and $\mathbf{q} \in \mathbb{Q}^d$. Since $(\widetilde{P})_{(\mathbf{q},j)} = \emptyset$ for $j < 0$, the statement that the fiber fans (1) and (2) coincide follows once we show that $\mathcal{N}(P, \pi(P))$ is a refinement of the normal fan of $(\widetilde{P})_{(\mathbf{q},0)}$ for all $(\mathbf{q}, 0) \in \widetilde{\pi}(\widetilde{P})$. Since $(\widetilde{P})_{(\mathbf{q},0)} = \mathrm{rec}(P)_{\mathbf{q}}$, we have $(\mathbf{q}, 0) \in \widetilde{\pi}(\widetilde{P})$ if and only if $\mathbf{q} \in \pi(\mathrm{rec}(P))$. Thus, it is enough to show that $\mathcal{N}(P, \pi(P))$ refines $\mathcal{N}(\mathrm{rec}(P)_{\mathbf{q}})$ for all $\mathbf{q} \in \pi(\mathrm{rec}(P))$. We show below that for all $\mathbf{q} \in \pi(\mathrm{rec}(P))$ there is $\lambda > 0$ such that $\lambda\mathbf{q} \in \pi(P)$, and $\mathcal{N}(P_{\lambda\mathbf{q}})$ refines $\mathcal{N}(\mathrm{rec}(P)_{\lambda\mathbf{q}})$. Since these normal fans equal the corresponding ones with $\lambda\mathbf{q}$ replaced by $\mathbf{q}$, this shows that $\mathcal{N}(P_{\mathbf{q}})$ and hence $\mathcal{N}(P, \pi(P))$ is a refinement of the normal fan of $\mathrm{rec}(P)_{\mathbf{q}}$.

Fix $\mathbf{q} \in \pi(\mathrm{rec}(P))$. The condition that $\lambda\mathbf{q} \in \pi(P)$ is satisfied for all $\lambda \gg 0$, or all λ if $\mathbf{0} \in P$. Write $P = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \geq \mathbf{b}\}$, where A is an $r \times n$ matrix, and $\mathbf{b} \in \mathbb{Q}^r$. Write A in block form as $A = (A'|A'')$, where A' is a $r \times (n-d)$ matrix, and A'' is a $r \times d$ matrix. Then $P_{\lambda\mathbf{q}} = \{\mathbf{x} \in \mathbb{Q}^{n-d} : A'\mathbf{x} \geq \mathbf{b} - A''\lambda\mathbf{q}\}$, while $\mathrm{rec}(P)_{\lambda\mathbf{q}} = \{\mathbf{x} \in \mathbb{Q}^{n-d} : A'\mathbf{x} \geq -A''\lambda\mathbf{q}\}$. We first consider the case that $A''\mathbf{q} \neq \mathbf{0}$. Since $\lambda\mathbf{q} \in \pi(P)$, $P_{\lambda\mathbf{q}}$ and $\mathrm{rec}(P)_{\lambda\mathbf{q}}$ are nonempty for all λ, so $\mathbf{b} - A''\lambda\mathbf{q}$ and $-A''\lambda\mathbf{q}$ lie in some chambers of the chamber complex of the Gale dual of A' (see [BFS90]). For most generic choices of $\mathbf{q}$ the vector $-A''\mathbf{q}$ lies in the interior of a top-dimensional chamber, so for $\lambda \gg 0$ the vectors $-A''\mathbf{q}$ and $\mathbf{b} - A''\lambda\mathbf{q}$ lie in the same chamber, and thus $P_{\lambda\mathbf{q}}$ and $\mathrm{rec}(P)_{\lambda\mathbf{q}}$ are normally equivalent. If $\mathbf{q}$ is not generic, then for $\lambda \gg 0$ the vector $\mathbf{b} - A''\lambda\mathbf{q}$ either lies in the same chamber as $-A''\lambda\mathbf{q}$, or in a larger dimensional chamber that has the chamber of $-A''\lambda\mathbf{q}$ as a face. In the first case $P_{\lambda\mathbf{q}}$ and $\mathrm{rec}(P)_{\lambda\mathbf{q}}$ are normally equivalent, while in the second the normal fan of $P_{\lambda\mathbf{q}}$ is a refinement of that of $\mathrm{rec}(P)_{\lambda\mathbf{q}}$. Finally, if $A''\mathbf{q} = \mathbf{0}$, then $\mathrm{rec}(P)_{\lambda\mathbf{q}} = \mathrm{rec}(P_{\lambda\mathbf{q}})$ is the recession cone of $P_{\lambda\mathbf{q}}$, so the refinement is automatic in this case.

Finally, to see that fan (2) equals fan (3), apply Corollary 2.13 with $C = \widetilde{P}$ and $\widetilde{\pi}$ in place of π. □

Definition 2.16. Let P be a polyhedron in $\mathbb{Q}^n$, and let $\pi\colon \mathbb{Q}^n \to \mathbb{Q}^d$ be a linear map. The *subdivision induced by* $\pi(P)$ is the subdivision of $\pi(P)$ whose cells are the intersections $\cap_{\mathbf{q}\in\pi(F)}\pi(F)$ as $\mathbf{q}$ varies over the set $\pi(P)$, where the intersection is over faces F of P with $\mathbf{q} \in \pi(F)$.

When P is a cone the subdivision induced by $\pi(P)$ is the fan induced by $\pi(P)$ of Definition 2.11.

Lemma 2.17. *Let $\pi\colon P \to Q$ be a linear projection of polyhedra. Then $\widetilde{\pi}(\widetilde{P}) = \widetilde{Q}$, so $\widetilde{\pi}(\widetilde{P})$ is the cone over Q. Moreover, the fan induced by $\widetilde{\pi}(\widetilde{P})$ is obtained by taking the cone over each cell in the subdivision induced by $\pi(P)$. This implies that the subdivision induced by $\pi(P)$ is a polyhedral complex. Also, the union of the $(d-1)$-dimensional faces of the subdivision induced by $\pi(P)$ is $\cup_{\dim(F)=d-1}\pi(F)$ where F is a $(d-1)$-dimensional face of P.*

Proof. It is easy to see that $\pi(\mathrm{rec}(P)) = \mathrm{rec}(\pi(P)) = \mathrm{rec}(Q)$, so the equality of sets $\widetilde{\pi}(\widetilde{P}) = \widetilde{Q}$ follows from the fact that $\widetilde{\pi}(\lambda\mathbf{x}, \lambda) = (\lambda\pi(\mathbf{x}), \lambda)$ for $\mathbf{x} \in P$. For the second statement, note first that the faces of $\widetilde{P}$ divide naturally into two disjoint sets: those arising from faces of the recession cone of P; and those of the form $\widetilde{E}$ for some face $E \subseteq P$. The faces in the former set lie in $\mathbb{Q}^n \oplus \{0\} \subset \mathbb{Q}^n \oplus \mathbb{Q}$, so it is enough to check that $\cap_{\mathbf{q}\in\pi(F)}\pi(F)$, where $\mathbf{q} \in Q$, is the intersection of the slice $\mathbb{Q}^d \oplus \{1\} \subset \mathbb{Q}^d \oplus \mathbb{Q}$ with $\cap_{(\mathbf{q},1)\in\widetilde{Q}}\widetilde{\pi}(\widetilde{F})$. This follows since for every face $E \subseteq P$, the intersection of $\mathbb{Q}^d \oplus \{1\}$ with the cone $\widetilde{\pi}(\widetilde{E}) \subseteq \widetilde{Q}$ is the cell $\pi(E) \subseteq Q$. This shows that the subdivision induced by $\pi(P)$ is the intersection of a fan with a hyperplane, and is thus a polyhedral complex.

Finally, it is straightforward to see that the image of the $(d-1)$-dimensional faces of P is contained in the union of the $(d-1)$ dimensional faces of $\pi(P)$. The fact that every point in a $(d-1)$ dimensional face of $\pi(P)$ is in the image of some $(d-1)$-dimensional face of $\pi(P)$ follows by considering a face of P corresponding to a vertex of the fiber of π over the point. □

Proposition 2.18. *Let $\pi\colon P \to Q$ be a linear projection of polyhedra, and set $C := (\widetilde{P})^\vee$. Then the fiber fan $\mathcal{N}(C, \widetilde{\pi}^\vee(C))$ is the fan induced by $\widetilde{\pi}(\widetilde{P})$, and thus is obtained by taking the cone over each cell in the subdivision induced by $\pi(P)$.*

Proof. Applying Corollary 2.13 to the map $\widetilde{\pi}^\vee\colon C \to \widetilde{\pi}^\vee(C)$ gives $\mathcal{N}(C, \widetilde{\pi}^\vee(C)) = (\widetilde{\pi}^\vee)^\vee(C^\vee)$. Since $\widetilde{P}$ is a polyhedral cone, we have $C^\vee = ((\widetilde{P})^\vee)^\vee = \widetilde{P}$. The first statement follows from the fact that $(\widetilde{\pi}^\vee)^\vee = \widetilde{\pi}$. The second statement follows from Lemma 2.17. □

3. On quotients of toric varieties by subtori

3.1. **GIT quotients of quasiprojective toric varieties.** We first recall the polyhedral construction of quasiprojective toric varieties that are projective over affine toric varieties (see Thaddeus [Tha94, §2.8]). We work over an algebraically closed field $\Bbbk$.

Let $P \subseteq \mathbb{Q}^n$ be an n-dimensional lattice polyhedron and let $\widetilde{P}$ be the closure in $\mathbb{Q}^n \oplus \mathbb{Q}$ of the cone over P as given in Definition 2.14. The semigroup algebra $R := \Bbbk[\widetilde{P} \cap (\mathbb{Z}^n \oplus \mathbb{Z})]$ defines the affine toric variety $\operatorname{Spec} R$. The second projection determines a grading of the ring R. Let R_j be the $\Bbbk$-vector subspace spanned by the elements of R of degree $j \in \mathbb{N}$. Then $X_P := \operatorname{Proj} \bigoplus_{j \geq 0} R_j$ is the n-dimensional quasiprojective toric variety whose fan is $\mathcal{N}(P)$, the inner normal fan of the polyhedron P. Note that $X_{\widetilde{P}} \cong \operatorname{Spec} R$ is the affine cone over X_P. Also, since R_0 is isomorphic to the semigroup algebra $\Bbbk[\operatorname{rec}(P) \cap \mathbb{Z}^n]$ determined by the recession cone $\operatorname{rec}(P) \subseteq \mathbb{Q}^n$, we see that X_P is projective over the affine toric variety $\operatorname{Spec} R_0$. Thus, X_P is projective if P is a polytope.

Remark 3.1. Any quasiprojective toric variety X that is projective over an affine toric variety arises in this way once a relatively ample torus-invariant divisor D is chosen on X, in which case $R_j \cong H^0(X, \mathcal{O}_X(jD))$ for all $j \in \mathbb{N}$.

Let $(\Bbbk^*)^n = \operatorname{Spec} \Bbbk[\mathbb{Z}^n]$ denote the dense torus of X_P, and let $T_L := \operatorname{Spec} \Bbbk[L]$ be a d-dimensional algebraic subtorus of $(\Bbbk^*)^n$. Choose generators for $L \cong \mathbb{Z}^d$ and write $\pi_{\mathbb{Z}} \colon \mathbb{Z}^n \to \mathbb{Z}^d$ for the map of character lattices induced by the inclusion $T_L \hookrightarrow (\Bbbk^*)^n$. Composing $\pi_{\mathbb{Z}}$ with the $\mathbb{Z}^n$-grading of the ring R arising from the action of $(\Bbbk^*)^n$ on X_P determines a $\mathbb{Z}^d$-grading of R. Given a character $\mathbf{v} \in \mathbb{Z}^d$ of the torus T_L, the GIT quotient of X_P by the action of T_L linearized by $\mathbf{v}$ is the quotient of the affine variety $\operatorname{Spec}(R)$ by the lift of T_L. Specifically, begin by defining $\widetilde{\pi_{\mathbb{Z}}} \colon \mathbb{Z}^n \oplus \mathbb{Z} \to \mathbb{Z}^d \oplus \mathbb{Z}$ by sending $(\mathbf{x}, \lambda)$ to $(\pi_{\mathbb{Z}}(\mathbf{x}), \lambda)$. This lattice map induces an inclusion of the $(d+1)$-dimensional torus $T_{\widetilde{L}} := \operatorname{Spec} \Bbbk[\widetilde{L}]$ into the dense torus $(\Bbbk^*)^{n+1} = \operatorname{Spec} \Bbbk[\mathbb{Z}^n \oplus \mathbb{Z}]$ of $X_{\widetilde{P}}$, where $\widetilde{L} = L \oplus \mathbb{Z}$. Characters $\mathbf{v} \in \mathbb{Z}^d$ of T_L give rise to characters $\widetilde{\mathbf{v}} := (\mathbf{v}, 1) \in \mathbb{Z}^d \oplus \mathbb{Z}$ of $T_{\widetilde{L}}$. Let $R_{j\widetilde{\mathbf{v}}}$ denote the $j\widetilde{\mathbf{v}}$-graded piece of the ring R with respect to the $\mathbb{Z}^d \oplus \mathbb{Z}$-grading of R. Then

$$X_P \mathbin{/\!\!/}_{\mathbf{v}} T_L = X_{\widetilde{P}} \mathbin{/\!\!/}_{\widetilde{\mathbf{v}}} T_{\widetilde{L}} := \operatorname{Proj} \bigoplus_{j \geq 0} R_{j\widetilde{\mathbf{v}}}. \tag{3.1}$$

We now adopt the notation of Section 2.4. Thus, we write $\pi \colon \mathbb{Q}^n \to \mathbb{Q}^d$ and $\widetilde{\pi} \colon \mathbb{Q}^n \oplus \mathbb{Q} \to \mathbb{Q}^d \oplus \mathbb{Q}$ for the linear maps obtained from $\pi_{\mathbb{Z}}$ and $\widetilde{\pi_{\mathbb{Z}}}$ respectively by extending scalars. The following result is due to Kapranov–Sturmfels–Zelevinsky [KSZ92] when P is a polytope, and to Thaddeus [Tha94] in general.

Lemma 3.2. *For $\mathbf{v} \in \operatorname{int}(\pi(P))$, the GIT quotient $X_P \mathbin{/\!\!/}_{\mathbf{v}} T_L$ is the toric variety with fan $\mathcal{N}(P_{\mathbf{v}})$.*

Proof. We know that $X_P \mathbin{/\!\!/}_{\mathbf{v}} T \cong X_{\widetilde{P}} \mathbin{/\!\!/}_{\widetilde{\mathbf{v}}} T_{\widetilde{L}}$. The left-most vertical arrow in the diagram (2.1) sends the polyhedron $\widetilde{P}_{\widetilde{\mathbf{v}}}$ isomorphically onto $P_{\mathbf{v}}$. Since $\mathbf{v} \in \operatorname{int}(\pi(P))$ and hence $\widetilde{\mathbf{v}} \in \operatorname{int}(\widetilde{\pi}(\widetilde{P}))$, the polyhedra $\widetilde{P}_{\widetilde{\mathbf{v}}}$ and $P_{\mathbf{v}}$ have full dimension in $\ker(\widetilde{\pi})$ and $\ker(\pi)$ respectively, and the left-most vertical map in the dual diagram (2.2) identifies the normal fans $\mathcal{N}(\widetilde{P}_{\widetilde{\mathbf{v}}})$ and $\mathcal{N}(P_{\mathbf{v}})$. The result follows from the fact that for $j \in \mathbb{N}$, the graded piece $R_{j\widetilde{\mathbf{v}}}$ is isomorphic to the $\Bbbk$-vector space spanned by the lattice points of the slice $\widetilde{P} \cap \widetilde{\pi}^{-1}(j\widetilde{\mathbf{v}})$. $\square$

Remark 3.3. If $\mathbf{v}$ lies on the boundary of $\pi(P)$, then the statement of Lemma 3.2 holds if one regards $P_{\mathbf{v}}$ as a polyhedron in its affine hull. The normal fan is then necessarily nondegenerate.

Proposition 3.4. *For $\mathbf{v} \in \operatorname{int}(\pi(P))$, the fan $\mathcal{N}(P_{\mathbf{v}})$ of $X_P /\!\!/_{\mathbf{v}} T$ is isomorphic to the $\widetilde{\mathbf{v}}$-induced $\widetilde{\pi}^\vee$-coherent subdivision of the cone $\widetilde{\pi}^\vee(\widetilde{P}^\vee) \subseteq \ker(\widetilde{\pi})^*$.*

Proof. Corollary 2.10 shows that $\mathcal{N}(\widetilde{P}_{\widetilde{\mathbf{v}}})$ is the $\widetilde{\mathbf{v}}$-induced $\widetilde{\pi}^\vee$-coherent subdivision of the cone $\widetilde{\pi}^\vee(\widetilde{P}^\vee)$. The result follows since $\mathcal{N}(P_{\mathbf{v}})$ is isomorphic to $\mathcal{N}(\widetilde{P}_{\widetilde{\mathbf{v}}})$. □

Definition 3.5. A character $\mathbf{v} \in \pi(P)$ is *generic* if every $\mathbf{v}$-semistable point of X_P is $\mathbf{v}$-stable in the sense of GIT (see Dolgachev–Hu [DH98, (0.2.2)]). For generic characters $\mathbf{v}, \mathbf{v}' \in \pi(P)$, we set $\mathbf{v} \sim \mathbf{v}'$ if every $\mathbf{v}$-stable point of X_P is $\mathbf{v}'$-stable and vice-versa. This equivalence relation gives a polyhedral decomposition of $\pi(P)$, called the (*polarized*) *GIT chamber decomposition* of the T_L-action on X_P.

Remark 3.6. The GIT chamber decomposition is the subdivision of Q induced by $\pi(P)$ in the sense of Definition 2.16 and Lemma 2.17 (see [BP90], [Tha94]). For analogous results for general GIT quotients and for more on the GIT decomposition of the space of all T-ample linearizations, see Thaddeus [Tha96], Dolgachev–Hu [DH98] and Ressayre [Res00].

3.2. **The toric Chow quotient.** In this section we define the toric Chow quotient of a quasiprojective toric variety by a subtorus. This coincides with the traditional Chow quotient when the variety is projective, and generalizes the toric Chow quotient introduced by Haiman–Sturmfels [HS04, §5]. We first introduce a subcategory of the category of $\Bbbk$-schemes. Fix $T := (\Bbbk^*)^n$ for $n \in \mathbb{N}$.

Definition 3.7. Consider the category $\mathcal{C}_T$ whose objects are Noetherian $\Bbbk$-schemes X with a T-action $\sigma_X \colon T \times X \to X$ and an irreducible T-invariant component $X_0 \subseteq X$, such that the restriction of the T-action to X_0, denoted $\sigma_{X_0} \colon T \times X_0 \to X_0$, gives X_0 the structure of a not-necessarily normal toric variety with dense torus T. The morphisms of $\mathcal{C}_T$ are T-equivariant proper morphisms $f \colon X \to X'$ over $\Bbbk$ that induce a birational morphism $f|_{X_0} \colon X_0 \to X'_0$ between the toric components.

Lemma 3.8. *Fiber products exist in the category $\mathcal{C}_T$. Moreover, the fiber product in $\mathcal{C}_T$ coincides with the fiber product in the category of $\Bbbk$-schemes.*

Proof. For an object S in $\mathcal{C}_T$, let $f \colon X \to S$ and $f' \colon X' \to S$ be morphisms in $\mathcal{C}_T$. We must show that the fiber product in the category of $\Bbbk$-schemes $Z := X \times_S X'$ is an object of $\mathcal{C}_T$, and that the canonical projections $p \colon Z \to X$ and $q \colon Z \to X'$ are morphisms in $\mathcal{C}_T$.

To construct the component $Z_0 \subseteq Z$, note that the schemes X, X' and S contain open subvarieties T_X, $T_{X'}$ and T_S respectively, each isomorphic to the torus T. The fiber product $T_Z := T_X \times_{T_S} T_{X'}$ is canonically isomorphic to the subscheme $p^{-1}(T_X) \cap q^{-1}(T_{X'})$ of Z by [GD60, Chap. I, Coro 3.2.3], so it is open in Z. Furthermore, the T-equivariant birational morphisms $f|_{X_0} \colon X_0 \to S_0$ and $f'|_{X'_0} \colon X'_0 \to S_0$ restrict to give isomorphisms $T_X \cong T_S$ and $T_{X'} \cong T_S$, so T_Z is isomorphic to $T \times_T T \cong T$ and hence is irreducible. In particular, the n-torus $T_Z \subseteq Z$ is dense in some component of Z that we denote Z_0.

We next show that Z_0 is reduced. Since this is a local question, and $T_Z \subseteq Z_0$ is reduced, we can reduce to the case where $Z_0 = \operatorname{Spec}(A)$ is an irreducible Noetherian $\Bbbk$-scheme with a nonempty open subscheme W for which the induced scheme structure on W is reduced. We may assume that $W = \operatorname{Spec}(A_f)$ for some

$f \in A$. Writing $A = \Bbbk[x_1, \dots, x_m]/I$ for some I, we note that I is primary since Z_0 is irreducible, and since W is nonempty $f \notin \sqrt{I}$. Suppose Z_0 is not reduced, so there exists $g \notin I$ with $g^k \in I$ for some I. Then $(g/1)^k \in I_f$, so since W is reduced we have $g/1 \in I_f$. But then $g/1 = y/f^l$ for some l and some $y \in I$. This means that $gf^l \in I$, which contradicts I being primary, $g \notin I$, and $f \notin \sqrt{I}$. We thus conclude that Z_0 is reduced.

We now describe the torus action. The fiber product Z embeds as a closed subscheme of the product $X \times_{\Bbbk} X'$. Since the morphisms f and f' are T-equivariant, it can be shown that the product T-action on $X \times_{\Bbbk} X'$ restricts to give an action $\sigma_Z \colon T \times Z \to Z$. This restricts to give an action $T \times T_Z \to T_Z$ that coincides with the multiplicative structure on the torus T_Z after identifying $T_Z \cong T$. Thus, $\sigma_Z \colon T \times Z \to Z$ extends the natural multiplicative structure of T_Z. To see that Z_0 is T-invariant we argue by continuity as follows. The component $Z_0 \subseteq Z$ is closed and contains T_Z, so $\sigma_Z^{-1}(Z)$ is a closed subscheme of $T \times Z$ that contains $\sigma_Z^{-1}(T_Z) = T \times T_Z$. In particular, $\sigma_Z^{-1}(Z)$ contains the closure $T \times Z_0$ of $T \times T_Z$, so the image of $T \times Z_0$ under σ_Z lies in Z_0. The resulting action $\sigma_{Z_0} \colon T \times Z_0 \to Z_0$ extends the multiplicative structure of T_Z on itself by the above, so Z_0 is a not-necessarily-normal toric variety.

It remain to prove that the projections p and q are morphisms in $\mathcal{C}_T$. Properness of p and q follows from properness of f and f' by base extension. The T-action on Z was constructed to ensure that p and q are T-equivariant. Their restrictions $p|_{T_Z} \colon T_Z \to T_X$ and $p'|_{T_Z} \colon T_Z \to T_{X'}$ are isomorphisms, so both p and q are birational on the toric components as required. □

We now return to the GIT set-up from Section 3.1, where $\pi \colon P \to Q$ is a linear projection of polyhedra, with $L \cong \mathbb{Z}^d$ the image of the corresponding map $\pi_{\mathbb{Z}} \colon \mathbb{Z}^n \to \mathbb{Z}^d$. Set $M := \ker(\pi_{\mathbb{Z}})$. For $\mathbf{v} \in \mathrm{int}(Q)$, the GIT quotient $X_P /\!\!/_{\mathbf{v}} T_L = X_{P_{\mathbf{v}}}$ is the toric variety with dense torus T_M defined by the fan $\mathcal{N}(P_{\mathbf{v}})$.

Let $\mathcal{P}$ be the face poset of the subdivision of Q induced by $\pi(P)$, with the faces on the boundary of Q removed. Thus $\tau \prec \sigma$ if τ is a face of σ. To each $\sigma \in \mathcal{P}$ we associate the toric variety $X_\sigma := X_{P_{\mathbf{v}}}$ for any $\mathbf{v} \in \sigma$. If τ is a face of σ, then there is a proper, birational toric morphism from X_σ to X_τ by Thaddeus [Tha94, Theorem 3.11 and Corollary 3.12]. Therefore $\mathcal{P}$ gives a directed system of $\Bbbk$-schemes. We are interested in the inverse limit of this system in the category of $\Bbbk$-schemes.

Proposition 3.9. *Let T_M be the dense torus in X_σ for $\sigma \in \mathcal{P}$. Then the inverse limit in the category of $\Bbbk$-schemes*

$$Z := \varprojlim_{\sigma \in \mathcal{P}} X_\sigma \tag{3.2}$$

exists and is an object of the category $\mathcal{C}_{T_M}$. In particular, Z has an irreducible component Z_0 that is a not-necessarily-normal toric variety with dense torus T_M.

Proof. For each $\sigma \in \mathcal{P}$, the toric variety X_σ is an object of the category $\mathcal{C}_{T_M}$. The proof is by induction on the number of elements of $\mathcal{P}$. The base case is when $\mathcal{P}$ has only one maximal element σ. Then $Z = X_\sigma$ exists and is an object of $\mathcal{C}_{T_M}$. Suppose now that the inverse limit exists, and is an object in $\mathcal{C}_{T_M}$, for any such poset with fewer elements than $\mathcal{P}$ whose elements are toric varieties with dense torus T_M, and whose maps are proper birational toric morphisms. Fix a maximal element σ of $\mathcal{P}$, and let $\mathcal{P}' = \mathcal{P} \setminus \sigma$, and let $\mathcal{P}''$ be the subposet of all $\tau \in \mathcal{P}$ with $\tau \prec \sigma$. Let $Z_{\mathcal{P}'}$ and $Z_{\mathcal{P}''}$ be the corresponding inverse limits. Define

$$Z' := Z_{\mathcal{P}'} \times_{Z_{\mathcal{P}''}} X_\sigma.$$

Lemma 3.8 and induction show that Z' is an object in $\mathcal{C}_{T_M}$. It remains to show that Z' is the inverse limit (3.2) in the category of $\Bbbk$-schemes. Let Y be any $\Bbbk$-scheme with maps f_τ to each X_τ for $\tau \in \mathcal{P}$ that commute appropriately with the maps in $\mathcal{P}$. By the universal property of the inverse limits $Z_{\mathcal{P}'}$ and $Z_{\mathcal{P}''}$ we get a unique map from $Z_{\mathcal{P}'}$ to $Z_{\mathcal{P}''}$ that commutes with the maps in $\mathcal{P}$, and a unique map from Y to $Z_{\mathcal{P}'}$ and $Z_{\mathcal{P}''}$ whose compositions with the maps from $Z_{\mathcal{P}'}$ to $Z_{\mathcal{P}''}$ and from X_σ to $Z_{\mathcal{P}''}$ agree. Then the universal property of fiber products gives a unique map ϕ from Y to Z' such that the composition of ϕ with the maps from Z' to each X_τ equals the corresponding f_τ, so Z' is the inverse limit. $\square$

Definition 3.10. The *toric Chow quotient* of the toric variety X_P by the subtorus T_L is the not-necessarily-normal toric variety $X_P/T_L := Z_0$ arising as the toric component of the inverse limit Z as in Proposition 3.9.

Corollary 3.12 below establishes that X_P/T_L coincides with Kapranov's definition of the Chow quotient of X_P by T_L whenever the latter is defined, i.e., when P is a polytope. The terminology *toric Chow quotient* was introduced by Haiman–Sturmfels [HS04, §5] for the case $X_P = \mathbb{A}^n_\Bbbk$. First, we explain the link between the toric Chow quotient and fiber fans.

Proposition 3.11. *Let $\overline{X_P/T_L}$ be the normalization of the toric Chow quotient. Then $\overline{X_P/T_L}$ is the toric variety with fan $\mathcal{N}(P, \pi(P))$.*

Proof. Write $\overline{Z_0} = \overline{X_P/T_L}$. For each $\sigma \in \mathcal{P}$ there is a proper, birational toric morphism $\overline{Z_0} \to X_\sigma$, so the fan of $\overline{Z_0}$ refines that of X_σ by [Ful93, Proposition 2.4]. In particular, the fan of $\overline{Z_0}$ refines the common refinement $\mathcal{N}(P, \pi(P))$ of these fans.

The proposition follows once we show that $\mathcal{N}(P, \pi(P))$ refines the fan of $\overline{Z_0}$. It is enough to construct a proper, birational toric morphism $X_{CR} \to \overline{Z_0}$, where X_{CR} is the toric variety with fan $\mathcal{N}(P, \pi(P))$. Since $\mathcal{N}(P, \pi(P))$ refines the fan of each X_σ, there is a proper, birational toric morphism $X_{CR} \to X_\sigma$. The universal property of the inverse limit Z in the category $\mathcal{C}_{T_M}$ then gives a proper T_M-equivariant morphism $\phi\colon X_{CR} \to Z$ that is birational onto Z_0. Since X_{CR} is normal, the universal property of normalization gives a morphism $\overline{\phi}\colon X_{CR} \to \overline{Z_0}$ that is both proper (since ϕ is proper and the normalization is separated) and birational (since ϕ is birational onto Z_0 and the normalization is finite). It remains to show that $\overline{\phi}$ is a toric morphism.

Since ϕ is a dominant birational morphism of not-necessarily-normal toric varieties, it is given locally by an inclusion of subsemigroups $S \to S'$ of M, where $\operatorname{Spec} \Bbbk[S]$ is a chart on Z_0, $\operatorname{Spec} \Bbbk[S']$ is a chart on X_{CR} and $T_M = \operatorname{Spec} \Bbbk[M]$. Note that S' is saturated since X_{CR} is normal. The normalization $\overline{Z_0} \to Z_0$ is given locally by the inclusion of S in its saturation S_{sat}. Since S_{sat} is the smallest saturated subsemigroup of M, the inclusion $S \hookrightarrow S'$ factors through S_{sat}. The induced semigroup morphism $S_{\text{sat}} \to S'$ gives the local description of the induced map $\overline{\phi}\colon X_{CR} \to \overline{Z_0}$, so $\overline{\phi}$ is a toric morphism. $\square$

Corollary 3.12. *Let P be a polytope. Then the toric Chow quotient X_P/T_L is equal to the Chow quotient in the sense of Kapranov* [Kap93, Section 0.1].

Proof. Write $\mathrm{Chow}(X_P/T_L)$ for the Chow quotient. For a polytope P, Kapranov-Sturmfels-Zelevinsky [KSZ92, Proposition 2.3] proved that the normalization of the Chow quotient is the toric variety defined by the fan $\mathcal{N}(P, \pi(P))$, so X_P/T_L and $\mathrm{Chow}(X_P/T_L)$ share the same normalization. To see that they coincide, recall from [KSZ92, Corollary 4.3] that $\mathrm{Chow}(X_P/T_L)$ is a subvariety of the inverse limit Z defined in equation (3.2). Since $\mathrm{Chow}(X_P/T_L)$ is irreducible and contains the dense torus T_Z, it is a subvariety of the component $Z_0 = X_P/T_L$ of Z. If this inclusion $\mathrm{Chow}(X_P/T_L) \subseteq X_P/T_L$ were strict, $\mathrm{Chow}(X_P/T_L)$ and X_P/T_L could not share the same normalization, hence $\mathrm{Chow}(X_P/T_L) = X_P/T_L$ as claimed. $\square$

Remark 3.13. The statement of Corollary 3.12 was known to the authors of [KSZ92, Section 4], though they did not supply a proof. These same authors also observed that the inverse limit Z constructed in Proposition 3.9 may have more than one irreducible component. If P is the positive orthant, then the fiber fan $\mathcal{N}(P, \pi(P))$ is the secondary fan of the configuration $\{\pi(\mathbf{e}_i) : 1 \leq i \leq n\}$, and there will be more than one irreducible component whenever this configuration has a *non-regular* (or *non-coherent*) triangulation. For an example, see Billera–Filliman–Sturmfels [BFS90, Example 2.4].

3.3. **Proof of the main theorems.** We now prove the main theorems. Recall that the fan of a not-necessarily-normal toric variety, such as the toric Chow quotient, is the fan of its normalization.

Theorem 3.14. *Let $P \subseteq \mathbb{Q}^n$ be an n-dimensional polyhedron and let $\pi\colon P \to Q$ be a surjective linear projection of polyhedra. Set $P' := (\widetilde{P})^\vee$ and $L' := \mathrm{im}_{\mathbb{Z}}(\widetilde{\pi}^\vee)$. The cone over the GIT chamber decomposition arising from the T_L-action on X_P is the fan of the toric Chow quotient $X_{P'}/T_{L'}$.*

Proof. The fan defining the toric Chow quotient $X_{P'}/T_{L'}$ is $\mathcal{N}(P', \widetilde{\pi}^\vee(P'))$ by Proposition 3.11. Now Proposition 2.18 implies that the fan defining the toric Chow quotient $X_{P'}/T_{L'}$ is equal to the cone over the subdivision of Q induced by $\pi(P)$. The result follows from Remark 3.6, since the subdivision of Q induced by $\pi(P)$ is the GIT chamber decomposition arising from the T_L-action on X_P. $\square$

The case when P is a polyhedral cone (so X_P is affine) is of particular interest. For example, Cox [Cox95] showed that every simplicial, projective toric variety can be constructed as a GIT quotient of affine space by a subtorus. In the affine case, Theorem 3.14 can be strengthened as follows:

Theorem 3.15 (Duality for affine quotients)**.** *Let $P \subseteq \mathbb{Q}^n$ be an n-dimensional polyhedral cone and $\pi\colon P \to Q$ a surjective linear projection. Set $P' := P^\vee$ and $L' := \mathrm{im}_{\mathbb{Z}}(\pi^\vee)$. Then*

(i) *the fan of the toric Chow quotient X_P/T_L is equal to the GIT chamber decomposition arising from the action of $T_{L'}$ on $X_{P'}$; and*

(ii) *the GIT chamber decomposition arising from the T_L-action on X_P is the fan of the toric Chow quotient $X_{P'}/T_{L'}$.*

Proof. For part (i), note that the fan of X_P/T_L is the fiber fan $\mathcal{N}(P, \pi(P))$ by Proposition 3.11. Corollary 2.13 shows that $\mathcal{N}(P, \pi(P))$ equals the fan induced by $\pi^\vee(P^\vee)$. This is precisely the GIT chamber decomposition arising from the action of $T_{L'}$ on $X_{P^\vee}$ according by Remark 3.6.

For part (ii), note that the fan of the toric Chow quotient $X_{P'}/T_{L'}$ is the fiber fan $\mathcal{N}(P^\vee, \pi^\vee(P^\vee))$. This equals $(\pi^\vee)^\vee((P^\vee)^\vee) = \pi(P)$ by applying Corollary 2.13 with $C = P^\vee$ and $\pi^\vee$ playing the role of π. $\square$

Remark 3.16. If X is a nonnormal quasiprojective toric variety then one can give similar combinatorial descriptions of both GIT quotients by subtori $X /\!\!/_{\mathbf{v}} T$, and of the normalization of the toric Chow quotient; normality played no essential role in either construction. However the description of the GIT chamber complex in Definition 3.5 is now a priori only a coarsening of the true GIT chamber complex, as the quotient toric varieties are no longer determined solely by their fans.

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