MATH 216B HOMEWORK 1

SPRING 2004

(1) Show that every face τ of a polyhedral cone σ is a face of some facet. Conclude that every face is contained in some chain $\tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau \subsetneq \cdots \subsetneq \tau_k = \sigma$.

(2) Show that if $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subseteq \mathbb{Z}^n$ generate $\operatorname{span}(\mathbf{u}_1, \dots, \mathbf{u}_k) \cap \mathbb{Z}^n$ as an additive group, then there is $M \in SL(n, \mathbb{Z})$ such that $M\mathbf{u}_i = \mathbf{e}_i$, where $\mathbf{e}_i$ is the standard basis vector.

(3) Let $\sigma = \operatorname{pos}((0,1),(4,-1))$. Write $\mathbb{C}[S_\sigma]$ as R/I, where R is a polynomial ring and I is an ideal.

(4) Repeat the previous question for

$$\sigma = \operatorname{pos}((-6,1,1),(1,-1,0),(1,0,-1),(-2,-1,3),(-3,1,0)).$$

You will (probably) need to use software to compute this. (Warning: I haven't told you how to do the last step yet with software).

(5) Recall that $I_\sigma = \ker(\phi : \mathbb{C}[x_1, \dots, x_k] \to \mathbb{C}[S_\sigma] = \mathbb{C}[t^{u_1}, \dots, t^{u_k}]$. Show that $I_\sigma = \langle x^u - x^v : \phi(x^u) = \phi(x^v)$.

(6) Show that $V(I_\sigma) \cap (\mathbb{C}^*)^k$ is isomorphic to $(\mathbb{C}^*)^n$.

(7) List (with justification) the orbits of the torus action for the example of Question 2, thought of as a subvariety of $\mathbb{C}^5$. (Note the hint for Question 2 embedded in this question!)

(8) Prove that for the polytope $\operatorname{conv}(0, \mathbf{e}_1, \dots, \mathbf{e}_n) \subseteq \mathbb{R}^n$ we have $X_P = \mathbb{P}^n$.