MIDTERM SOLUTIONS

MATH 171, SPRING 2003

(1) Let the limit of the convergent sequence x_n be x. Since x_n converges to x, given $\epsilon > 0$ we can find N such that if $n \geq N$, $|x_n - x| < \epsilon/2$. Since $A_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$, we can find $m(n) \geq n$ such that $0 \leq A_n - x_{m(n)} < \epsilon/2$. Thus for $n \geq N$, $|A_n - x| \leq |A_n - x_{m(n)}| + |x_{m(n)} - x| < \epsilon/2 + \epsilon/2 = \epsilon$, and so A_n converges to x. Similarly, we can find $m'(n) \geq n$ such that $0 \leq x_{m'(n)} - B_n < \epsilon/2$, so for $n \geq N$, $|B_n - x| \leq |B_n - x_{m'(n)}| + |x_{m'(n)} - x| < \epsilon$, so B_n converges to x.

(2) We first show that $\mathrm{bd}(A \cup B) \subseteq \mathrm{bd}(A) \cup \mathrm{bd}(B)$. Let $x \in \mathrm{bd}(A \cup B)$. Then for every $\epsilon > 0$ the ball $D(x, \epsilon)$ contains a point in $A \cup B$ and a point in $M \setminus (A \cup B)$. Suppose there is some ϵ' for which $D(x, \epsilon')$ contains no points of A. Then for every $\epsilon \leq \epsilon'$, and thus for every $\epsilon > 0$, $D(x, \epsilon)$ contains a point of B and a point of $M \setminus B \supseteq M \setminus (A \cup B)$, so $x \in \mathrm{bd}(B)$. Otherwise for every $\epsilon > 0$ the ball $D(x, \epsilon)$ contains a point of A and a point of $M \setminus A \supseteq M \setminus (A \cup B)$, and thus $x \in \mathrm{bd}(B)$. So in either case $x \in \mathrm{bd}(A) \cup \mathrm{bd}(B)$.

We now show that $\mathrm{bd}(A) \cup \mathrm{bd}(B) \subseteq \mathrm{bd}(A \cup B) \cup A \cup B$. Let $x \in \mathrm{bd}(A) \cup \mathrm{bd}(B)$. If $x \in A$ or $x \in B$ then $x \in \mathrm{bd}(A \cup B) \cup A \cup B$, so we may assume that $x \in M \setminus (A \cup B)$. Let $\epsilon > 0$ be given. If $x \in \mathrm{bd}(A)$ then $D(x, \epsilon)$ contains a point of A, while if $x \in \mathrm{bd}(B)$ then $D(x, \epsilon)$ contains a point of B. Thus for all $\epsilon > 0$ the ball $D(x, \epsilon)$ contains a point of $A \cup B$, and a point $(x!)$ of $M \subseteq (A \cup B)$, so $x \in \mathrm{bd}(A \cup B)$. Thus $\mathrm{bd}(A) \cup \mathrm{bd}(B) \subseteq \mathrm{bd}(A \cup B) \cup A \cup B$.

To show that both inclusions can be proper, consider $A = [0, 2]$, and $B = [1, 3]$. Then $\mathrm{bd}(A \cup B) = \{0, 3\} \subsetneq \mathrm{bd}(A) \cup \mathrm{bd}(B) = \{0, 1, 2, 3\} \subsetneq \mathrm{bd}(A \cup B) \cup A \cup B = [0, 3]$.

(3) (a) Let x_k be a Cauchy sequence in M. To show that M is complete we need to show that x_k converges to a point of M. We consider two cases:

- Case I: $\{x_n\}$ is finite. Let $\epsilon = \min\{d(x_i, x_j) : x_i \neq x_j\}$. This minimum exists, as the set is finite. We have $\epsilon > 0$ since $x_i \neq x_j$ means $d(x_i, x_j) \neq 0$. Since x_n is Cauchy, there exists $N > 0$ such that $k, l \geq N$ implies that $d(x_k, x_l) < \epsilon$. By the construction of ϵ, for $k \geq N$ we must have $x_k = x_N$, so x_n converges to x_N.
- Case II: $\{x_n\}$ is infinite. Then by the hypothesis it has an accumulation point x. Since x_n is Cauchy, given $\epsilon > 0$ there exists $N > 0$ such that $k, l \geq N$ implies $d(x_k, x_l) < \epsilon/2$. Let $\epsilon' = \min\{\epsilon/2, d(x, x_i) : 1 \leq i \leq N-1\}$. Since x is an accumulation point for $\{x_n\}$, there exists an m such that $x_m \in D(x, \epsilon') \cap \{x_n\}$. By the construction of ϵ', we know $m \geq N$. Now by the triangle inequality, for $k \geq N$,

$$\begin{aligned} d(x, x_k) &\leq d(x, x_m) + d(x_m, x_k) \\ &\leq \epsilon' + \epsilon/2 \\ &\leq \epsilon, \end{aligned}$$

so x_n converges to x.

In both cases we have shown that x_n converges, so it follows that every Cauchy sequence in M converges, and thus M is complete.

(b) Let $M = \mathbb{Z}$, and let d be the discrete metric. We showed in class that any set with the discrete metric is complete (as every Cauchy sequence is eventually constant). Let $x_n = n$. Then $A = \{x_n\}$ is an infinite set in a complete metric space, but A has no accumulation points, as no set has any accumulation points in the discrete metric.

(4) (a) We first show that $d_\infty(x,y) \leq d_2(x,y) \leq \sqrt{2} d_\infty x, y(x,y)$. Indeed, $d_\infty(x,y)^2 = (\max_{i=1,2}(|x_i - y_i|))^2 \leq (x_1 - y_1)^2 + (x_2 - y_2)^2 = d_2(x,y)^2$, so since $d_\infty(x,y)$ and $d_2(x,y)$ are both nonnegative, we have $d_\infty(x,y) \leq d_2(x,y)$. Also $d_2(x,y)^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq 2(\max_{i=1,2}|x_i - y_i|)^2 = 2d_\infty(x,y)^2$, so $d_2(x,y) \leq \sqrt{2} d_\infty(x,y)$.
We now show that d_2 and d_∞ are equivalent metrics. Let U be any open set in the d_∞ metric. Then for any $x \in U$ there exists $\epsilon > 0$ such that $D_\infty(x,\epsilon) \subseteq U$. If $y \in D_2(x,\epsilon)$ then $d_2(x,y) < \epsilon$, so $d_\infty(x,y) < d_2(x,y) < \epsilon$, and so $y \in D_\infty(x,y)$. Thus $D_2(x,\epsilon) \subseteq U$, which shows that U is open in the d_2 topology, and thus d_2 gives a stronger topology than d_∞.
For the other direction, let U be an open set in the d_2 metric. Then for any $x \in U$ there exists $\epsilon > 0$ for which $D_2(x,\epsilon) \subseteq U$. Now since $d_2(x,y) \leq \sqrt{2} d_\infty(x,y)$, $D_\infty(x, \epsilon/\sqrt{2}) \subseteq D_2(x,\epsilon)$, and thus $D_\infty(x,\epsilon/\sqrt{2}) \subseteq U$, so U is open in the d_∞ metric. Thus d_∞ gives a stronger topology than d_2.

(b) We first show that d_∞ gives a stronger topology than d_2. Let U be an open set in the d_2 metric. Then for all $f \in U$ there exists $\epsilon > 0$ such that $D_2(f,\epsilon) \subseteq U$. Let $g \in D_\infty(f,\epsilon)$. Then

$$\begin{aligned} d_2(f,g)^2 &= \int_0^1 (f(x) - g(x))^2 dx \\ &\leq \int_0^1 \sup_{x \in [0,1]} (f(x) - g(x))^2 dx \\ &= (\sup_{x \in [0,1]} (f(x) - g(x)))^2 dx \\ &= d_\infty(f,g)^2 \\ &< \epsilon^2, \end{aligned}$$

so $D_\infty(f,\epsilon) \subseteq D_2(f,\epsilon) \subseteq U$, and thus U is open in the d_∞ metric.
We finish by showing that there are open sets in the d_∞ metric which are not open in the d_2 metric, so d_2 does not give a stronger metric than d_∞. Consider the set $U = D_\infty(0,1)$, where 0 is the zero function on $[0,1]$. This is an open set in the d_∞ metric, and we will show that it is not an open set in the d_2 metric, by showing that for every $\epsilon > 0$ there is some point of $D_2(0,\epsilon)$ which does not lie in U. Given $\epsilon > 0$, choose n so that $1/\sqrt{2n+1} < \epsilon$, and let $f(x) = x^n$. Then $d_2(0,f) = 1/\sqrt{2n+1} < \epsilon$, so $f \in D_2(0,\epsilon)$. However $d_\infty(0,f) = 1$, so $f \notin U$, and thus U is not open in the d_2 metric.