HOMEWORK 6 SOLUTIONS

MATH 171, SPRING 2003

(1) **Section 4.4, problem 4** Let $g : [0,1] \to \mathbb{R}$, $g(t) = f(c(t))$. Then g is continuous and $[0,1]$ is compact, hence g assumes minimum and maximum values on $[0,1]$. But this is the same thing as f assumes minimum and maximum values on the curve.

(2) **Section 4.5, problem 3** Consider $g(x) = f(x) - x$. Then g is a continuous function on the connected set $[0,1]$ with $g(0) \geq 0$, and $g(1) \leq 0$, so by the Intermediate Value Theorem there exists $c \in [0,1]$ with $g(1) \leq g(c) = 0 \leq g(0)$. For that c we have $f(c) - c = 0$, so $f(c) = c$.

(3) **Section 4.6, problem 3** No. Consider $f(x) = \sin(x^2)$. If f were uniformly continuous then we could find a $\delta > 0$ for which $|y - x| < \delta$ meant that $|f(y) - f(x)| < 1$. Suppose that this is the case, and choose x with $\sin(x^2) = 0$ for which the closest y with $\sin(y^2) = \pm 1$ has $|y - x| < \delta$. But then $f(y) - f(x) = 1$, contradicting our choice of δ. Choosing $x > \pi/4\delta$ suffices, as then $(x + \delta)^2 - x^2 > \pi/2$, so there is a y with $y^2 = x^2 + \pi/2$ and $y - x < \delta$.

(4) **Section 4.6, problem 7**

   (a) We showed in the first question that $\sqrt{x}$ is a continuous function, so since $[0,1]$ is closed and bounded and thus compact we know that it is uniformly continuous.

   (b) The function f does not have bounded derivative on $(0,1]$, and is not differentiable at 0. So it is necessary to have bounded derivative to be uniformly continuous. However having bounded derivative does guarantee that the function is uniformly continuous.

(5) **Section 4.7, problem 1** Let $f(x) = \sum_{i=1}^{n} |x - x_i|$. Then f is the sum of continuous functions, so is continuous. Let $g_i(x) = |x - x_i|$, so $f(x) = \sum_{i=1}^{n} g_i(x)$. Note that $g_i(x) = x - x_i$ for $x > x_i$, and $g_i(x) = x_i - x$ for $x < x_i$, so $g(x)$ is differentiable for $x \neq x_i$. However for all $n > 0$ $(g_i(x_i - 1/n) - g_i(x_i))/(-1/n) = -1$ while $(g_i(x_i + 1/n) - g_i(x_i))/(1/n) = 1$,

so $\lim_{h\to 0}(g_i(x_i+h)-g_i(x_i))/h$ does not exist, and so g_i is not differentiable at x.

Since the sum of differentiable functions is differentiable, it follows that f is differentiable for $x \neq x_i$ for any i. If f were differentiable at x_i for some i, then $f - \sum_{j\neq i} g_j$, which is the sum of functions which are all differentiable at x_i, would be differentiable at x_i. But this function is g_i, which we just showed is not differentiable at x_i. So f fails to be differentiable exactly at the points x_i.

(6) **Section 5.1, problem 2 Section 5.1, problem 2:** The seqence f_n doesn't converge uniformly. In fact f_n converges pointwise to the function $f(x)$ given by :

$$f(x) = x \; for \; x < 1 \; and \; f(1) = 0$$

This function $f(x)$ is not continuous at $x = 1$, so the convergence can't be uniform (see Proposition 5.1.4).

(7) **Chapter 5 exercises, problem 20**

(a) (Picture omitted).

(b) Set $g_k(x) = (1/4^{k-1})g(4^{k-1}x)$. Note that $|g_k(x)| \leq 1/(2\cdot 4^{k-1})$ and $\sum_{k=1}^{\infty} 1/(2\cdot 4^{k-1})$ converges to $2/3$, so $f(x) = \sum_{k=1}^{\infty}$ converges uniformly by the Weierstrass M-test. Since the space of bounded continuous functions is complete (See section 5.5), we know that f is continuous.

(c) Let $f_k = \sum_{n=1}^{k} g_n(x)$. Fix a point $x \in \mathbb{R}$. We will show that f is not differentiable at x. We first define two sequences which converge to x. On any bounded interval $g_k(x)$ is differentiable except at a finite number of points. Let x_k be the greatest such point less than or equal to x, and let y_k be the least point greater than x. For example, if $x = 1/3$, $x_1 = 0$, $y_1 = 1/2$, $x_2 = 1/4$, $y_2 = 3/8$, …Note that $0 \leq x - x_k < 1/(2\cdot 4^{k-1})$, $0 < y_k - x \leq 1/(2\cdot 4^{k-1})$, and $y_k - x_k = 1/(2\cdot 4^{k-1})$.

Note also that $g_l(x_k) = g_l(y_k) = 0$ for $l > k$, so $f_k(x_k) = f(x_k)$ and $f_k(y_k) = f(y_k)$.

If $l \leq k$ then x_k, x_{k+1}, y_k, and y_{k+1} all lie on the same linear "branch" of $g_l(x)$, so $(g_l(x_k) - g_l(y_k))/(x_k - y_k) = (g_l(x_{k+1}) - g_l(x_{k+1}))/(x_{k+1} - y_{k+1})$. This ratio is either $+1$ or -1, depending on which slope of the branch of g_l. Set $h_k = (f_k(x_k) - f_k(y_k))/(x_k - y_k)$. We just showed that $|h_{k+1} - h_k| = |(g_{k+1}(x_{k+1}) - g_{k+1}(y_{k+1}))/(x_{k+1} - y_{k+1})| =$

1, so the sequence h_k is not Cauchy so does not converge. Note that by the previous paragraph $h_k = (f(x_k) - f(y_k))/(x_k - y_k)$.
To complete the proof we need the following lemma:

Lemma 1. *If a function $h : \mathbb{R} \to \mathbb{R}$ is differentiable at y, then given $\epsilon > 0$ there is $\delta > 0$ for which if $x < y < z$ with $y - x, z - y < \delta$ then*

$$|\frac{h(z) - h(x)}{z - x} - h'(y)| < \epsilon.$$

Proof. Since $h'(y)$ exists, given $\epsilon > 0$ there is a $\delta > 0$ for which if $0 < |y' - y| < \delta$, then

$$|\frac{h(y') - h(y)}{y' - y} - h'(y)| < \epsilon,$$

so if $y' > y$ we have $h'(y)(y'-y) - \epsilon(y'-y) < h(y') - h(y) < h'(y)(y' - y) + \epsilon(y' - y)$. If $y' < y$ after rearranging we get $h'(y)(y - y') - \epsilon(y - y') < h(y) - h(y') < h'(y)(y - y') + \epsilon(y - y')$. Substituting in $y' = z$ to the first equation and $y' = x$ into the second, and adding

$$h'(y)(z - x) - \epsilon(z - x) < h(z) - h(x) < h'(y)(z - x) + \epsilon(z - x),$$

since $z - x = (z - y) + (y - x)$. Dividing through by $z - x$, which is positive, gives the desired result. □

Now suppose that f is differentiable at x. Given $\epsilon > 0$, pick δ as in the lemma, and chose N such that $1/(2 \cdot 4^{k-1}) < \delta$ for $k > N$. Then the lemma says that $|h_k - f'(x)| < \epsilon$ for $k > N$. This shows that h_k converges to $f'(x)$, which contradicts our assertion that it is not even Cauchy. From this we conclude that f is not differentiable at x.
The idea of this proof is due to Alessandro Magnani.

(8) **Chapter 5 exercises, problem 23** No. Let $f(x) = 1$ for $x \neq 0$, and let $f(0) = 2$. Then $f \circ f(x) = 1$ for all $x \in \mathbb{R}$, so $f \circ f$ is continuous, but f is not continuous.