MATH 171 - HOMEWORK 2 SOLUTIONS

We first show that $\mathbb{R}$ has the *Archimedean property*: For any $x \in \mathbb{R}$, there is an integer n with $n > x$. To see this, note that the Dedekind cut corresponding to x is not all of $\mathbb{Q}$, so there exists $q \in \mathbb{Q}$ with $q \notin x$. We may assume $q > 0$ (by taking $q = 1$ if our first choice of q is non-positive). Write $q = a/b$, with $a, b \in \mathbb{N}$. Then $2a > q$, so $x \subsetneq \overline{2a}$, and thus $2a$ is an integer greater than x.

(1) (a) For $r = 1/2$, $d_0 = 0$, $d_1 = 4$, and $d_k = 9$ for $k \geq 2$. For $r = -1/3$, $d_0 = -1$, and $d_k = 6$ for $k \geq 1$.

(b) To construct each d_k we just need to find the largest integer less than a given number, so we need only show that this always exists. In other words, we show that if s is any real number, there is a largest integer less than s.
If s is positive, let S be the set of all natural numbers greater than s. We know that S is nonempty because of the Archimedean property. Since $\mathbb{N}$ is well ordered, S has a smallest element, d. Because $s > 0$, we know that $d-1 \geq 0$, which means that $d - 1 < s$. We now have that $d - 1$ is the greatest integer less than s. If s is negative, let S be the set of all natural numbers greater than $-s$. Again, S is nonempty, and contains a smallest element d. Note that $-d < s$, and $-d + 1 > s$, so $-d$ is the greatest integer less than s.

(c) Let $r'_k = r - r_k$. Note that $r'_k = r - r_k = r - r_{k-1} - d_k/10^k = r'_{k-1} - d_k/10^k$, and d_k is chosen so that $d_k < 10^k r'_{k-1}$. This means that $r'_k > 0$ for all $k \geq 0$. This in turn means that $d_{k+1} \geq 0$, as it is the largest integer less than a strictly positive number. Now suppose that $d_k \geq 10$ for some $k \geq 1$. This means that $10 < 10^k r'_{k-1}$, so $10^{k-1}(r - r_{k-1}) > 1$. Now $10^{k-1}(r - r_{k-1}) = 10^{k-1}(r - r_{k-2} - d_{k-1}/10^{k-1}) = 10^{k-1} r'_{k-2} - d_{k-1}$, so the fact that this quantity is greater than one contradicts the choice of d_{k-1} as the greatest integer less than $10^{k-1} r'_{k-2}$. This means that $d_k < 10$ for all $k > 1$.

(d) Given $\epsilon > 0$, choose $N > 0$ such that $10^N > \lceil 1/\epsilon \rceil$. This is possible because $10^k > k$ for all $k > 0$ (check!), and the Archimedean property guarantees the existence of an integer greater than $\lceil 1/\epsilon \rceil$. Then if $k > N$, $r - r_k = r'_k < (d_{k+1}+1)/10^{k+1} < 10/10^{k+1} < 1/10^k < \epsilon$, so r_k converges to r.

(e) Let $s_k = \sum_{l=0}^{k} c_l/10^l$ (this was terrible notation to also call this r_k!) Then s_k is an increasing sequence, so to show that it converges it suffices to show that it is bounded above by c_0+1. Now $s_k \leq c_0 + \sum_{l=1}^{k} 9/10^l = c_0 + 1 - 1/10^k$, where the last inequality follows from the formula for summing a geometric series. This shows that the increasing sequence s_k is bounded above, and thus converges, since $\mathbb{R}$ is complete. The converse requires the following lemma:

Lemma 1. *If x_n converges to x, and there is some $M > 0$ such that $x_n \leq B$ for $n > M$, then $x \geq B$.*

Proof. Suppose that $x > B$, and set $\epsilon = B - x$. Then there is some $N > 0$ for which $|x - x_n| < \epsilon$ for all $n > N$. But for $n > \max(N, M)$, $|x - x_n| \geq |x - B| = \epsilon$, so N does not exist, and so we conclude that $x \geq B$.

□

Suppose that there is no $N > 0$ for which $c_l = 0$ for all $l > N$, so $s_k < r$ for all $k \geq 0$. We now show that $c_k = d_k$ for all k. The proof is by induction on k. Lemma 1 says that $s_k < c_0 + 1$ for all k. It now follows from the construction of d_0 that $d_0 = c_0$. Suppose that $d_{l-1} = c_{l-1}$ for some $l > 1$. Recall that d_k was chosen so that d_k was the largest integer less than $10^k(r - r_{k-1})$. If $c_k > d_k$, then $c_k > 10^k(r - r_{k-1}) = 10^k(r - s_{k-1})$, so $s_k = s_{k-1} + c_k/10^k > r$, which contradicts the fact that s_k is an increasing sequence converging to r. So we conclude that if $c_k \neq d_k$, we must have $c_k < d_k$. But then for $l > k$, we have $s_{k-1} + c_k/10^k + \sum_{j=k+1}^{l} 9/10^j < s_{k-1} + (c_k+1)/10^k \leq s_{k_1} + d_k/10^k = r_k$, where the first inequality again arises from summing the geometric series. Now Lemma 1 says that $r < r_k$, and thus $r < r_l$ for all $l \geq k$, since r_k is an increasing sequence. Let $x_k = r - r_k$. We just showed that $x_l < r - r_k < 0$ for $k \geq l$, so its limit should be less than or equal to $r - r_k$. But x_k converge to zero. From this contradiction we conclude that $c_k = d_k$, which completes the induction step.

(2) **Section 1.2, problem 3:** Let $\epsilon > 0$ be given. The Archimedean property says that we can choose an integer $N > (1 - \epsilon^2)/2\epsilon$. Then for $n > N$, we have $1 - \epsilon^2 < 2n\epsilon$, so $n^2 + 1 < (\epsilon + n)^2$.

Thus $\sqrt{n^2+1} < \epsilon + n$, and so $x_n < \epsilon$. As $x_n > 0$ for all n, this shows that x_n converges to 0.

(3) **Section 1.7, problem 1:** For the sup norm, $d(f,g) = \|f - g\| = \sup\{|f(x)| : x \in [0,1]\} = 1$.

For the norm of Example 1.7.7, $d(f,g) = \|f-g\| = \sqrt{\langle f-g, f-g\rangle} = \sqrt{\int_0^1 (1-x)^2 dx} = 1/\sqrt{3}$.

(4) **End of Chapter 1 exercises, problem 10:** Let d be metric on a set M, and define $\rho(x,y) = d(x,y)/(1+d(x,y))$. Then $d(x,y) \geq 0$ for all x and y, so $\rho(x,y)$ is the quotient of a nonnegative number by a strictly positive number, and is thus nonnegative. If $\rho(x,y) = 0$, then $d(x,y)/(1+d(x,y)) = 0$, so $d(x,y) = 0$. Next, note that $\rho(x,y) = d(x,y)/(1+d(x,y)) = d(y,x)/(1+d(y,x)) = \rho(y,x)$, since $d(x,y)$ is symmetric. Finally, we show that ρ satisfies the triangle inequality. Since d satisfies the triangle inequality, we have

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &\leq d(x,y) + d(y,z) + 2d(x,y)d(y,z) + d(x,y)d(x,z)d(y,z) \end{aligned}$$

since $d(u,v)$ is always nonnegative. Now adding the same thing to both sides we get

$$\begin{aligned} &d(x,z) + d(x,z)d(x,y) + d(x,z)d(y,z) + d(x,y)d(x,z)d(y,z) \\ &\leq d(x,y) + d(x,y)d(x,z) + d(x,y)d(y,z) + d(x,y)d(x,z)d(y,z) \\ &+d(y,z) + d(x,y)d(y,z) + d(y,z)d(x,z) + d(x,y)d(x,z)d(y,z), \end{aligned}$$

which means

$$\begin{aligned} &d(x,z)(1+d(x,y))(1+d(y,z)) \leq d(x,y)(1+d(x,z))(1+d(y,z)) \\ &+d(y,z)(1+d(x,z))(1+d(x,y)). \end{aligned}$$

The triangle inequality for $\rho(x,y)$ now follows from dividing both sides by $(1+d(x,y))(1+d(x,z))(1+d(y,z))$.