

# Math 171: ‘Writing in the Major’ Assignment

## 1. Overview

The aim of this assignment is to give you practice writing mathematical prose that is both readable and precise.

You are being asked to write a proof of an important theorem in analysis, the Arzèla-Ascoli Theorem. Below is the statement of the theorem to be proved, as well as an outline of the suggested method of proof. If you have any questions about these, please ask for help. The mathematical content of this assignment is not intended to be the primary challenge; it will be provided in order to allow you to concentrate on the writing.

There are several rounds of revision for this assignment. The first is to give a copy of your assignment to another student in the class. Part of your grade will be on the quality of the feedback you give to your partner in this phase. The papers will be then be turned in a first time to me, at which point, comments and suggestions for improvement will be made. After the papers are returned, you will be required to resubmit *edited or rewritten* versions. Editing is an integral part of the writing process.

### Important dates:

Tuesday, May 13. First draft given to peer in class for review.

Thursday, May 15. Comments returned to peer.

Tuesday, May 20. Complete first draft due to me.

Tuesday, May 27. Assignments returned with comments.

Tuesday, June 3. Final rewritten versions due.

## 2. Parts of the paper

In your paper you must do the following things:

- Introduce the theorem by mentioning its context or subject matter and by describing its main result.
- Prepare the reader for the proof by describing succinctly the main idea or technique used in the proof.
- State the theorem explicitly.
- Prove the theorem not only correctly but *clearly*.
- Discuss a counterexample where only hypothesis (a) of the theorem fails and the conclusion fails. Discuss a different counterexample where only hypothesis (b) fails and the conclusion fails.

Your preparatory sections should *not* be identical to the actual statement and proof of the theorem. Their purpose is to convey the main ideas in words, using as few symbols

as possible. As an example, a prologue to a different theorem might include: “This theorem proves that irrational numbers exist by showing that the square root of 2 is an example of an irrational number. The proof assumes that  $\sqrt{2}$  is a fraction and uses facts about the parity of squared integers in order to derive a contradiction.”

It is also important to focus the reader’s attention on the key aspects of the *statement* of the theorem and explain why and how the result might be used, at least in a mathematical context. For example, one of the main points of this theorem is that it provides conditions for compactness of subsets of the space of continuous functions that, compared to the general conditions of completeness and total boundedness, are easier to check. (The Heine-Borel Theorem does this for  $\mathbf{R}^n$ .)

When you state the theorem you may use the exact statement found below, or you may replace it by any *equivalent* rephrasing you prefer. When you prove the theorem, you are encouraged to use the method of proof outlined below. You must fill in the details and the reasons. Use a paragraph form similar to that of proofs found in your textbook.

### 3. Stylistic guidelines for writing

Use full sentences. The only sentence fragments that are acceptable are the headings “Theorem.” and “Proof.” Words that indicate the logic, such as “therefore” and “which implies,” are good but you should write them out. Do *not* use symbols such as  $\therefore$  or  $\Rightarrow$ , or abbreviations such as “s.t.” or “iff.” In many cases, statements involving these can be read in a number of different ways, creating confusion. It is a good idea to read your paper out loud to see if the prose is understandable.

In the proof you may use other math notation, such as “ $x \in [a, b]$ .” Indeed, these are often more concise and clearer than wordy explanations. On the other hand, in the explanatory sections that precede the statement of the theorem, it is best to avoid symbols and technicalities and rather to rely on ordinary words that convey the underlying ideas. Throughout the paper, avoid strings of equalities or inequalities on one line. Remember that the goal of this assignment is to present an argument that is accurate and understandable; readability is necessary for both. You are *strongly* encouraged to type your paper. Math majors will want to use this as an excuse to learn L<sup>A</sup>T<sub>E</sub>X, and sample files will be available on the webpage shortly.

Your intended audience for this paper should *not* be a grader who already knows the material and needs only to be minimally convinced that you know it too. Instead, pretend that you are writing this for a colleague in Math 171 who has asked you for help with this theorem.

### 4. Preliminaries to the Theorem

Recall the following definitions and results from class. Let  $I = [0, 1]$  and define  $\mathcal{C}(I) = \{f : f \text{ is a continuous real-valued function on } I\}$ . Then  $(\mathcal{C}(I), d)$  is a metric space, when equipped with the sup metric  $d$ , given by  $d(f, g) \equiv \|f - g\| = \sup\{|f(x) - g(x)| : x \in I\}$

## 5. Statement of the Theorem

**Theorem 5.1** *Let  $\Phi$  be a subset of the space,  $\mathcal{C}(I)$ , of continuous real-valued functions on  $I = [0, 1]$ , equipped with the sup metric. Suppose that*

(a) *there is some  $L > 0$  such that  $|\phi(x)| \leq L$  for all  $x \in I$  and all  $\phi \in \Phi$*

*and*

(b) *for every  $\epsilon > 0$  there is a  $\delta > 0$  such that  $|\phi(x) - \phi(y)| < \epsilon$  for all  $\phi \in \Phi$  whenever  $|x - y| < \delta$ , and  $x, y \in I$ .*

*Then the closure of  $\Phi$  is compact. Equivalently, every sequence of functions in  $\Phi$  has a subsequence that converges in  $\mathcal{C}(I)$ .*

## 6. Outline of a Proof of the Theorem

You may use the following results without proving them here:

- (1) A subset of a metric space is sequentially compact if and only if it is compact.
- (2) A metric space is compact if and only if it is both complete and totally bounded.
- (3)  $\mathcal{C}(I)$  is a complete metric space.
- (4) A closed subset of a complete metric space is complete.

You *must* explain why the two conclusions of the theorem are equivalent. Use the facts above to show that the only remaining thing to prove here is that the closure of  $\Phi$  is a totally bounded subset of  $\mathcal{C}(I)$ . Prove that the closure of a set is totally bounded if the set itself is totally bounded; hence, it suffices to show that  $\Phi$  is a totally bounded.

To prove that  $\Phi$  is a totally bounded, suppose that  $\epsilon > 0$  is given. We need to cover  $\Phi$  by finitely many balls of radius  $\epsilon$ . Let  $L$  be as in hypothesis (a) of the theorem and let  $\delta > 0$  correspond to the given  $\epsilon$ , as in hypothesis (b).

Divide  $I$  into subintervals, each of length less than  $\delta$ , using finitely many points of subdivision:  $0 = x_0 < x_1 < \cdots < x_n = 1$ . Also divide  $[-L, L]$  into subintervals of length less than  $\epsilon$  using finitely many points of subdivision:  $-L = y_0 < y_1 < \cdots < y_p = L$ . Then the rectangle  $I \times [-L, L]$  is divided into  $np$  smaller rectangles, each with width less than  $\delta$  and height less than  $\epsilon$ .

Show that it is possible to associate to each  $\phi \in \Phi$  a continuous, piecewise linear function,  $y = \psi(x)$ , whose graph has vertices all of which are of the form  $(x_k, y_l)$  for some  $k \in \{1, \dots, n\}, l \in \{1, \dots, p\}$  and such that  $|\psi(x_k) - \phi(x_k)| < \epsilon$  for all  $k \in \{1, \dots, n\}$ .

Use hypothesis (b) and the triangle inequality to show that  $|\psi(x_k) - \psi(x_{k+1})| < 3\epsilon$  for all  $k \in \{1, \dots, n-1\}$ . Use the piecewise linearity of  $\psi$  to show that  $|\psi(x_k) - \psi(x)| < 3\epsilon$  whenever  $x_k \leq x \leq x_{k+1}$ . If  $x \in I$ , then let  $x_k$  be the subdivision point nearest  $x$  on the left. Use the triangle inequality and the preceding results to show that  $|\phi(x) - \psi(x)| < \epsilon + \epsilon + 3\epsilon$ . (You may be able to do better than this.)

Conclude that  $\Phi$  can be covered by balls of radius  $5\epsilon$  whose centers are such functions  $\psi$ . Show that there exist only finitely many such functions, leading to the desired result.