MATH 111 MIDTERM, WINTER 2004

DUE AT THE START OF CLASS ON FRIDAY, FEBRUARY 13

You may consult the text-book, your notes, and your old homework while attempting the exam. You may also use any computer system, as long as the use is acknowledged and relevant printouts are attached. You may not discuss the exam with anyone else during the exam period. Turning in the exam will be taken as agreement with these terms, and the terms of the Stanford Honor Code.

Show all working on all questions. Answers without full justification will receive little or no credit.

Throughout this exam k is an arbitrary field.

(1) Let $S = k[x]$, and let $I = \langle x^3 - 3x + 2, x^4 + 2x^3 - 4x^2 - 2x + 3 \rangle$.
  (a) Find a single polynomial $f \in S$ for which $I = \langle f \rangle$.
  (b) Does $x^2 + 4x - 5$ lie in I?
  (c) What is $V(I)$? $I(V(I))$?

(2) Let $S = k[x, y, z]$, and $I = \langle x^2, y^2, xy + yz \rangle$. Compute by hand the reduced Gröbner basis for I with respect to the reverse lexicographic order with $x > y > z$. Show all working.

(3) Let $S = k[x_1, \dots, x_n]$.
  (a) Show that a monomial ideal $I \subseteq S$ is radical if and only if it has a squarefree generating set (that is, one where each generator is not divisible by the square of any variable).
  (b) Show that if J is any ideal in S, and $\langle LT(J) \rangle$ is radical, then J is radical.
  (c) Is the converse true? Prove or give a counterexample. You may (or may not) want to experiment with Macaulay 2 for this. The command to compute the radical of an ideal `I` is `radical I`.

(4) In the homework you showed that we could form a term order $\prec_{w,\sigma}$ from a vector $w \in \mathbb{Z}^n$ and another term order $\prec_\sigma$ by setting $x^u \prec_{w,\sigma} x^v$ if $w \cdot u < w \cdot v$, or if $w \cdot u = w \cdot v$ and $x^u \prec_\sigma x^v$.
  (a) Show that there is no nonzero vector w, and term order σ for which $\prec_{w,\sigma}$ is the lexicographic term order.
  (b) Let $S = k[x, y, z]$, and let $I = \langle x^2 - 3y^5z + 4y^2z^3, y^6 - 8y^5z - 111y^2z^{10} + z^{12} \rangle$. Let $\prec_\sigma$ be the reverse lexicographic term order with $x > y > z$. Find a vector nonzero $w \in \mathbb{Z}^3$ for

which the reduced Gröbner basis with respect to the term order $\prec_{w,\sigma}$ is the same as the reduced Gröbner basis with respect to the lexicographic term order with $x > y > z$.

(c) Why do the two parts of this question not contradict each other?