MATH 111 HOMEWORK 4, WINTER 2004

DUE WEDNESDAY, FEBRUARY 18

(1) This exercise explains how we can use Gröbner bases to solve polynomial equations.

(a) Let $S = \mathbb{C}[x_1, \dots, x_n]$ have the lexicographic term order. Let I be an ideal in S. Show that if I contains any polynomials containing only powers of x_n, then there must be one in the reduced Gröbner basis for I.

(b) Let I be such that $V(I)$ is a finite set. Show that $I(V(I))$ must contain a polynomial only containing only powers of x_n.

(c) We will show later in the course that when we work over the complex numbers then $I(V(I))$ is the radical of I (this is the Hilbert Nullstellensatz). Assuming this, show that I contains a polynomial containing only powers of x_n.

(d) If we know a polynomial in I containing only powers of one variable, we can solve for the roots of this polynomial (symbolically or numerically), and use this to reduce to a simpler problem. Use this idea to solve the system of equations:

$$\begin{aligned} x^2 - 3xy + y^2 &= 0 \\ x^3 - 8x + 3y &= 0 \\ x^2y - 3x + y &= 0 \end{aligned}$$

Give your answer symbolically (that is, in terms of radicals).

(2) Let $S = k[a, b, c, d]$. Let $I = \langle c^5 - b^3d^2, a^2d - bc^2, a^2c^3 - b^4d, a^4c - b^5 \rangle$.

(a) Show that the given generating set is a Gröbner basis for I with respect to the reverse lexicographic order with $a > b > c > d$.

(b) Compute the lexicographic Gröbner basis for I with respect to this ordering of the variables.

(c) Find a term order for which $\langle c^{11}, bc^2, b^3d^2, b^4d, b^5, a^2b^2d^3, a^4bd^4\rangle$ is the lead term ideal of I. (Hint: What would the reduced Gröbner basis be?)

(3) CLO 2.5 #7
(4) CLO 2.5 #10
(5) CLO 2.6 #4
(6) CLO 2.6 #6
(7) CLO 2.7 #6
(8) CLO 2.7 #7