TCC: TROPICAL GEOMETRY

HW 6

These questions are designed to help you make sense of the lectures. You do not need to write up solutions to *all* of them to get credit for this module; check with me if you are unsure whether what you have done suffices.

These questions are for the last three lectures of the module.

(1) (For people who have not seen toric varieties before). Write down all torus orbits of $\mathbb{P}^3$ and $\mathbb{P}^2 \times \mathbb{P}^2$.

(2) Describe the tropical toric variety $\text{trop}(\mathbb{P}^1 \times \mathbb{P}^1)$.

(3) Consider the following hypersurfaces $Y \subseteq (\mathbb{C}^*)^n$. Let $\overline{Y}$ be the closure of Y in $\mathbb{P}^n$. For a hypersurface it suffices to take the defining equation f of I, multiply by the smallest monomial that makes f a polynomial (instead of a Laurent polynomial) with no monomial factors, and homogenize this with respect to x_0. The corresponding projective hypersurface is the projective closure of Y. Verify that $\overline{Y} \cap \mathcal{O}_\sigma \neq \emptyset$ if and only if $\text{trop}(Y) \cap \text{relint}(\sigma) \neq \emptyset$ for these varieties.
   (a) $f = x + y + x^2y + xy^2 + x^2y^2 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$,
   (b) $f = x^2y^{-1} + 3x + x^3 + 2x^2y - x^3y \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$,
   (c) $f = x + y + z + 1 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$,
   (d) $f = x - yz \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

(4) (For people who know something about toric varieties). Give an example to show that the tropical compactification $\overline{Y}$ depends on the choice of fan structure on $\text{trop}(Y)$. Hint: Think about blow-ups of toric varieties.

(5) This exercise will discuss tropicalizing hyperplane arrangement complements. Let $\mathcal{A} = \cup_{i=1}^{d} \{[x] \in \mathbb{P}^n : \mathbf{a}_i \cdot x = 0\}$, where $\mathbf{a}_i \in \mathbb{C}^{n+1}$, and let $Y = \mathbb{P}^n \setminus \mathcal{A}$. We embed Y into $(\mathbb{C}^*)^{d-1}$ by the map $x \mapsto [\mathbf{a}_1 \cdot x : \cdots : \mathbf{a}_d \cdot x]$. Write A for the $n+1 \times d$ matrix with columns $\mathbf{a}_i$. Show that $Y = V(\sum_j b_j x_j : (b_1, \ldots, b_d) \in \ker(A)) \subseteq (\mathbb{C}^*)^d / \mathbb{C}^*$.

(6) (Tropical Riemann Roch) If you have never seen the Riemann Roch theorem before, spend some time with an introduction to algebraic curves. One such reference is Fulton's *Algebraic Curves*, which is freely available from his webpage. If you have seen Riemann Roch before, look at one of the papers on tropical curves. For the Specialization Lemma see: arxiv.0701075. For tropical Riemann Roch one place to start is arxiv.0608360.

(7) (Enumerative Geometry) Choose five general enough points in the plane. Verify that there is a unique tropical rational curve of degree two passing through all five points. What is the "general enough" condition that makes this true? (Harder) Repeat this for eight points and degree three rational curves.

(8) (Another enumerative geometry question will be added after the lecture).