**TCC: TROPICAL GEOMETRY**

HW 5

These questions are designed to help you make sense of the lectures. You do not need to write up solutions to *all* of them to get credit for this module; check with me if you are unsure whether what you have done suffices.

(1) For each of the following linear varieties compute $\operatorname{trop}(X)$. For a maximal cone $\sigma \in \operatorname{trop}(X)$, and $w$ in the interior of this cone, compute $\operatorname{in}_w(I(X))$ and verify that the multiplicity is one and that $\operatorname{trop}(\operatorname{in}_w(I(X)) = \operatorname{star}_{\operatorname{trop}(X)}(\sigma)$. For a codimension-one cone in $\operatorname{trop}(X)$ check that $\operatorname{trop}(X)$ is balanced. (You need only do this for one or two examples of such cones).
   (a) $X = V(x_1 + x_2 + x_3 + x_4, x_1 + 2x_2 + 4x_3 - x_4) \subset (\mathbb{C}^*)^4$;
   (b) $X = V(x_1 + x_2 + x_3 + x_4 + x_5, x_1 - x_2 + 3x_3 + 4x_4 + 7x_5) \subset (\mathbb{C}^*)^5$;
   (c) $X = V(x_1 + x_2 + x_3 + x_4 + x_5, x_1 + x_2 + x_3 + 3x_4 - x_5) \subset (\mathbb{C}^*)^5$.

(2) Let $X \subset (\mathbb{C}^*)^n$ be defined by linear equations. Show that the multiplicity of each cone in $\operatorname{trop}(X)$ is one.

(3) Notice that the algorithm given to construct $\operatorname{trop}(X)$ does not always give the coarsest possible fan structure. For example, consider $X = V(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + 2x_3 + 3x_4 + 4x_5)$. The algorithm given gives a fan with 20 three-dimensional cones, but these can be pair-wise amalgamated to give a fan with 10 three-dimensional cones. Verify this statement. Can you give a general description of the coarsest possible fan structure on the tropicalization of a linear variety?

(4) A *circuit* of a linear space $\ker(A)$, where $A$ is a $(n-d)\times n$ matrix of rank $n-d$, is a vector in the row space of $A$ with minimal support (ie a $v \in \operatorname{row}(A)$ such that there is no $v' \in \operatorname{row}(A)$ with $\{i : v'_i \neq 0\} \subsetneq \{i : v_i \neq 0\}$. Show that there are only a finite number of circuits up to scaling. Let $I = \langle \sum_{j=1}^n a_{ij}x_j : 1 \leq i \leq n-d \rangle$. Show that the circuits of $A$ give a tropical basis for $\operatorname{trop}(V(I))$. For hints, see Chapter 4 of the book.

(5) Check with `gfan` that the description of $\operatorname{trop}(G^0(2,5))$ is correct.

(6) If you have not seen it before, check that the equations given for the Plücker embedding of the Grassmannian are correct. In other words, check that every point in the variety of the Plücker relations is the vector of minors of a $r \times n$ matrix of rank $r$. This is essentially an advanced exercise in linear algebra. For hints, one source is Hassett's *Introduction to Algebraic Geometry*.

(7) (Harder). Show that the tropical Grassmannian parameterizes tropical linear spaces (tropoicalizations of varieties cut out by linear equations).