TCC: TROPICAL GEOMETRY

HW 4

These questions are designed to help you make sense of the lectures. You do not need to write up solutions to *all* of them to get credit for this module; check with me if you are unsure whether what you have done suffices.

(1) Consider $X = V(x^2+t+1) \subseteq (\mathbb{C}\{\{t\}\}^*)$. Compute $\text{trop}(X)$. Calculate $y \in X$ with $\text{val}(y) = w$ for all $w \in \text{trop}(X)$. Hint: To really do this explicitly, you should look at the proof that the field of Puiseux series is algebraically closed; that algorithmically constructs a root for any algebraic equations term by term. You can check your answer with (for example) `Maple`, using the `puiseux` package.

(2) Go back to Q4 of HW1 and draw the tropical curves using the regular subdivision trick. Compare this with your previous answers.

(3) Let $X = V(x^2 + y^2 + x + y) \subseteq (K^*)^2$. Consider the change of coordinates $\phi : (K^*)^2 \to (K^*)^2$ given by $\phi(x) = xy$, $\phi(y) = x^2y^3$. Compute $\phi(X)$, and $\text{trop}(\phi(X))$. Observe that this has the predicted relationship with $\text{trop}(X)$.

(4) Show that if σ is a d-dimensional Γ-rational polyhedron in $\mathbb{R}^n$, then there is $A \in \text{GL}(n, \mathbb{Z})$ with $A\sigma \subset \text{span}(\mathbf{e}_1, \ldots, \mathbf{e}_d)$, where $\mathbf{e}_i$ is the ith standard basis vector. Conclude that if $X \subseteq (K^*)^n$ and $w \in \text{trop}(X)$ then there is a monomial change of coordinates $\phi : (K^*)^n \to (K^*)^n$ for which $\text{trop}(\phi)(w)$ is in the relative interior of a polyhedron of $\text{trop}(X)$ contained in $\text{span}(\mathbf{e}_1, \ldots, \mathbf{e}_d)$.

(5) The definition of the star of a polyhedron σ in a polyhedral complex Σ depended on the choice of $w \in \text{relint}(\sigma)$. Show that the fan $\text{star}_\Sigma(\sigma)$ is in fact independent of the choice of w.

(6) You now know the definitions to do Q2 and Q3 from HW2.

(7) (This is a follow-up on the introductory lecture, and a warm-up for the lecture in a few weeks). Show that there is a unique conic through five general points in $\mathbb{P}^2$. Here "general" means "outside a Zariski-closed subset of $(\mathbb{P}^2)^5$. Can you describe this subset?

Concretely, this means that if $u_1, \ldots, u_5 \in \mathbb{P}^2$ there is a unique (up to scaling) polynomial $F = a*x^2 + b*xy + c*y^2 + d*xz + e*yz + f*z^2$ with $F(u_i) = 0$ for $1 \leq i \leq 5$.

(8) Install `gfan` and do some examples. Compare the gfan output with the examples you have already computed. Caveats: gfan uses the max convention (so the answers will be the negative of what you computed) and requires the input be homogeneous (so to compute the tropical variety $\text{trop}(V(x+y+1))$ you need to homogenize, and ask gfan for $\text{trop}(V(x+y+z))$). Gfan is available from: `http://home.imf.au.dk/jensen/software/gfan/gfan.html`.