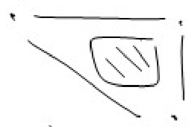
Last Time:

Toric varieties

Tropical toric varieties



$Y \subseteq X_\Sigma$, $Y \cap \mathcal{O}_\sigma \neq \emptyset$ if & only if $\operatorname{trop}(Y \cap T) \cap \operatorname{relint}(\sigma) \neq \emptyset$

torus orbit labelled by σ

$0 \in \operatorname{cl}(Y^\circ \subseteq \mathbb{A}^n)$ iff $\operatorname{trop}(Y^\circ) \cap \mathbb{R}^n_{>0} \neq \emptyset$

Tropical Compactifications

$Y^\circ \subseteq (\mathbb{C}^*)^n$ $\quad Y = \operatorname{cl}(Y^\circ \subseteq X_\Sigma)$

where $\operatorname{trop}(Y^\circ) = |\Sigma|$.

$Y \cap \mathcal{O}_\sigma$ is pure of codim $\dim(\sigma)$.

eg $M_{0,n}$ is the moduli space of smooth genus zero curves with n marked pts. = ways to arrange n distinct pts on $\mathbb{P}^1$ up to $\operatorname{Aut}(\mathbb{P}^1)$.

eg $M_{0,3} = pt$ (ex: There is a unique element of $\operatorname{Aut}(\mathbb{P}^1)$ taking any 3 distinct ordered pts to any other 3 distinct ordered pts

$M_{0,4} = \mathbb{P}^1 \setminus \{0,1,\infty\}$.

$M_{0,5} = (\mathbb{P}^1 \setminus \{0,1,\infty\})^2 \setminus \text{diag}$

$M_{0,n} = (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \text{diag}$

$$= \mathbb{P}^{n-3} \setminus \left\{ \begin{array}{ll} x_0 = 0 & x_i = x_0 \\ x_i = 0 & x_i = x_j \\ & 1 \le i < j \le n-3 \end{array} \right\}$$

$\binom{n-2}{1} + \binom{n-2}{2} = \binom{n-1}{2}$.

$M_{0,n} \hookrightarrow (\mathbb{C}^*)^{\binom{n-1}{2}-1}$

$x \longmapsto [x_i : \dots : x_i - x_j] \in (\mathbb{C}^*)^{\binom{n-1}{2}-1}$

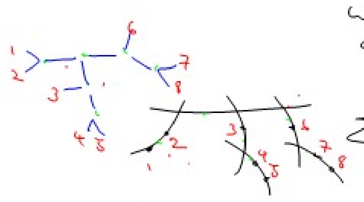
$[\underbrace{x_0 : x_1 : \dots : x_{n-3}}_{z_{-1i}} : x_1 - x_0 : x_2 - x_0 : \dots \; x_{n-3} - x_{n-4}]$

This is a closed subvariety:

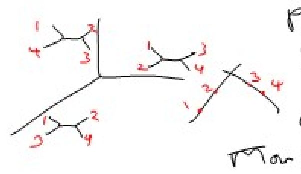
$M_{0,n} = V(z_{ij} - z_{-1j} + z_{-1i} : 0 \le i < j \le n-3)$

$\operatorname{trop}(M_{0,n})$ is an $(n-3)$-dim fan Σ in $\mathbb{R}^{\binom{n-1}{2}-1}$

The closure of $M_{0,n}$ in X_Σ is the Deligne-Mumford compactification $\overline{M}_{0,n}$, which is the moduli space of stable genus zero curves with n marked pts. = trees of $\mathbb{P}^1$'s with at least 3 nodes or marked pts on each branch.



$\Sigma = \operatorname{trop}(M_{0,n})$ is the space of phylogenetic trees. (cf $\operatorname{trop}(G^\circ(2,n))$)



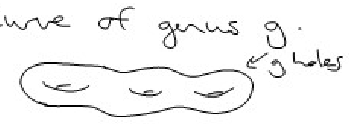
Cones in $\Sigma \leftrightarrow$ trees with n labelled leaves.

$\overline{M}_{0,n} \cap \mathcal{O}_\sigma$ = curves with dual graph ~~that~~ that tree

Tropical Curves + Riemann Roch

(References: Baker-Norine) 0608360
Baker 0701075)

Let $C \subseteq \mathbb{P}^n$ be a smooth projective curve of genus g.



A divisor on C is $D = \sum_{p \in C} a_p p$

$a_p \in \mathbb{Z}$
all but finitely many a_p are zero

The degree of D is $\sum_{p \in C} a_p$

D is effective ($D \geq 0$) if $a_p \geq 0$ $\forall p$

Let $K(C)$ be the function field of C. (if $C = V(I) \subseteq \mathbb{P}^n$,
$C \cap U_0 \subseteq \mathbb{A}^n$
$= V(I|_{x_0=1})$. Ring of fract. of $K[x_1, x_n] / I|_{x_0=1}$

eg $C = V(y^2z - x^3 + xz^2 - z^3)$
$C \cap U_z = \{(x,y) : y^2 = x^3 - x + 1\}$
$K(C) = K(x,y) / (y^2 = x^3 - x + 1)$

A rational function $f \in K(C)$ gives a divisor $\mathrm{div}(f)$ by

$$\mathrm{div}(f) = \sum_{p \in C} \underset{\substack{\uparrow \\ \text{order of vanishing of } f \text{ at } p}}{\mathrm{ord}_p(f)}$$

(valuation on $K(C)$.

D is linearly equivalent to D', $D \sim D'$ if $D' = D + \mathrm{div}(f)$ for some $f \in K(C)$.

$\sum a_p p$

$L(D) = \{ f \in K(C) : D + \mathrm{div}(f) \geqslant 0 \}$
ie. $a_p + \mathrm{ord}_p(f) \geqslant 0$ $\forall p$

Note: $L(D)$ is a vector space$/K$.
$\mathrm{ord}_p(f+g) \geqslant \min(\mathrm{ord}_p(f), \mathrm{ord}_p(g))$

$l(D) = \dim L(D)$. $r(D) = \dim \mathbb{P}(L(D))$
$= l(D) - 1$

One special divisor class is the canonical class.

Consider C as a complex manifold, and let ω be a meromorphic differential form (locally $\omega = \underset{\substack{\uparrow \\ \text{meromorphic fcn}}}{f}\, dz$)

$\mathrm{div}(\omega) \leftarrow$ locally $\mathrm{div}(f)$.

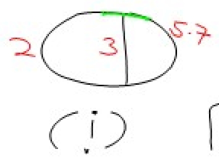
eg $\mathbb{P}^1$ chart 1: z $\quad \omega = dz$
chart 2: $u = \frac{1}{z}$ $\quad d\left(\frac{1}{u}\right) = \frac{-1}{u^2} du$
$\mathrm{div}(\omega) = -2\, pt(u=0)$.

$[K] = [\mathrm{div}(\omega)]$ for any meromorphic differential.
$\uparrow$ linear equiv class.

Theorem (Riemann-Roch).
$$l(D) - l(K-D) = \deg(D) + 1 - g$$
$l(D) \geqslant$

Tropical analogues

Defn A tropical abstract curve is a metric graph Γ.



ie a topological space determined by a graph G.

(!) $\quad \Gamma = \coprod_{e \in G} [0, \underset{\text{length}}{\ell(e)}] \amalg \coprod_{v \in G} pt \Big/ \sim$

glue ends of edges to vertices

The genus of Γ is

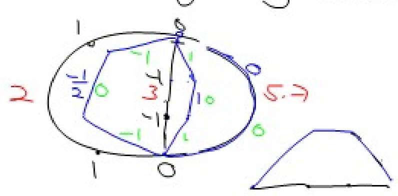
$\dim H^1(\Gamma, \mathbb{Q}) = \#\text{edges} - \#\text{vertices} + 1$

of underlying graph

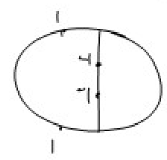
A divisor on Γ is $D = \sum_{p \in \Gamma} a_p p$ where $a_p \in \mathbb{Z}$ and is zero for all but finitely many p

$\deg(D) = \sum a_p$.

A tropical rational function on Γ is a piecewise linear function $\Gamma \to \mathbb{R}$ with integral slopes and finitely many break pts



$$\operatorname{div}(f) = \sum_{p \in \Gamma} \begin{pmatrix}\text{sum of}\\ \text{outgoing}\\ \text{slopes}\end{pmatrix} p$$



Two divisors D, D' are linearly equivalent if $D' = D + \operatorname{div}(f)$ for some tropical rational function f

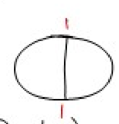
$r(D) := \max\{r : D - E \sim D' \geq 0$ for ~~all~~ all effective E with $\deg(E) = r\}$

(Note: This is a lemma classically – see Baker/Norine)

Note: If $D \geq 0$ $r(D) \geq 0$

$r(D) \geq 1 \Rightarrow \forall p \in \Gamma \ \exists D' \sim D$ with $D' = p + D''$, $D'' \geq 0$

$K = \sum_{p \in \Gamma} (\operatorname{val}(p) - 2) p$

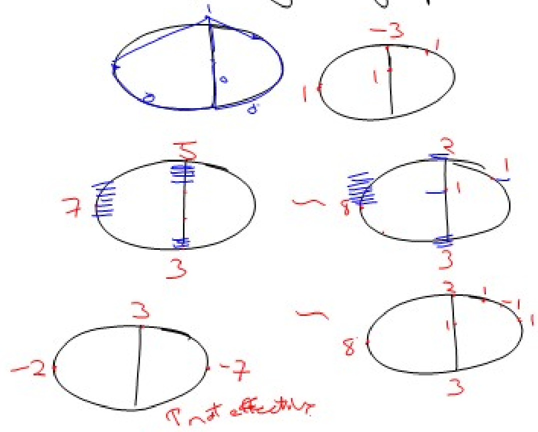


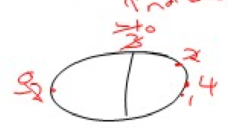
Theorem (tropical Riemann Roch).
(Baker–Norine, Gathmann–Kerber, Mikhalkin–Zharkov)

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

Pf is fairly straight forward (not geometric).

The linear equivalence relation can be formulated as "chip firing on graphs"





Chip firing predates tropical geometry!

Q/ What do the tropical calculations tell us about classical curves?

A/ Baker's Specialization Lemma.

Let C be a smooth projective curve $/K$

$C_{/K} \rightsquigarrow \mathcal{C}_{/R} \longrightarrow C_0{}_{/k}$ ← reducible curve 

R valuation ring of K

$\mathrm{Spec}(R) = \{0, m\}$

C_0 (fibre of $\mathcal{C}$ over m.)

G = dual graph of C.

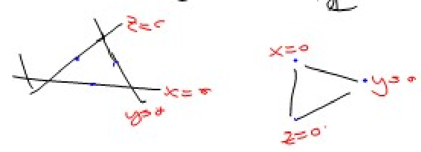
D a divisor on $C \longrightarrow \rho(D)$ a tropical divisor on G

$$\underset{\overset{\shortparallel}{l(D)-1}}{r(D)} \leqslant r(\rho(D))$$

eg $C = V(t(x^3+y^3+z^3)+xyz) \subseteq \mathbb{P}^2_{\mathbb{C}(t)}$

$\mathcal{C} = V(\quad \text{"} \quad) \subseteq \mathbb{P}^2_R$

$C_0 = V(xyz) \subseteq \mathbb{P}^2_{\mathbb{C}}$



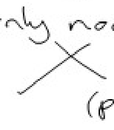
$C = V(I) \subseteq \mathbb{P}^n_K$

$I_R = I \cap R[x_0, x_n]$

$\mathcal{C} = \mathrm{Proj}(R[x_0, x_n]/I_R)$

$\mathrm{Spec}(R) = \{0, m\}$

Fibre over m is $V(\bar{I}_R) \subseteq \mathbb{P}^n_k$
$\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad V(\mathrm{in}_0(I))$

Assume: C_0 is reduced and has only nodes as singularities  "semistable" (possible by semistable reduction)

Take the dual graph G of C_0

vertices = irreducible components

edges = intersections

(Assume $\Gamma = \mathbb{Z}$ (R is a DVR)

Make G into a metric graph by setting $\ell(\text{edge joining } C^i, C^j) = \ell(C^i \cap C^j)$

C^i, C^j components of C

$\rightarrow$ all edges are length 1

ideal of a pt p.

Given $I_p \subseteq K[x_0, \; x_n]$,

look at $\dfrac{I_p \cap R[x_0, \; x_n]}{I_p \cap R[x_0, \; x_n]} \subseteq k[x_0, \; x_n]$

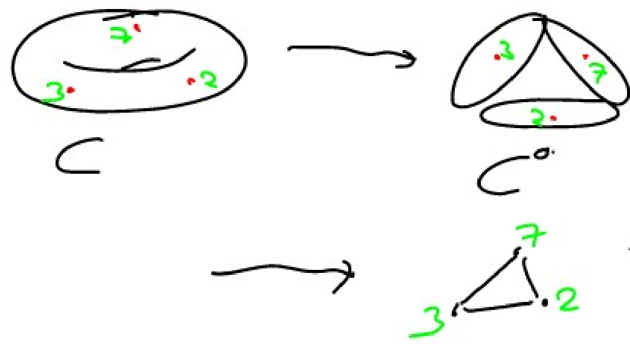
$\quad \overset{\shortparallel}{in_0}(I_p)$

So $p \in C \longrightarrow \rho(p) \in C_0$

a divisor D on $C \longrightarrow$ a divisor $\rho(D)$ on C_0

[assume that $\mathcal{C}/R$ is regular]

$\Rightarrow \rho(p)$ is a smooth pt on C_0



So $\rho(D)$ determines a tropical divisor on G.

Thm (Baker's specialization lemma)

$r(D) \leq r(\rho(D))$.