Last Time: Linear varieties

$$V\left(\sum_{j=1}^{n} a_{ij}x_j : 1 \le i \le n-d\right)$$

$\rightsquigarrow \hat{A} = a_{ij}$

& $\hat{B}$ $\quad AB^T = 0$

$(b_1 \; " \; b_n')$

Grassmannian

$\operatorname{Trop}(G^\circ(2,n))$ = space of phylogenetic trees

Today: Connections with toric varieties.

A (normal) toric variety is a normal variety X containing a dense copy of $T \cong (\mathbb{C}^*)^n$ and an action of T on X extending the action of T on itself.

Zariski topology

eg $X = (\mathbb{C}^*)^n$

eg $X = \mathbb{A}^n = \mathbb{C}^n \supseteq T = \{x : x_i \neq 0 \; \forall i\}$

eg $X = \mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim \quad \underline{v} \sim \lambda \underline{v}$

$\cup|$

$\{[x] : x_i \neq 0 \quad 0 \le i \le n\}$

$\cong (\mathbb{C}^*)^n$.

eg $\mathbb{P}^n \times \mathbb{P}^m$

eg $Bl_p(\mathbb{P}^2)$.

Goal for today:

We can tropicalize subvarieties of toric varieties.

A Toric variety is a finite union of T-orbits.

eg $\mathbb{P}^2 = (\mathbb{C}^*)^2 = \{[x_0 : x_1 : x_2] : x_i \neq 0\}$

$\cup \; \mathbb{C}^* = \{[0 : x_1 : x_2] : x_1, x_2 \neq 0\}$.

$\cup \; \mathbb{C}^* = \{[x_0 : 0 : x_2] : x_0, x_2 \neq 0\}$

$\cup \; \mathbb{C}^* = \{[x_0 : x_1 : 0] : x_0, x_1 \neq 0\}$

$\cup \; (\mathbb{C}^*)^0 = \{[1:0:0]\}$,

$\cup \; (\mathbb{C}^*)^0 \cup (\mathbb{C}^*)^0 \quad [0:1:0]$

$[0:0:1]$.

$(1 : 1 : \frac{x_2'}{x_2})$

$\cdot (0 : 1 : x_2)$

$= (0 : 1 : x_2')$

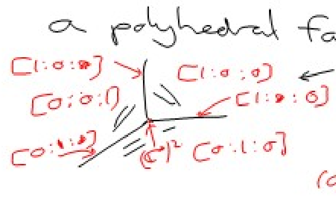
$(t_0 : t_1 : t_2) \cdot (x_0 : x_1 : x_2)$

$= (t_0x_0 : t_1x_1 : t_2x_2)$.

eg $\mathbb{A}^n = (\mathbb{C}^*)^n \amalg$ n copies of $(\mathbb{C}^*)^{n-1}$ $\amalg$ $\binom{n}{2}$ copies of $(\mathbb{C}^*)^{n-2}$

$\amalg \cdots \amalg \binom{n}{n} = 1$ pt. $\{0\}$.

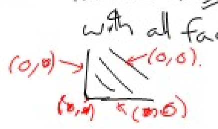
We record these orbits in a polyhedral fan.



$\leftarrow \mathbb{P}^2$

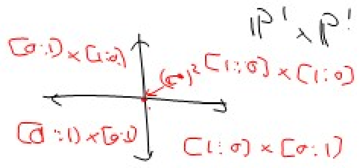
$\mathbb{A}^n$

fan is $\mathbb{R}^n_{\ge 0}$ with all faces.



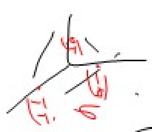
d dim cone $\sigma \in \Sigma$ $\longleftrightarrow$ codim d T-orbit $O_\sigma \subseteq X = X_\Sigma$

$\mathbb{P}^1 \times \mathbb{P}^1$



(More traditionally)

given a rational polyhedral fan Σ we can construct a toric variety X_Σ by constructing an affine toric variety U_σ for each $\sigma \in \Sigma$ and gluing according to the fan Σ.

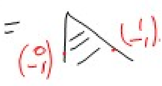


Specifically, given $\sigma \in \Sigma$

$\sigma^\vee = \{ u \in \mathbb{R}^n : u \cdot v \geq 0 \ \forall v \in \sigma \}$

$S_\sigma = k[x^u : u \in \sigma^\vee \cap \mathbb{Z}^n]$ — f.g by Gordan's lemma

U_σ is the affine variety with coordinate ring S_σ.

eg $\sigma^\vee =$  $S_\sigma = \mathbb{C}[y^{-1}, xy^{-1}]$ is $\mathbb{C}[z_1, z_2]$.

Cox, Little, Schenck, Fulton. $U_\sigma = \mathbb{A}^2$.

Tropicalizing subvarieties of toric varieties

Let $Y \subseteq \mathbb{A}^n$, $Y = V(I)$, $I \subseteq \mathbb{C}[x_1, x_n]$

Pick $K \supset \mathbb{C}$ with a nontrivial valuation eg $\mathbb{C}\{\{t\}\}$.

$Y(K) = \{ y \in \mathbb{A}^n_K : f(y) = 0 \ \forall f \in I \otimes K \}$, $I \otimes K \subseteq K[x_1, x_n]$

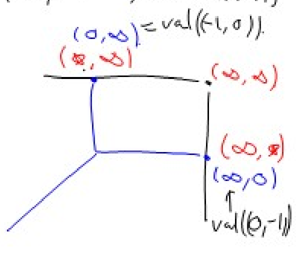
For $y \in \mathbb{A}^n_K$, $\mathrm{val}(y) \in (\underbrace{\mathbb{R} \cup \infty}_{\overline{\mathbb{R}}})^n = \overline{\mathbb{R}}^n$.

We'll give $\overline{\mathbb{R}}$ the topology with sub basis $\{(a,b) : a, b \in \mathbb{R}\} \cup \{ (a, \infty] : a \in \mathbb{R} \}$

Defn $\mathrm{trop}(Y) = \mathrm{cl}(\mathrm{val}(Y(K)))$

eg $Y = V(x+y+1) \subseteq \mathbb{A}^2$
$= \{ (a, -1-a) : a \in \mathbb{C} \}$

$K = \mathbb{C}\{\{t\}\}$. $Y(K) = \{ (a, -1-a) : a \in \mathbb{C}\{\{t\}\} \}$
$\ni (t, -1-t)$

$\mathrm{trop}(Y) =$ 

In general, we can write

$$X_\Sigma = \coprod_{\sigma \in \Sigma} \mathcal{O}_\sigma \leftarrow \text{torus orbits}$$

$$\mathrm{trop}(X_\Sigma) = \coprod_{\sigma \in \Sigma} \mathrm{trop}(\mathcal{O}_\sigma)$$

$$= \coprod_{\sigma \in \Sigma} \mathbb{R}^{n - \dim(\sigma)}$$

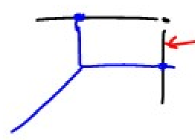
eg $\mathbb{A}^2 = (\mathbb{C}^*)^2 \sqcup \mathbb{C}^* \sqcup \mathbb{C}^* \sqcup \overset{(0,0)}{\downarrow}{pt}$

$\mathrm{trop}(\mathbb{A}^2) = \mathbb{R}^2 \sqcup \mathbb{R} \sqcup \mathbb{R} \sqcup pt$

$$\mathrm{trop}(Y) = \coprod \mathrm{trop}(Y \cap \mathcal{O}_\sigma)$$

$$\subseteq \mathrm{trop}(X_\Sigma).$$

eg $V(x+y+1)$



$\leftarrow \mathrm{trop}(\{(0,a) : a \neq 0\})$

$Y \cap \mathcal{O}_\sigma = V(x+y+1, x)$

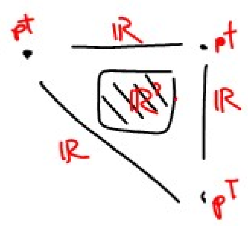
$\subseteq 0 \times \mathbb{C}^*$

$= \{(0,-1)\}.$

eg $\mathbb{P}^2 = (\mathbb{C}^3 \setminus \underline{0}) / \sim \quad \underline{v} \sim \lambda \underline{v}$

$\mathrm{trop}(\mathbb{P}^2) = (\mathbb{R}^3 \setminus (\infty, \infty, \infty)) / \sim$

$\underline{v} \sim (\lambda, \lambda, \lambda) + \underline{v}.$

$=$ 

eg $\mathrm{trop}(\mathbb{A}^2) = \mathbb{R}^2 \sqcup \mathbb{R} \sqcup \mathbb{R} \sqcup pt$

$= (\mathbb{R})^2$

In general, we can construct $\mathrm{trop}(X_\Sigma)$ as $\bigcup \mathrm{trop}(U_\sigma)$ (glue by Σ).

this gives a topology on $\mathrm{trop}(X_\Sigma)$.

Q, Given $Y \subseteq X_\Sigma$

which T-orbits does Y intersect?

eg $Y \subseteq \mathbb{P}^n$, which coordinate subspaces does Y intersect?

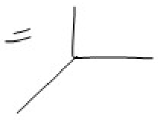
A, Theorem [Tevelev]

Y intersects the torus orbit $\mathcal{O}_\sigma$ if and only if

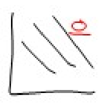
$\operatorname{trop}(Y \cap T) \cap \operatorname{relint}(\sigma) \neq \emptyset$.

eg $Y = V(x_0 + x_1 + x_2) \subseteq \mathbb{P}^2$

$Y \cap T = V(x_1 + x_2 + 1) \subseteq (\mathbb{C}^*)^2$

$\operatorname{trop}(Y \cap T) =$ 

<u>Prop</u> Let $Y^\circ \subseteq (\mathbb{C}^*)^n$ and let Y be the closure of Y° in $\mathbb{A}^n$. Then $0 \in Y$ if and only if $\operatorname{trop}(Y) \cap \mathbb{R}^n_{>0} \neq \emptyset$



"Pf": Let $I = I(Y) \subseteq \mathbb{C}[x_1, \ldots, x_n]$.

If: If $0 \notin Y$, then $\exists f \in I$ with $f = 1 + g$ for $g \in \langle x_1, \ldots, x_n \rangle$

Note: $f \in I(Y^\circ)$

But for $\omega \in \mathbb{R}^n_{>0}$, $\operatorname{in}_\omega(f) = 1$ so $\omega \notin \operatorname{trop}(Y^\circ)$.

"Only if" (Sketch) Suppose $0 \in Y$.

Then $\dim(Y) > 0$ (otherwise $Y = Y^\circ$)

We proceed by induction on $\dim(Y)$

We may assume Y° is irreducible.

If $\dim(Y^\circ) > 1$, pick a general linear form h, and let

$Y^{\circ new} = Y^\circ \cap V(h)$.

$Y^{new} = \operatorname{cl}(Y^{\circ new} \subseteq \mathbb{A}^n)$.

Then $0 \in Y^{new}$ & it suffices to show $\operatorname{trop}(Y^{\circ new}) \cap \mathbb{R}^n_{>0} \neq \emptyset$.

We may thus assume $\dim(Y^\circ) = 1$.

Let J be the integral closure I. $J \supseteq I$, and $V(J)$ is smooth,

$x^{j} + a_1 x^{j-1} + \cdots + a_j$ $\quad a_i \in I^i$

Then $R = \left(\mathbb{C}[x_1, \ldots, x_n] / J \right)_{\langle x_1, \ldots, x_n \rangle}$

So $R \subseteq \underset{\substack{\uparrow \\ \text{completion}}}{\hat{R}} \cong \mathbb{C}[[t]]$

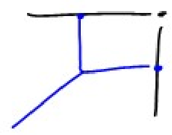
Let $p_i \in \mathbb{C}[[t]]$ be the image of x_i $\quad \in \langle t \rangle$

Then $(p_1, \ldots, p_n) \in Y^\circ(\overline{\mathbb{C}((t))})$ $\quad \uparrow$ Laurent series

$(\operatorname{val}(p_1), \ldots, \operatorname{val}(p_n)) \in \mathbb{R}^n_{>0}$ $\cap \operatorname{trop}(Y^\circ)$

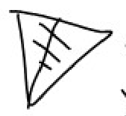
Cor Let $Y \subseteq X_\Sigma$.

then $\mathrm{trop}(Y) \subseteq \mathrm{trop}(X_\Sigma)$
is the closure of $\mathrm{trop}(Y \cap T)$ in $\mathrm{trop}(X_\Sigma)$.
$\mathrm{trop}(Y \cap T) \subseteq \mathbb{R}^n$



Tropical Compactifications

Given $Y^\circ \subseteq (\mathbb{C}^*)^n$, choose a fan structure Σ on $\mathrm{trop}(Y^\circ) \subseteq \mathbb{R}^n$



Consider the Toric variety X_Σ.

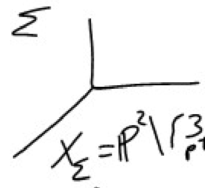
$$Y^\circ \subseteq (\mathbb{C}^*)^n \subseteq X_\Sigma.$$

Let Y be the closure of Y° in X_Σ.
Y is a tropical compactifation of Y°.

Theorem (Tevelev, Hacking).
The tropical compactification Y is a complete variety (projective)
& $Y \cap O_\sigma$ is pure of codim $\dim(\sigma)$.

eg $Y^\circ = V(x+y+1)$.

Σ



$X_\Sigma = \mathbb{P}^2 \setminus \{3 \text{ pts}\}$

$\mathrm{cl}(Y^\circ \subset X_\Sigma)$
$= V(x+y+z) \subseteq \mathbb{P}^2$.

(Warning X_Σ is not complete)

eg rays of $\Sigma \longrightarrow$ codim-one loci of Y. (divisors)

If the fan structure is finer then Y is nicer.

For $\sigma \in \Sigma$ with $\dim(\sigma) = \dim(Y)$
$Y \cap O_\sigma$ is zero-dim, and
Y is CM at these pts,
& $\ell(Y \cap O_\sigma) = \mathrm{mult}(\sigma)$.

One "nice" fan structure comes from the Gröbner fan.

$$Y \cap O_\sigma = V(\mathrm{in}_w(I)) / (\mathbb{C}^*)^{\dim(\sigma)}$$
$w \in \mathrm{relint}(\sigma)$