Last time:

X irred d-dim

trop(X) pure of dim d.

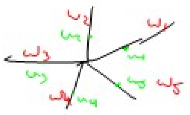
multiplicity: For σ d-dim polyhedron $w \in$ relint(σ), change coords so

$\mathrm{in}_w(I) \underset{\text{gens}}{\overset{\text{has}}{}} k[x_{d+1}^{\pm 1}, x_n^{\pm}]$

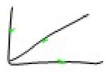
$J = \mathrm{in}_w(I) \cap k[x_{d+1}, x_n]$

$\mathrm{mult}(\sigma) = \dim_k k[x_{d+1}, x_n]/J$

Thm Trop(X) is balanced.



$\sum w_i u_i = \underline{0}$

eg  is not trop(X) for any X

Gfan eg:

TCC/gfaneg.in
.st_
.on

gfan_tropicalstartingcone
gfan_tropicaltraverse

Warnings: · gfan input must be homogeneous
· gfan uses max.

Linear varieties

eg $I = \langle x_1+x_2+x_3+x_4, x_2+2x_3+3x_4)$
$\subseteq \mathbb{C}[x_1^{\pm 1}, , x_4^{\pm 1}]$.
What is trop$(V(I))$?

In general $I = \langle \sum_{j=1}^{n} a_{ij} x_j : 1 \le i \le n-d)$

Then $V(I)$ is the intersection of a d-dim subspace with $(\mathbb{C}^*)^d$.
What is trop$(V(I))$?

Recipe: Write $A = (a_{ij}) \leftarrow (n-d) \times n$ matrix.

- Choose a $d \times n$ matrix B so

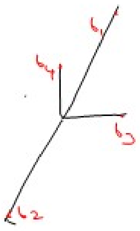
$$0 \rightarrow \mathbb{Z}^d \xrightarrow{B^T} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^{n-d} \rightarrow 0$$

$V(I) = \underset{\text{space}}{\text{row}}(B) \cap (\mathbb{C}^*)^n$

- Write $\underline{b}_1, , \underline{b}_n$ for the columns of B.

eg $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

$B = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{pmatrix}$

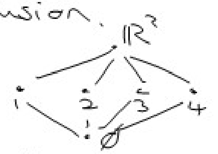


Note that: the $\underline{b}_i$ are defined up to $GL(d)$.

· If $\sum \alpha_j \underline{b}_j = 0$ then $\sum \alpha_j x_j \in I$
$(AB^T = 0)$

Let $\mathcal{L}$ be the lattice of flats of $\mathcal{B}$. This is the poset (partially ordered set) with elements $\mathrm{span}(b_i : i \in I) \leq \mathbb{R}^d$ for $I \subseteq \{1, \ldots, n\}$, ordered by inclusion.

eg



$\mathcal{L}$ is a lattice in the poset sense.
(reference: Stanley EC vol 1)

Defn The sets $\{\{i : b_i \in V\} : V \in \mathcal{L}\}$ are the flats of the matroid of $\mathcal{B}$.

A matroid is a combinatorial structure with several equivalent definitions.

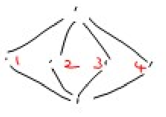
eg Axioms for flats:
$\mathcal{F} \subseteq 2^{\{1, \ldots, n\}}$ s.t.
1) $\{1, \ldots, n\} \in \mathcal{F}$
2) If $S, T \in \mathcal{F}$, $S \cap T \in \mathcal{F}$
3) If $S \in \mathcal{F}$, then $\{T \setminus S : \overset{T \in \mathcal{F}}{S \subseteq T}, \nexists\, U \in \mathcal{F}$ with $S \subsetneq U \subsetneq T\}$ partitions $\{1, \ldots, n\} \setminus S$

One definition of a matroid is a collection of subsets $\mathcal{F}$ satisfying these axioms.
Not all such sets come from some set $\{b_1, \ldots, b_n\}$ — such matroid are called representable.

- To each subspace V in $\mathcal{L}$ we associate the vector $e_V = \sum_{b_i \in V} e_i \in \mathbb{R}^n$ ← ith std basis vector.

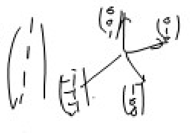
- "Construct the order complex of $\mathcal{L}$"
trop(V(I)) has a cone for each chain
$\emptyset \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{d-1} \subseteq V_d = \mathbb{R}^d$
The cone is $\mathrm{pos}(e_{V_1}, \ldots, e_{V_{d-1}}) + \mathrm{span}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

eg



$\mathrm{trop}(V(I)) = \bigcup_{i=1}^{4} \underline{\mathrm{pos}}(e_i) + \mathrm{span}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

(mod $\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$)

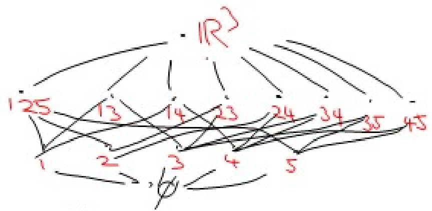


$\mathrm{pos}(\underline{u}_1, \ldots, \underline{u}_s) = \{\sum_{i=1}^{s} \lambda_i \underline{u}_i : \lambda_i \geq 0\}$ ← positive hull

eg $I = \langle x_1+x_2+x_3+x_4+x_5,$
$x_2+2x_3+2x_4+3x_5 \rangle$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 1 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}$$



trop(V(I)) has cones

$$\begin{pmatrix}4\\3\\0\\0\\3\end{pmatrix}, \begin{pmatrix}7\\6\\3\\3\\6\end{pmatrix} \in \text{pos}(e_1, e_1+e_2+e_5) + \text{span}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix},$$

$$\text{pos}(e_1, e_1+e_3) + \text{span}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix},$$

…… + 15 more 3-dim cones

Note: trop(V(I)) only depended on $\mathcal{L}$. (and really on $\mathcal{J} \subseteq 2^{[n]}$.

Details of why recipe works in Ch 4 of book draft.

- Show if $v \notin$ fan Δ constructed above we can find
  $x_j - \Sigma \alpha_i x_i$ with
  $\alpha_i \neq 0 \Rightarrow v_i > v_j$,
  so $\text{in}_v($ " $) = x_j$.
- Show if $v \in \Delta \cap \mathbb{Q}^n$,
  $\exists\, y \in (\mathbb{C}\{\{t\}\}^*)^n$ with
  $y \in V(I)$ and $\text{val}(y) = v$.

Grassmannians

The Grassmannian $G(r,n)$ parameterizes all r dim subspaces of K^n.

The Plücker embedding of $G(r,n)$ embeds it into $\mathbb{P}^{\binom{n}{r}-1}$

For an r-dim subspace V of K^n choose a basis $\underline{a}_1, \ldots \underline{a}_r$, and write these vectors as the rows of an $r \times n$ matrix

$$A_V = \begin{pmatrix} - a_1 - \\ - a_r - \end{pmatrix}$$

From A_V form the vector of $r \times r$ minors $\in K^{\binom{n}{r}}$.

eg $A_V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ $V \in G(2,4)$

$$\searrow \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 1 & 2 & 3 & 1 & 2 & 1 \end{pmatrix}$$

Choosing a different basis for V multiplies this vector by a nonzero scalar, so we get a well-defined map $G(r,n) \longrightarrow \mathbb{P}^{\binom{n}{r}-1}$

eg $A_V = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$

$$\downarrow$$

$$\begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \\ (1 & c & d & -a & -b & ad-bc) \end{array}$$

Note: $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$

$G(2,4) = V(\quad " \quad) \subseteq \mathbb{P}^5$

In general the vectors of minors satisfy the eqtns

$P_{I,J}$ for $I, J \subseteq \{1, \ldots n\}$
$|I| = r-1 \; |J| = r+1$

$$P_{I,J} = \sum_{j \in J} (-1)^{\text{sign}(j,I)} x_{I \cup j} x_{J \setminus j}$$

$\text{sign}(j, I) = \# \underset{j < i}{i \in I}$

eg $r=2, n=4$ $I = 1, J = \{2,3,4\}$

The Plücker ideal is

$I_{r,n} = \langle P_{I,J} : I, J \subseteq \{1, \ldots n\}, \; |I| = r-1, |J| = r+1 \rangle$

Thm $G(r,n) = V(I_{r,n}) \subseteq \mathbb{P}^{\binom{n}{r}-1}$

Let $T = \{x \in \mathbb{P}^{\binom{n}{r}-1} : x_I \neq 0 \; \forall I\}$

$G^0(r,n) = G(r,n) \cap T$

Q/ What can we say about $\text{trop}(G^0(r,n))$?

(aside: Not always a good question to say "$Y \subseteq \mathbb{P}^n$, what is $\text{trop}(\underline{Y \cap T^n})$?" For most cases the answer is boring.

Note: $\text{trop}(G^0(r,n))$

$$\subseteq \bigcap_{I,J} \text{trop}(P_{I,J})$$

Thm (Speyer-Sturmfels)

When $r = 2$ we have equality & $\text{trop}(G^0(2,n))$ is the space of phylogenetic trees

Aside: When $|J \setminus I| = 3$

$$P_{I,J} = \sum_{j \in J \setminus I} (-1)^{\bullet} x_{I \cup j} x_{J \setminus j}$$

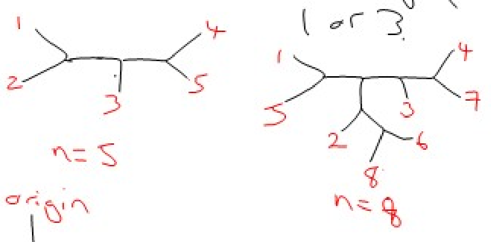
has 3 terms.

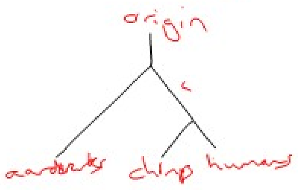
$$\underbrace{\bigcap_{|J \setminus I| = 3} \text{trop}(P_{I,J})}_{\text{"Dressian"}} \underset{\substack{\uparrow \\ \text{often proper}}}{\supseteq} \text{trop}(G^0(r,n))$$

Defn A phylogenetic tree is a trivalent tree with n labelled leaves.

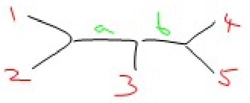
every vertex has degree 1 or 3.

eg





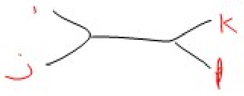
If we give lengths to the internal edges we get a "distance" on $\{1, \dots, n\}$.



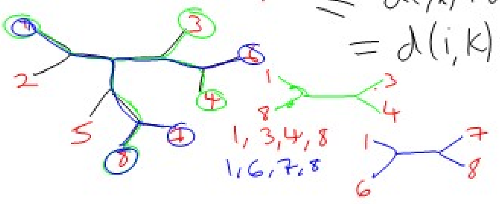
$d(1,2)=0$
$d(1,3)=d(2,3)=a$
$d(1,4)=d(2,5)=$
$=a+b$
$d(3,5)=b$

(also allow negative lengths on the leaves)

Note: For all i,j,k,l, up to reordering we have



$$d(i,j)+d(k,l) \leq d(i,l)+d(j,k) = d(i,k)+d(j,l).$$ ✡



$I_{2,n} = V(x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} : 1 \leq i < j < k < l \leq n$

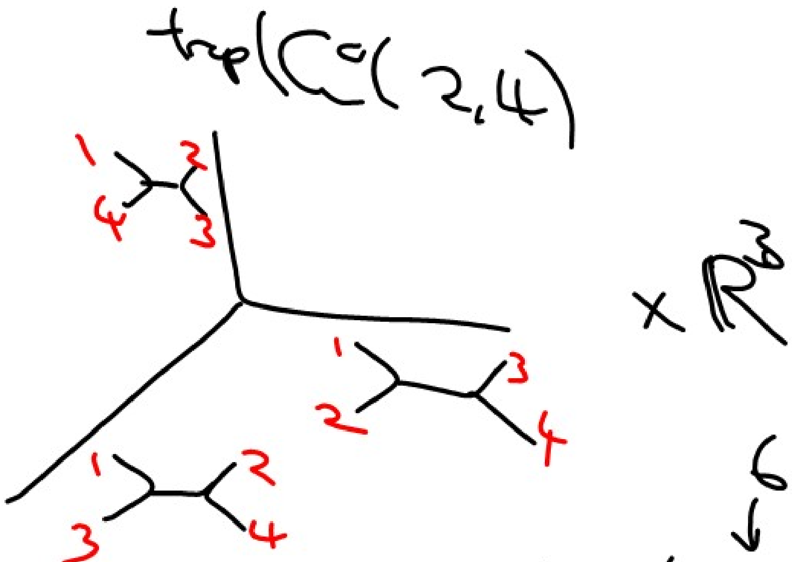
$\operatorname{trop}(\text{"}) = \min(x_{ij}+x_{kl}, x_{ik}+x_{jl}, x_{il}+x_{jk}).$ $x_{ij} \leftrightarrow -d(i,j)$

Thm (Buneman 4pt condition) $\{ d(i,j) \}$ is the distance vector a tree if & only if it satisfies the 4pt condition ✡

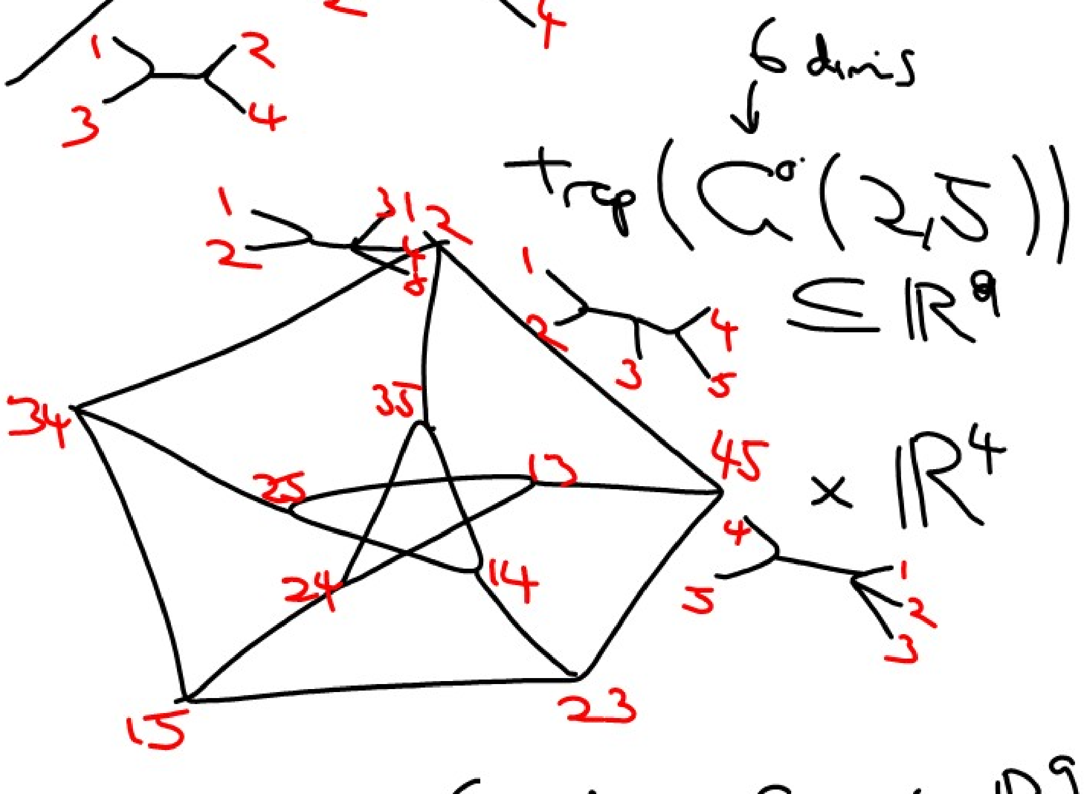
Since the $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}$ form a tropical basis, $w \in \operatorname{trop}(G^0(2,n))$ if & only if $\min(w_{ij}+w_{kl}, w_{il}+w_{jk}, w_{ik}+w_{jl})$ is achieved at least twice, so if & only if $-w$ is the distance vector for a tree.

Note: phylogenetic trees are (abstractly) tropicalizations of lines in $\mathbb{P}^{n-1}$. Concretely, given a line $L \subseteq \mathbb{P}^{n-1}$ $\operatorname{trop}(L \cap T^{n-1})$ is a tree with n leaves.

eg



eg



To draw a 6-dim fan in $\mathbb{R}^9$ with 4-dim lineality space, quotient by the lineality space to get a 2-dim fan in $\mathbb{R}^5$, intersect with the 4-sphere to get a graph.