Last time:

Hypersurface case of fundamental theorem

($\omega \in \mathrm{trop}(X) \cap \Gamma^n \Rightarrow \exists y \in X$ with $\mathrm{val}(y) = \omega$)

Changing coordinates:

$\varphi : (K^*)^n \longrightarrow (K^*)^n \quad \varphi^*(x_i) = x^{\underline{a}_i}$, $\underline{a}_i \in \mathbb{Z}^n$

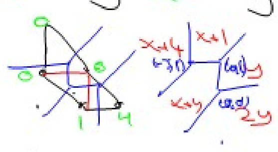
$\mathrm{trop}(\varphi(X)) = \underbrace{\mathrm{trop}(\varphi)}_{A}(\mathrm{trop}(X))$

Drawing tropical plane curves | $\varphi : (K^*)^n \to (K^*)^n$, $\mathrm{trop}\,\varphi : \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$

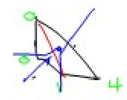
Defn The regular subdivision of $\mathrm{Newt}(f)$ wrt $(\mathrm{val}(c_u))$ is the projection to $\mathbb{R}^n$ of the lower faces of $\mathrm{conv}\left(\begin{pmatrix} u \\ \mathrm{val}(c_u) \end{pmatrix} : c_u \neq 0\right)$

($f = \sum c_u x^u$)

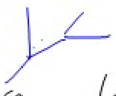
$f = 16x^2 + y^2 + 2x + y + xy$ 2-adic val



$\leftarrow \mathrm{trop}(V(f))$

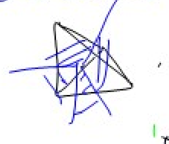


Prop $\mathrm{trop}(V(f)) =$

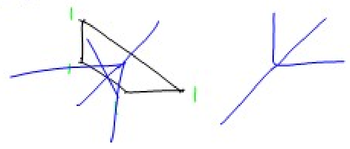


$\{\omega \in \mathbb{R}^n : \mathrm{face}_{(\omega,1)}\left(\mathrm{conv}\begin{pmatrix} u \\ \mathrm{val}(c_u) \end{pmatrix}\right)$ is not a vertex $\}$

1 vertex
4 rays
6 2-dim cones



= projection to $\mathbb{R}^2$ of the codim-one faces of the normal fan of $\mathrm{conv}(\ )$



Recall: $\mathrm{trop}(X)$ is the support of a Γ-rational polyhedral complex.

Now: X irred d-dim, $\mathrm{trop}(X)$ is pure d-dim

Prop Let Σ be a polyhedral complex with support $\mathrm{trop}(X)$. Then for all $\sigma \in \Sigma$, $\dim(\sigma) \leq \dim(X)$

Pf Let $I = I(X)$.

Now $\dim(I) = \dim(I_{\mathrm{proj}}) - 1$

$= \dim(\mathrm{in}_{(0,\omega)}(I_{\mathrm{proj}})) - 1$

$= \dim(\mathrm{in}_{(0,\omega)}(I_{\mathrm{proj}})|_{x_0=1})$

$\geq \dim(\mathrm{in}_\omega(I))$ (actually $=$, but pf is harder.)

Fix $\sigma \in \Sigma$ with $\dim(\sigma) = \lambda$. After a change of coords, we may assume that $\mathrm{aff}(\sigma) - \omega \subseteq \mathrm{span}(\underline{e}_1, \ldots, \underline{e}_\lambda)$, $\omega \in \sigma$ (ex: this exists)

Choose $w \in \operatorname{relint}(\sigma)$

Then $\forall v \in \operatorname{span}(e_1, e_l)$ $\exists \varepsilon > 0$

s.t. $\operatorname{in}_v(\operatorname{in}_w(I)) = \operatorname{in}_{w+\varepsilon' v}(I)$

$\forall 0 < \varepsilon' < \varepsilon$

$= \operatorname{in}_w(I)$.

(since $w + \varepsilon' v \in \sigma$)

Choose a generating set $\{f_i\}$ for $\operatorname{in}_w(I)$ so no summand of f_i lies in $\operatorname{in}_w(I)$

Then $\operatorname{in}_v(f_i) = f_i$ $\forall i$, for all $v \in \operatorname{span}(e_1, e_l)$.

[$f = 3x_1^2 + 5x_1^2x_2 + 7x_1^3 + 8x_1^5x_2 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$

$\operatorname{in}_{(1,0)}(f) = 3x_1^2 + 5x_1^2x_2$

Thus $f_i = x^u f_i'(x_{l+1}, x_n)$.

Thus $\operatorname{in}_w(I)$ has a generating set in $x_{l+1}, \dots, x_n$.

Thus $\dim(\operatorname{in}_w(I)) \geq l$,

so $l \leq \dim(X)$

This was $\dim(\operatorname{trop}(X)) \leq \dim(X)$.

To show equality.

first show $=$ on previous page.

$(\dim(I) = \dim(\operatorname{in}_w(I)))$

Idea in general: if σ is a maximal polyhedron of Σ, then as before assume $\operatorname{in}_w(I)$ is generated in x_{l+1}, x_n.

Let $J = \operatorname{in}_w(I) \cap \Bbbk[x_{l+1}^{\pm 1}, x_n^{\pm 1}]$

Then $\operatorname{trop}(V(J)) \subseteq \mathbb{R}^{n-l}$

$= \{0\}$ (otherwise there would be a polyhedron in Σ containing σ

Then (requires fundamental thm)

$V(J)$ is finite, so

$\dim(\operatorname{in}_w(I)) = l \Rightarrow l = d$.

(Really needed:

$\varphi : (K^*)^n \longrightarrow (K^*)^m$ $m < n$

and $X \subseteq (K^*)^n$ then

$\operatorname{trop}(\overline{\varphi(X)}) = \operatorname{trop}(\varphi)(\operatorname{trop}(X))$

$\subseteq$

Defn Let $\Sigma \subseteq \mathbb{R}^n$ be a (rational) polyhedral complex, & let σ be a polyhedron in Σ.

The star of σ in Σ is a (rational) polyhedral fan with cones indexed by $\tau \in \Sigma$ with σ a face of τ. Fix $w \in \operatorname{relint}(\sigma)$.

$\overline{\tau} = \{v \in \mathbb{R}^n : \exists \varepsilon > 0$ s.t. $w + \varepsilon' v \in \tau$ for all $0 < \varepsilon' < \varepsilon\}$

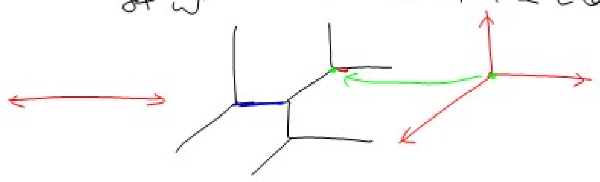
$+ \operatorname{aff}(\sigma) - w$

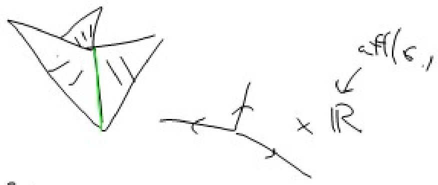
$\bar{\tau} = \{ v \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ s.t. } w + \varepsilon' v \in \tau$ for $0 < \varepsilon' < \varepsilon \}$

$\underbrace{\text{span}(v - w : v \in \tau)}$ $+ \text{aff}(\sigma) - w$

Minkowski sum.

This is independent of the choice of w





Lemma

Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, & let $w \in \text{trop}(V(I)) \cap \Gamma^n$.

Let Σ be a polyhedral complex with support $\text{trop}(V(I))$ & let $\sigma \in \Sigma$ be the smallest polyhedron containing w

Then

$$\text{trop}(V(\text{in}_w(I))) = |\text{star}_\Sigma(\sigma)|$$

support

Idea: $\text{in}_v(\text{in}_w(I)) = \langle 1 \rangle$

if & only if

$\text{in}_{w+\varepsilon' v}(I) = \langle 1 \rangle$

for small ε'.

Fix X irreducible, d-dim.

$w \in \text{relint}(\sigma)$, σ d-dim polyhedron in $\text{trop}(X)$. (with Gröbner complex structure).

As before, assume $\text{in}_w(I(X))$ is generated in $x_{d+1}, \dots, x_n$.

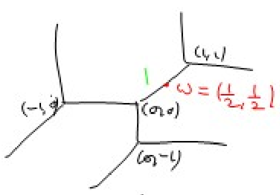
$J = \text{in}_w(I) \cap k[x_{d+1}, \dots, x_n]$.

$V(J)$ is a finite set of pts.

The multiplicity of σ is the number of those pts, counted with multiplicity.

(ie $\dim_k \dfrac{k[x_{d+1}, \dots, x_n]}{J}$)

eg $f = 2x^2 + xy + 2y^2 + x + y + 2$



$\text{in}_w(f) = x + y$

$V(x+y) = \{(a, -a) : a \in k^*\}$

$\cong 1$ copy of k^*

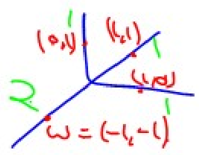
Change coords φ^*: $z_2 = x + y$

$z_1 = y$

$\varphi^{*-1}(\text{in}_{(\frac{1}{2}, \frac{1}{2})}(f)) = z_2$

$\dim_k k[z_2] / \langle z_2 \rangle = 1$.

eg $f = x^2 + y^2 + x + y \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$



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$\mathrm{in}_{(-1,-1)}(f) = x^2 + y^2$

$V(x^2+y^2) \subseteq (\mathbb{C}^*)^2$

$= V(x+iy) \cup V(x-iy).$

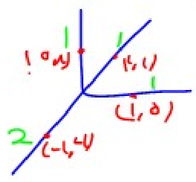
2 copies of $\mathbb{C}^*$ $\Big\}$ $V(z_2(z_2-1))$

mult $= 2$.

$w = (1,0)$ $\mathrm{in}_w(f) = y^2 + y$

$V(y^2+y) = V(y+1) = \{(a,-1) : a \in \mathbb{C}^*\}$

mult $= 1$.



Point:

$1\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} + 2\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$

<u>Defn</u> Let Σ be a one-dimensional weighted rational polyhedral fan. Then Σ is <u>balanced</u> if

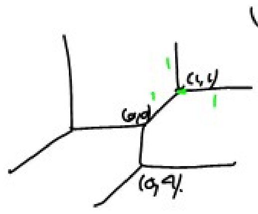
$$\sum \underbrace{w_i}_{\text{weight on ith ray}} \underbrace{u_i}_{\text{first lattice pt on the ith ray}} = \underline{0} \leftarrow \mathbb{Z}^n$$

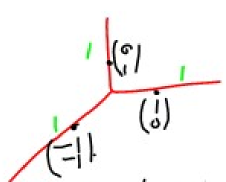
If Σ is a Γ-rational polyhedral complex pure of dim d, then Σ is balanced at a $(d-1)$-dim polyhedron σ if

$$\mathrm{star}_\Sigma(\sigma) \Big/ \underbrace{\mathrm{aff}(\sigma) - w}_{\text{lineality space}} \quad w \in \sigma$$

is balanced when we inherit weights from Σ.

Σ is balanced if it is balanced for all $(d-1)$-dim σ.





$1\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} + 1\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$

<u>Theorem:</u>

With these multiplicities $\mathrm{trop}(X)$ is balanced.