Last time:

Fundamental theorem:

TFAE:

1) $w \in \operatorname{trop}(X)$

2) $w \in \operatorname{cl}(\{w \in \Gamma^n : \operatorname{in}_w(I) \neq \langle 1 \rangle\})$

3) $w \in \operatorname{cl}(\operatorname{Val}(X(K)))$

We showed $1 \Leftrightarrow 2$.

$3 \Rightarrow 1$.

Structure Theorem:

$\operatorname{trop}(X)$ is the support of a polyhedral complex

(Pf: $I \rightsquigarrow \underset{\substack{\uparrow \\ \text{depends on the} \\ \text{choice of coords}}}{I_{proj}} \subseteq K[x_0, \ldots, x_n]$)

Idea of proof

1) First prove for hypersurfaces

2) Project general case to hypersurface case.

Hypersurface proof

Let $X = V(f) \subseteq (K^*)^n$, and let $\underset{(k^*)^n}{\alpha} \in V(\operatorname{in}_w(f))$. We want to show that $\exists y \in X$ with $\operatorname{val}(y) = w$ and $\overline{t^{-w} y} = \alpha$

$(\overline{t^{-w_1} y_1}, \ldots, \overline{t^{-w_n} y_n})$

The proof is by induction on n.

Suppose first $n = 1$.

Write $f = \sum_{\substack{i=0 \\ c_0, c_s \neq 0}}^{s} c_i x^i = \prod_{j=1}^{s} (a_j x - b_j)$

So $\operatorname{in}_w(f) \underset{\substack{\uparrow \\ \text{easy} \\ \text{ex.}}}{=} \prod_{j=1}^{s} \operatorname{in}_w(a_j x - b_j)$

So if $w \in \operatorname{trop}(V(f)) \cap \Gamma$

then $\operatorname{in}_w(a_j x - b_j) = \overline{a_j t^{-\mathrm{v}}} x - \overline{t^{-\mathrm{v}} b_j}$ for some j.

Since $\alpha \in V(\operatorname{in}_w(f))$, we can choose j with $\overline{a_j t^{-\mathrm{v}}} \alpha = \overline{t^{-\mathrm{v}} b_j}$

For this j we must have

$\operatorname{val}(a_j) + w = \operatorname{val}(b_j)$

Set $y = \frac{b_j}{a_j}$. Then $y \in V(f)$ and $\operatorname{val}(y) = w$ and $\overline{t^{-w} y} = \alpha$

Now assume that $n > 1$ & the Theorem is true for $n-1$.

<u>Case 1</u>: $\operatorname{in}_w(f)\big|_{x_n = \alpha_n} \neq 0$

ex: always poss. Pick $\underset{y_n}{y} \in K$ with $\operatorname{val}(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$.

Set $g(x_1, \ldots, x_{n-1}) \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ to be $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, y_n)$

$g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, y_n)$. $f = \sum c_u x^u$

Then $g = \sum_{u' \in \mathbb{Z}^{n-1}} d_{u'} x^{u'}$

$$d_{u'} = \underbrace{\sum_j c_{(u',j)} y_n^j}_{\text{Case I: These are not all zero}}$$

Let $w' = (w_1, \ldots, w_{n-1})$.

Then (Ex!)

$$in_{w'}(g) = in_w(f)(x_1, \ldots, x_{n-1}, \alpha_n)$$

So $in_{w'}(g)(\alpha_1, \ldots, \alpha_{n-1}) = 0$

Since $(\alpha_1, \ldots, \alpha_{n-1}) \in (k^*)^{n-1}$, $in_{w'}(g)$ is not a monomial, so $w' \in trop(V(g))$, so by induction $\exists\, y_1, \ldots, y_{n-1}$ with $val(y_i) = w_i$, $\overline{y_i t^{-w_i}} = \alpha_i$, and $g(y_1, \ldots, y_{n-1}) = 0$

So y has the desired form.

Case II $in_w(f)(x_1, \ldots, x_{n-1}, \alpha_n) = 0$.

Idea: Change coords so that this is not the case.

Q, How does changing coords affect $\mathrm{trop}(X)$?

A, Let $\varphi : (K^*)^n \longrightarrow (K^*)^n$ be an automorphism, so

$\varphi^* : K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \longrightarrow K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

is an isomorphism where

$\varphi^*(x_i) = x^{a_i}$ $\quad a_i \in \mathbb{Z}^n$

(If $x_i \longmapsto f \leftarrow$ not mono, then for $\alpha \in (K^*)^n$ $f(\alpha) = 0$ we would have $\varphi(\alpha)_i = 0$)

Claim: $\mathrm{trop}(\overline{\varphi(X)}) = A\,\mathrm{trop}(X)$

$\begin{pmatrix} - a_1 - \\ \vdots \\ - a_n - \end{pmatrix}$

eg $f = x + y + 1$.

$\varphi : (K^*)^2 \longrightarrow (K^*)^2$

given by $\varphi^* : K[x^{\pm 1}, y^{\pm 1}] \to K[x^{\pm 1}, y^{\pm 1}]$

$\varphi(x,y) = (x^3y^4, x^2y^3)$ $\quad \varphi^*(x) = x^3y^4 \quad \varphi^*(y) = x^2y^3 \quad A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \in GL(2, \mathbb{Z})$

(φ is an automorphism: $\varphi^{*-1}(x) = x^3/y^4 \quad \varphi^{*-1}(y) = y^3/x^2$)

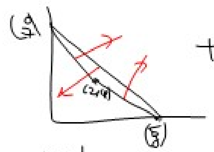
$\overline{\varphi(X)} = V(\varphi^{*-1}(f)) = \{y \in (K^*)^2 : \varphi^{*-1}(f)(y) = 0$, i.e. $f \circ \varphi^{-1}(y) = 0\} = \{y : \varphi^{-1}(y) \in V(f)\} = \{y : y \in \varphi(X)\}$

$= V\left(\frac{x^3}{y^4} + \frac{y^3}{x^2} + 1\right)$

$\varphi^* f = f \circ \varphi$

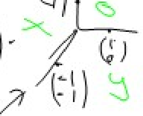
$\overline{\varphi(X)} = V(x^5 + y^7 + x^2y^4)$

Hassett algebraic Geometry

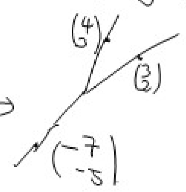


$\mathrm{trop}(V(x^5 + y^7 + x^2y^4))$

$=$



$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \cdot \longrightarrow$



$\mathrm{trop}(x^5 + y^7 + x^2y^4) = \min(5x, 7y, 2x + 4y)$

$x + y + 1$

$\mathrm{trop}(x + y + 1) = \min(x, y, 0)$

Note:

$A = \mathrm{trop}(\varphi)$

Claim: $\mathrm{trop}(\varphi(X)) = A\,\mathrm{trop}(X) = \mathrm{trop}(\varphi)(X)$

Pf Since φ^{-1*} is given by

$\varphi^{-1}(x)_i = \prod_{j=1}^{n} x_j^{(A^{-1})_{ij}}$

it suffices to prove $\supseteq$.

For this, it suffices to show that if $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ then

$\varphi^*(\mathrm{in}_{Aw}(\varphi^{*-1}(I)) \subseteq \mathrm{in}_w(I)$

Then if $\mathrm{in}_w(I) \neq \langle 1 \rangle$, then $\mathrm{in}_{Aw}(\varphi^{*-1}(I)) \neq \langle 1 \rangle$.

This follows ~~from~~

$\varphi^*(\mathrm{in}_{Aw}(f)) = \mathrm{in}_w(f)$

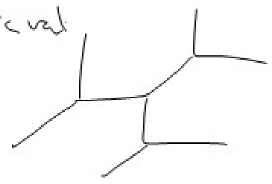
$f \in I$

Drawing plane curves

(and hypersurfaces)

eg $f = 4x^2 + 5xy + 6y^2 + 3x - y + 8 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$

eg 2-adic val

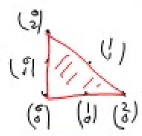


<u>Defn</u> Let $f = \sum c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

The Newton polytope of f is

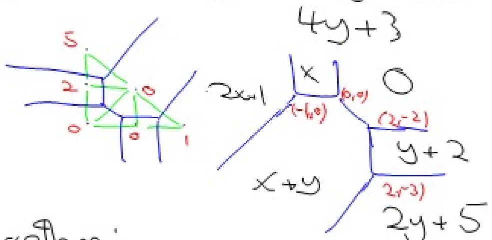
$\mathrm{conv}(u : c_u \neq 0) \subseteq \mathbb{R}^n$

eg $\mathrm{Newt}(f) =$



---

eg $2x^2 + xy + 32y^2 + x + 4y + 3$



Algorithm:

1) Form $\mathrm{conv}((u, \mathrm{val}(c_u)) \subseteq \mathbb{R}^3$

2) Project the lower faces to get a polyhedral complex with support $\mathrm{Newt}(f)$

3) Draw the dual graph
   vertex for each triangle
   edge for each edge

4) Flip $(x, y) \mapsto (-x, -y)$

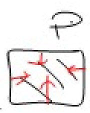
<u>Prop</u> Output is $\mathrm{trop}(V(f))$.

<u>Defn</u> For a polyhedron $P \subseteq \mathbb{R}^n$, The normal fan $\mathcal{N}(P)$ is the polyhedral fan is the can for which $\sigma \in \mathcal{N}(P)$, $w, w' \in \mathrm{relint}(\sigma)$, then $\mathrm{face}_w(P)$

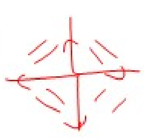
$= \mathrm{face}_{w'}(P)$

$= \{y \in P : w' \cdot y \leq w' \cdot x \ \forall x \in P\}$

eg



$\mathcal{N}(P)$



<u>Defn</u> Fix $\underline{u}_1, \dots, \underline{u}_s \in \mathbb{R}^n$, $v \in \mathbb{R}^s$.

The regular subdivision of $\mathrm{conv}(u_i)$ with respect $\underline{v}$ is the projection to $\mathbb{R}^n$ of the lower faces of $\mathrm{conv}((\underline{u}_i, v_i) : 1 \leq i \leq s) \subseteq \mathbb{R}^{n+1}$

$\mathrm{face}_{(w,1)}(\ \text{"}\ )$