graduate-studies@maths.ox.ac.uk

D.Maclagan@warwick.ac.uk

Tropical geometry is algebraic geometry over the tropical semiring

$\mathbb{R}\cup\{\infty\}$

$a \oplus b = \min(a,b)$ $3 \oplus 5 = 3$

$a \otimes b = a+b$ $5 \otimes 7 = 12$

Tropical addition/mult is assoc, distributative:

$3 \otimes (5 \oplus 7) = 3\otimes 5 \oplus 3 \otimes 7$
$= 8$

0 is the mult. identity

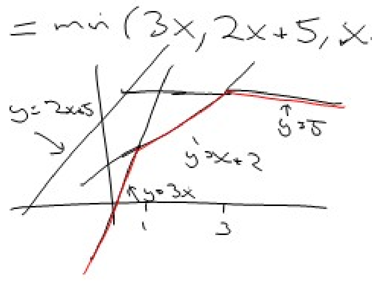
∞ is the add. identity

ie we have all the ring axioms except subtraction: semiring.

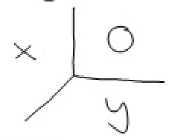
(older notation: max-plus semiring)

Tropical polynomials are piecewise linear fcns

eg $x^3 \oplus 5 \otimes x^2 \oplus 2 \otimes x \oplus 5$
$= \min(3x, 2x+5, x+2, 5)$



eg $x \oplus y \oplus 0 = \min(x,y,0)$



In classical algebraic geometry we start with hypersurfaces

$X = V(f) = \{x : f(x) = 0\}$

Problem: What does "$=0$" mean?

eg $3 \oplus x = 5$ no soln?

Solution: for a tropical polynomial f, the tropical hypersurface

$V(f) =$ the nonlinear locus of f.

eg $f = x^3 \oplus 5 \otimes x^2 \oplus 2 \otimes x \oplus 5$

$V(f) = \{1, 3\}$

$(x \oplus 1)^2 \otimes (x \oplus 3)$
$= x^3 \oplus x^2 \oplus 2 \otimes x \oplus 5$

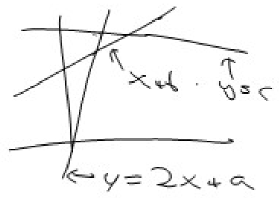
$= f$ as a fcn from $\mathbb{R} \to \mathbb{R}$

Tropical quadratic formula

$a \otimes x^2 \oplus b \otimes x \oplus c$

$x^2 = x \otimes x$

$a \otimes x^2 \oplus b \otimes x \oplus c$



Solns:

$(c-b, b-a)$

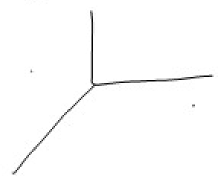
if

$2b \leq a+c$

$\frac{c-a}{2}$ o/w

eg $f = x \oplus y \oplus 0$

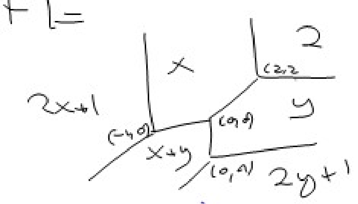
$V(f)$



eg $F = 1 \otimes x^2 \oplus x \otimes y \oplus 1 \otimes y^2$

$\oplus x \oplus y \oplus 2$

$V(F) =$

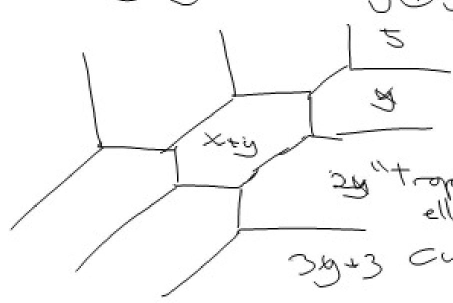


F=

eg $3 \otimes x^3 \oplus x^2 \otimes y \oplus x \otimes y^2$

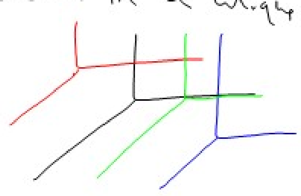
$\oplus 3 \otimes y^3 \oplus x^2 \oplus (-1) \otimes x \otimes y$

$\oplus y^2 \oplus x \oplus y \oplus 5$

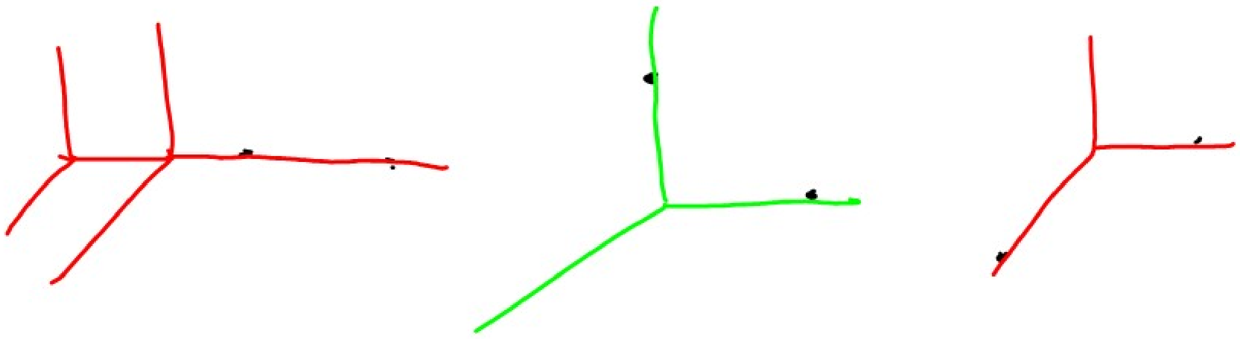


"tropical elliptic curve"

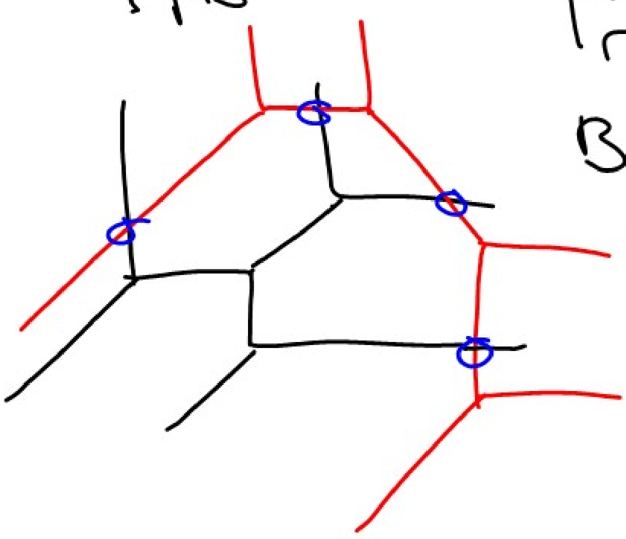
Two general tropical lines intersect in a unique pt.



There is a unique tropical line
through two general pts



(general)
Two tropical quadrics intersect
in 4 pts



Tropical
Bézout

Why?

Tropical geometry is a combinatorial shadow of classical algebraic geometry.

One can compute invariants of a variety from its tropicalization.

Next hour:

1) More motivation (enumerative geometry)
2) Tropicalizing varieties.

Why?

Fix $3d-1$ general pts $p_1, \dots, p_{3d-1}$ in $\mathbb{P}^2$ and ask how many rational curves of degree d pass through $p_1, \dots, p_{3d-1}$?

eg $d=1$ How many lines pass through 2 general pts?

Let N_d be the # of such curves.

$N_1 = 1$

important $N_2 = 1$ there is conics through 5 general pts.

Ex

$N_3 = 12$ (Steiner 1848)

$N_4 = 620$ (Zeuthen 1873)

$N_5 = 87304$

1994 Kontsevich

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}$$

Mikhalkin (2005?)

The numbers N_d can be computed tropically.

$\parallel$ They equal the number of $=$ tropical rational curves of degree d passing through $3d-1$ general pts in $\mathbb{R}^2$, counted with tropical multiplicity

Other applications / Connections

- Mirror symmetry
- Analytic geometry / Berkovich spaces
- Moduli spaces
- Computational aspects

- Combinatorics
- Real algebraic geometry
- Other applications to biology, statistics, ...

Cox, Little, O'Shea
Ideals, varieties & algorithms

Fix an alg. closed field K with a valuation $\mathrm{val}: K^* \to \mathbb{R}$, where $K^* = K \setminus \{0\}$

1) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$
2) $\mathrm{val}(a+b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$

Ex: If $\mathrm{val}(a) \neq \mathrm{val}(b)$ then $\mathrm{val}(a+b) = \min(\mathrm{val}(a), \mathrm{val}(b))$

eg $K = \mathbb{C}$ $\quad \mathrm{val}(a) = 0 \quad a \neq 0$

eg $K = \overline{\mathbb{Q}}$ $\quad$ p-adic val on $\mathbb{Q}$ $\quad \mathrm{val}_p(p^n \frac{a}{b}) = n$ $\quad p \nmid a, b$

$\mathrm{val}_2(4) = 2$
$\mathrm{val}_3(\frac{7}{6}) = -1$

eg $K = \overline{\mathbb{C}(t)}$ $\quad \mathrm{val}(\frac{f}{g}) = \text{lowdeg } f - \text{lowdeg } g$
$= n$ if $\frac{f}{g} = t^n \frac{\tilde{f}}{\tilde{g}}$ ← non zero constant term

eg $K = \mathbb{C}\{\{t\}\}$ Puiseux series
$= \bigcup_{n \geq 1} \mathbb{C}((t^{\frac{1}{n}}))$ ← Laurent series

Elements are Laurent series with rational exponents with a common denom

$a = 3t^{-\frac{1}{2}} + 5t^2 + 8t^{\frac{11}{3}} + \cdots$

$\mathrm{val}(a) = -\frac{1}{2}$

Ex: Check these are valuations

Notation: $\Gamma = \mathrm{im}\, \mathrm{val} \subseteq \mathbb{R}$

If val is not trivial, then we'll assume $1 \in \Gamma$.

<u>Defn</u> Let $S = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of Laurent polynomials

$f = \sum_{u \in \mathbb{Z}^n} c_u x^u \quad c_u \in K, \; x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$

eg $3x_1^3 + 2\frac{x_8}{x_7^2} + 8x_9^9$

<u>Defn</u> The tropicalization of $f = \sum c_u x^u \in S$ is

$$\mathrm{trop}(f)(w) = \min\left(\mathrm{val}(c_u) + w \cdot u\right)$$

$w \cdot u = w_1u_1 + w_2u_2 + \dots + w_nu_n$

addition $\rightsquigarrow$ trop. add.

mult $\rightsquigarrow$ trop mult

coeff $\rightsquigarrow$ valuation

eg $f = 6x^2 + 7xy - 4y^2 - 3x + 11y + 8 \in \overline{\mathbb{Q}}[x^{\pm1}, y^{\pm1}]$

2-adic val.

$$\mathrm{trop}(f) = \min(2x+1, x+y, 2y+2, x, y, 3)$$

eg $f = t^3x^3 + (5t + 3t^2)y^2 + t^{11} \in \mathbb{C}\{\{t\}\}[x^{\pm1}, y^{\pm1}]$

$$\mathrm{trop}(f) = \min(3x+3, 2y+1, 11)$$

<u>Defn</u> The tropical hypersurface

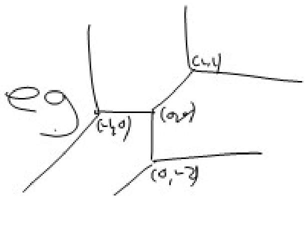
$$\mathrm{trop}(V(f)) := V(\mathrm{trop}(f))$$

(classical hypersurface: $V(f) = \{y \in (K^*)^n : f(y) = 0\}$)

ie $\mathrm{trop}(V(f))$ = nonlinear locus of $\mathrm{trop}(f)$

= locus where the min is achieved at least twice

eg



<u>Defn</u> for $I \subseteq K[x_1^{\pm1}, \dots, x_n^{\pm1}]$

$$V(I) = \{y \in (K^*)^n : f(y) = 0 \ \forall f \in I\}$$

<u>Defn</u> For a variety $X = V(I)$ the tropicalization is

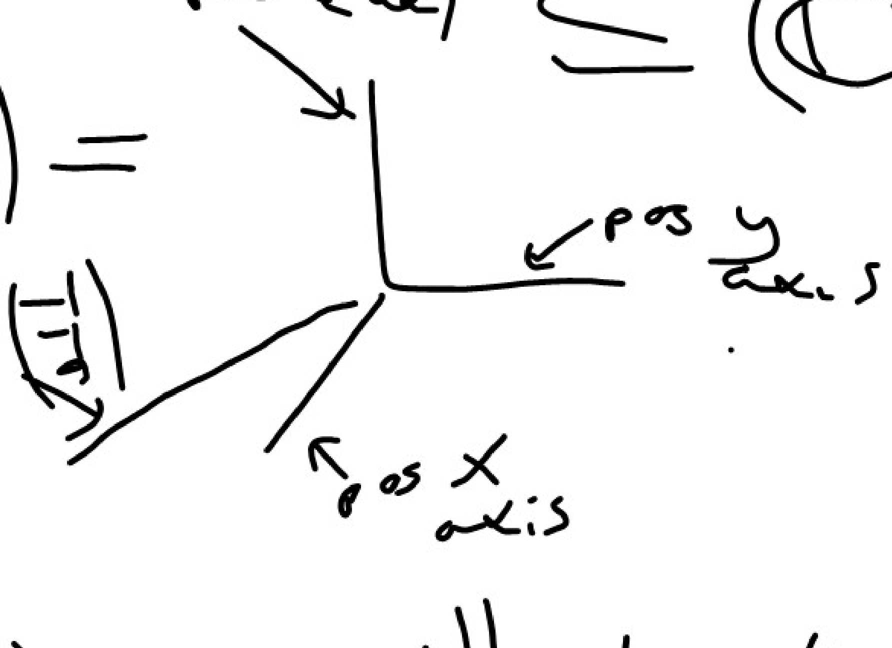
$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f))$$

Warning:

If $I = \langle f_1, \dots, f_r \rangle$, we might not have $\mathrm{trop}(X) = \bigcap_{i=1}^{r} \mathrm{trop}(V(f_i))$

eg $X = V(x+y+z+1,$
$x+2y+3z)$

$\mathrm{trop}(X) = \quad \subseteq (\mathbb{C}^*)^3$



$\mathrm{trop}(V(x+y+z+1)) \cap \mathrm{trop}(V(x+2y+3z))$

$\ni \quad (-1,-1,0) \notin \mathrm{trop}(X)$