MA5Q6 GRADUATE ALGEBRA - HOMEWORK 8

DUE WEDNESDAY, 19 DECEMBER

Hand in the first five questions **to my pigeon-hole**. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(1) Let G be a finite group and let $\mathbb{k}$ be an algebraically closed field with char($\mathbb{k}$) not dividing $|G|$. The *regular representation* of G has act on $\mathbb{k}^{|G|}$ by permutation of basis elements (ie $g \cdot \mathbf{e}_h = \mathbf{e}_{gh}$). Give the decomposition of $\mathbb{C}^{|G|}$ into simple $\mathbb{k}[G]$-modules.

(2) Let G be the dihedral group of order 8.
   (a) How many conjugacy classes does G have?
   (b) How many irreducible representations does G have over $\mathbb{C}$?
   (c) Compute the character table of G.

(3) Repeat the previous question for the quaterion group Q_8.

(4) Compute the Galois group Gal$(L/\mathbb{Q})$ where L is the splittingfield of $x^4 - 2$. Draw the lattices of subgroups and intermediate fields, and note which ones are normal. Be sure to justify that anything you claim is an automorphism of L.

(5) Compute the Galois group Gal$(L/\mathbb{Q})$ where L is the splittingfield of $x^4 - 3x^2 + 1$. Draw the lattices of subgroups and intermediate fields, and note which ones are normal. Be sure to justify that anything you claim is an automorphism of L.

(6) (Not to be handed in) It follows from the formula $\sum_{i=1}^{r} n_i^2 = |G| = r$ that all irreducible representations of an abelian group over an algebraically closed field are one-dimensional. Prove this directly (Hint: commutating matrices can be simultaneously diagonalized).

(7) (Not to be handed in). Read the proofs that the rows and columns of the character table are orthogonal in an appropriate sense. Use this to help you compute the character table of S_4. Can you determine the corresponding irreducible representations?